Dual parametrization of GPDs v.s. the Mellin-Barnes transform approach and the $J = 0$ fixed pole saga

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Outline

- Introduction
- Conformal PW expansion: Mellin-Barnes techniques and the dual parametrization
- Basic facts on the dual parametrization and MB techniques.
- Abel transform tomography and the Froissart-Gribov projection.
- Analyticity and the story of $J = 0$ "fixed pole" contribution
- Conclusions and outlook

M. Polyakov, A. Shuvaev, arXiv: hep-ph/0207153 (2002)

M. Polyakov, Phys. Lett. B **659**, 542 (2008)

K. S., Eur. Phys. J. A **36**, 303 (2008)

M. Polyakov, K. S.,Eur. Phys. J. A **40**, 181 (2009)

K. S., Eur. Phys. J. A **45**, 217,2 (2010).

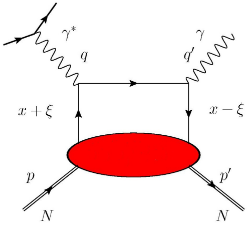
D. Müller, M. Polyakov and K.S., JHEP 1503, 052 (2015).

Deeply Virtual Compton Scattering and GPDs

Unpolarized nucleon quark GPDs

$$\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{i\lambda x P^+} \langle N_p(P+\frac{\Delta}{2}) | \bar{\psi}_q(-\lambda n/2) n_\mu \gamma^\mu \, [-\lambda n/2, \, \lambda n/2] \, \psi_q(\lambda n/2) | N_p(P-\frac{\Delta}{2}) \rangle$$

$$= \frac{1}{P^+} \bar{U}(P+\frac{\Delta}{2}) \left[H^q(x,\xi,t;\,\mu^2) n_\mu \gamma^\mu + \frac{1}{2M_N} E^q(x,\xi,t;\,\mu^2) i\sigma^{\mu\nu} n_\mu \Delta_\nu \right] U(P-\frac{\Delta}{2}) \, .$$



- The theoretical opportunity to access GPDs experimentally is provided by the collinear factorization theorems for high-Q^2 electroproduction processes.
- DVCS is one of the simplest processes that can be described in terms of GPDs.
- Kinematics: $Q^2 = -q^2$; $P = \frac{p+p'}{2}$; $\Delta = p' - p$; $t = \Delta^2$;

$$x_{Bj} = \frac{Q^2}{2p \cdot q}; \quad \xi = -\frac{\Delta \cdot n}{2P \cdot n}$$

- Studies of GPDs are the subject of experiments (HERA, HERMES, JLab Hall A, B, COMPASS, PANDA, EIC).

Motivation to study GPDs

1. Unique information of the energy-momentum tensor in QCD

$$\langle N(p')|\bar{\psi}\gamma_{\{\mu}(i\overleftrightarrow{\partial} + gA)_{\nu\}}\psi + \frac{1}{4}F^a_{\mu\alpha}F^a_{\nu\alpha}|N(p)\rangle$$

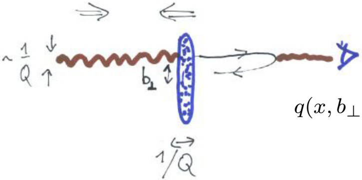
- Ji's sum rule (Ji'97):

$$\int_{-1}^{1} dx x(H^q(x,\xi,t=0) + E^q(x,\xi,t=0)) = 2J^q(0)\,.$$

- Shear forces inside nucleon (M. Polyakov'02): d_1

$$\int_{-1}^{1} dx x H^q(x,\xi,t) = M_2^q(t) + \frac{4}{5}d_1^q\xi^2.$$

2. Hadron imaging in the transverse plane (Burkardt'00):



$$q(x,b_\perp) = \lim_{\xi\to 0}\int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{i\Delta_\perp\cdot b_\perp} H(x,\xi,\Delta_\perp)$$

A note on GPD parametrization

- GPD modelling can be done in various representations: (DD representation, conformal PW expansions, expansions over orthogonal polynomials,...)

List of non-trivial requirements:

- polynomiality
- hermiticity
- T-invariance
- positivity

Other sources of inspiration:

- evolution properties
- relation to PDFs and FFs
- analyticity
- Regge theory insight

- Should be possible to map one representation to another (as long as basic properties are satisfied).
- "Which representation is better is not a meaningful question!" (see K. Kumerički & D. Müller'09).
- The hope: get more insight from considering various GPD properties within different representations.

Conformal PW expansion for GPDs I

- Idea: expand GPDs over the conformal basis (factorization of functional dependencies)
- Main advantage: trivial solution of the LO evolution equations.

- Conformal moments of quark GPDs are defined with respect to $c_n(x,\eta) = N_n \times \eta^n C_n^{\frac{3}{2}}\left(\frac{x}{\eta}\right)$; Normalization: $\lim_{\eta \to 0} c_n(x,\eta) = x^n$.

$$H_n(\eta,t) = \int_{-1}^{1} dx\, c_n^{\frac{3}{2}}\left(\frac{x}{\eta}\right) H(x,\eta,t)\,.$$

- $c_n(x,\eta)$ form a complete basis in $[-\eta,\ \eta]$ with the weight $\left(1-\frac{x^2}{\eta^2}\right)$.
- $p_n(x,\eta)$ include the weight and θ to ensure the support:

$$p_n(x,\eta) = \eta^{-n-1}\theta\left(1-\frac{x^2}{\eta^2}\right)\left(1-\frac{x^2}{\eta^2}\right) N_n^{-1}\frac{(n+1)(n+2)}{2n+3} C_n^{\frac{3}{2}}\left(-\frac{x}{\eta}\right)\,.$$

- Orthogonality of the basis: $$\int_{-1}^{1} dx\, p_n(x,\eta) c_n(x,\eta) = (-1)^n \delta_{mn}$$

Conformal PW expansion for GPDs II

Conformal PW expansion for GPDs:

$$H(x,\eta,t) = \sum_{n=0}^{\infty} p_n(x,\eta) H_n(\eta,t)\,.$$

- Allows to factorize x, η and t dependence of GPDs.
- Scale dependence of the conformal moments is simply multiplicative:

$$H_n(\eta,t,\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{\gamma_{0n}}{2\beta_0}} H_n(\eta,t,\mu_0)\,.$$

- Conformal moments are reproduced by this series.
- Restricted support property $\nRightarrow$ GPD vanishes in the outer region.
- The expansion is to be understand as an ill-defined sum of generalized functions.

Different ways to assign meaning to conformal PW expansion

1. Sommerfeld-Watson transform + Mellin-Barnes integral techniques D. Müller and A. Schäfer'05; A. Manashov, M. Kirch and A. Schafer'05;
2.
   - Shuvaev transform A. Shuvaev'99, J. Noritzsch'00;
   - Dual parametrization of GPDs M. Polyakov and A. Shuvaev'02;

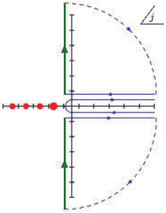
Inverse Mellin transform

$$M(n)=\int_0^\infty dx x^n f(x);\quad f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds x^{-n-1} M(n).$$

- Sommerfeld-Watson transform:

$$H(x,\eta,t)=\frac{1}{2i}\oint_{(0)}^{(\infty)} dj\,\frac{1}{\sin\pi j}\,p_j(x,\eta)\,H_j(\eta,t)\,.$$

- Residue theorem leads to conformal P.W. expansion $(\mathrm{Res}_{j=n}\frac{1}{\sin\pi j}=\frac{(-1)^j}{\pi})$.



- For $\eta=0$ p_j form the integral kernel for the inverse Mellin transform
- Asymptotic behavior of $p_j(x,\eta)$, H_j -?
- Integral over the large arc must vanish.

Main ingredients

- Schläfli integral for conformal PWs

$$p_j(x,\eta) = -\frac{\Gamma(5/2+j)}{\Gamma(1/2)\Gamma(2+j)} \frac{1}{2i\pi} \oint_{-1}^{1} du \, \frac{(u^2-1)^{j+1}}{(x+u\eta)^{j+1}} \,,$$

$p_j(x,\eta)$ are expressed through ${}_2F_1$ hypergeometric function.
Asymptotic behavior of for $j \to \infty$ is under control.

- Carlson's theorem

- Mellin-Barnes integral representation for GPDs:

$$H(x,\eta,t) = \frac{i}{2} \int_{c-i\infty}^{c+i\infty} dj \, \frac{1}{\sin \pi j} p_j(x,\eta) H_j(\eta,t) \,.$$

Starting point for D. Müller et al.

The basis for the Shuvaev transform & the dual parametrization

- How to restore $f(x)$ from its Mellin moments $M_n = \int dx x^n f(x)$?

- Formal solution:

$$f(x) = \sum_{n=0}^{\infty} M_n \delta^{(n)}(x) \frac{(-1)^n}{n!} \,.$$

✓ A trick: $\delta^{(n)}(x) = \frac{(-1)^n n!}{2\pi i} \left[\frac{1}{(x - i\epsilon)^{n+1}} - \frac{1}{(x + i\epsilon)^{n+1}} \right].$

Define $F(z) = \sum_{n=0}^{\infty} \frac{M_n}{z^{n+1}}$; then $f(x) = \frac{1}{2\pi i} \left[F(x - i\epsilon) - F(x + i\epsilon) \right]$.

Idea of the Shuvaev transform (see A. Shuvaev'99, J. Noritzsch'00):

- Introduce $f_\eta(y)$ whose Mellin moments generate Gegenbauer moments of GPD:

$$\int_0^1 dy y^n f_\eta(y) = H_n(\eta)$$

- One can explicitly construct the kernel $K(x, \eta; y)$ such that

$$H(x, \eta) = \int_0^1 dy \, K(x, \eta; y) \, f_\eta(y) \,.$$

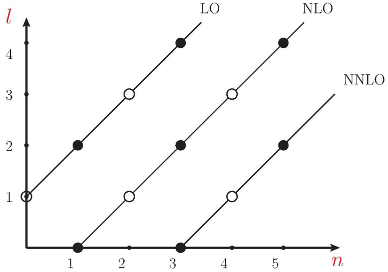
Dual Parametrization (M. Polyakov, A. Shuvaev'02):

- Mellin moments expanded in a set of suitable orthogonal polynomials. E.g. partial waves of the t-channel (t-channel refers to $\bar{h}h \to \gamma^*\gamma$):

$$N_n^{-1}\frac{(n+1)(n+2)}{2n+3}H_n(\eta,t) = \eta^{n+1}\sum_{l=0}^{n+1} B_{nl}(t) P_l\left(\frac{1}{\eta}\right)$$

Conformal PW expansion is then rewritten as:

$$H(x,\eta,t) = \sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{\substack{l=0\\ \text{even}}}^{n+1} B_{nl}(t)\,\theta\left(1-\frac{x^2}{\eta^2}\right)\left(1-\frac{x^2}{\eta^2}\right) C_n^{\frac{3}{2}}\left(\frac{x}{\eta}\right) P_l\left(\frac{1}{\eta}\right)$$



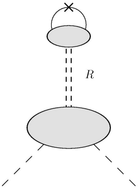
- Polynomiality implemented via Wigner-Ekkart theorem ($l \leq n+1$).
- Discrete symmetries (C, T) through the selection rules for l^{PC} (X. Ji, R. Lebed'01).
- Generalized FFs $B_{nl}(t)$ are renormalized multiplicatively.

t-channel point of view and duality

- Conformal PW expansion converges for $\eta > 1$.
- By means of the crossing relation one gets conformal PW expansion for two particle GDAs.

$$\frac{x}{\eta} \leftrightarrow 1-2z; \quad \frac{1}{\eta} \leftrightarrow 1-2\zeta; \quad t \leftrightarrow W^2$$

- Duality in the spirit of R. Dolen, D. Horn, C. Schmid'67. GPDs are presented as infinite series of t-channel Regge exchanges M. Polyakov'98:



$$\langle \pi(p')| \hat{O} |\pi(p)\rangle \sim \text{ Crossing of } \sum_{R_J} \sum_{\substack{\text{polarization}\\ \text{of } R_J}} \frac{1}{t-M_{R_J}^2}$$

$$\times \underbrace{\langle \pi(p')\pi(-p)|R_J\rangle}_{R_J\pi\pi \text{ effective vertex}} \underbrace{\langle R_J| \hat{O} |0\rangle}_{\text{F.T. of DA of } R_J} .$$

- Expansion in the t-channel PW:

$$\cos\theta_t = \frac{s-u}{\sqrt{1-\frac{4\,m^2}{t}}\,(\mathcal{Q}^2+t)} = -\frac{1}{\eta\sqrt{1-\frac{4m^2}{t}}} + O(\frac{1}{\mathcal{Q}^2}),$$

Dual parametrization: summing up the formal series

- Same idea as the Shuvaev transform: Mellin moments of $Q_k(y,t)$ generate the generalized F.Fs. B_{nl}.

$$B_{n\ n+1-2\nu}(t) = \int_0^1 dy y^n Q_{2\nu}(y,t)\,.$$

Then $\quad H(x > -\eta, \eta, t) = \sum_{\nu=0}^{\infty} \int_0^1 dy K_{2\nu}(x,\eta,y) y^{2\nu} Q_{2\nu}(y,t)\,.$

How to construct the convolution kernels?

- M. Polyakov and A. Shuvaev'02 (see also M. Polyakov and KS'08):

$$K_{2\nu}(x,\eta,y) = \mathrm{disc}_{z=x} F^{(2\nu)}(z,\eta,y)\,, \qquad \text{where}$$

$$F^{(2\nu)}(z,\eta,y) = \frac{1}{y^{2\nu+1}} \left(1 + y\frac{\partial}{\partial y}\right) \int_{-1}^{1} ds \eta^k \frac{z_s^{1-k}}{\sqrt{z_s^2 - 2z_s + \eta^2}}\,, \qquad z_s \equiv \frac{2(z-\eta s)}{(1-s^2)y}$$

Two ways to compute the discontinuity:

1. Expand in powers of $\frac{1}{z_s}$ and employ Rodriguez formula for Gegenbauer polynomials $\Rightarrow$ formally recover conformal PWE for GPD.
2. Consider the discontinuity due to the cut $1-\sqrt{1-\eta^2} < z_s < 1+\sqrt{1-\eta^2}$ (and from poles at $z_s = 0$ for $k \geq 2$) $\Rightarrow$ analytical expressions for the convolution kernels in terms of elliptic integrals.

- GPDs satisfy polynomiality property and the support property.
- The D-term is the natural ingredient of the dual parametrization.
- Scale dependence of $Q_k(x)$ is given by DGLAP equations.
- $Q_0(x)$ is fixed in terms of (t-dependent) PDFs:

$$Q_0(x) = q(x) + \bar{q}(x) - \frac{x}{2} \int_x^1 \frac{dy}{y^2} \left(q(y) + \bar{q}(y)\right);$$

- $Q_2(x)$ contains FFs of the EMT (J^q, shear forces)
- x-dependence of forward like functions should implement the insight from the Regge theory
- A principle allowing to take into account only a finite number of conformal PWs (i.e. Q_ks)?

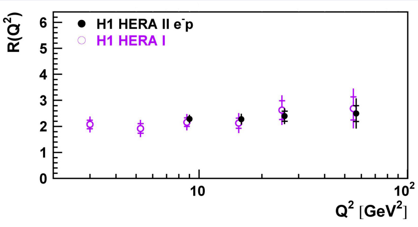
Minimalist model and skewness effect

- Consider the "minimalist model": include just $Q_0(x)$
- Assume that $q(x) \sim 1/x^{\alpha^q}$.

Skewness effect in the "minimalist" dual model equals conformal ratio (K. Kumericki, D. Mueller and K. Passek-Kumericki'08, 09)

$$r^q_{Q_0} \equiv \frac{H^q(\xi,\xi)}{H^q(\xi,0)}\bigg|_{\xi\sim 0} \simeq \frac{2^{\alpha^q}\Gamma(\alpha^q+\frac{3}{2})}{\Gamma(\frac{3}{2})\Gamma(2+\alpha^q)} \approx 3/2 \quad \text{for} \quad \alpha^q \approx 1\,;$$

Skewness effect from H1:



$$\mathbf{R} = 2^{\alpha_q} r^q \sim \frac{\sqrt{\sigma_{DVCS}}}{\sigma_{DIS}}$$

The observable ratio $R(\mathrm{Q}^2)$ for fixed $W = 82$ GeV. The Figure is taken from H1'07.

N.B. A. Shuvaev, Martin et al.'99.

Some lessons

- In order to describe the data the dual parametrization model should include some additional forward like functions $Q_{2\nu}$ with $\nu > 0$.
- These functions should be singular enough in order to make influence on the small ξ asymptotic behavior of $\mathrm{Im} A(\xi)$:

$$Q_{2\nu}(x) \sim \frac{1}{x^{2\nu+\alpha}} \, .$$

Seems to be a problem:

- This leads to divergencies of generalized form factors require regularization

$$B_{2\nu-1\;0} = \mathrm{Reg} \int_0^1 \frac{dx}{x} x^{2\nu} Q_{2\nu}(x) = \int_{(0)}^1 \frac{dx}{x} x^{2\nu} Q_{2\nu}(x) + B^{\mathrm{f.p.}}_{2\nu-1\;0}$$

Analytical regularization

- Compute for large positive j. Then analytically continue to $j = -1$
- This is precisely a so-called analytic (or canonical) regularization ($1 < \alpha < 2$):

$$\int_{(0)}^1 dx \, \frac{f(x)}{x^{1+\alpha}} = \int_0^1 dx \, \frac{1}{x^{1+\alpha}} \left[f(x) - f(0) - x f'(0) \right] - \frac{f(0)}{\alpha} - \frac{f'(0)}{\alpha - 1} \, .$$

Mostly question of normalization

$$H_n(\eta,t) = \sum_{\nu=0}^{(n+1)/2} \eta^{2\nu} H_{n,n+1-2\nu}(t)\hat{d}^{n+1-2\nu}(\eta), \quad \text{for} \quad \text{odd} \quad n,$$

where $\hat{d}^l_{00}(\eta) = \frac{\Gamma(\frac{1}{2})\Gamma(1+J)}{2^J\Gamma(\frac{1}{2}+J)}\eta^l P_l\left(\frac{1}{\eta}\right)$ is the reduced Wigner function.

Double PW expansion employed within MB approach

$$H(x \geq -\eta,\eta,t) \;\; = \;\; \sum_{\nu=0}^{\infty} \frac{1}{2i} \int_{c+2\nu-i\infty}^{c+2\nu+i\infty} dj \, \frac{p_j(x,\eta)}{\sin(\pi[j+1])} \, H_{j,j+1-2\nu}(t)\, \eta^{2\nu}\, \hat{d}^{j+1-2\nu}_{00}(\eta)$$

$$- \sum_{\nu=1}^{\infty} \eta^{2\nu} \, p_{2\nu-1}(x,\eta) H_{2\nu-1,0}(t)\,.$$

Establishing equivalence

Relation between dPWAs (MB approach) and generalized FFs (dual parametrization)

$$H_{n,n+1-2\nu}(t) = \frac{\Gamma(3+n)\Gamma\left(\frac{3}{2}+n-2\nu\right)}{2^{2\nu}\Gamma\left(\frac{5}{2}+n\right)\Gamma(2+n-2\nu)} B_{n,n+1-2\nu}(t)$$

Forward-like functions from the dPWAs

Inversion of the Mellin transform

$$y^{2\nu} Q_{2\nu}(y,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dj\, y^{-j-1}\, \frac{2^{2\nu}\Gamma(5/2+j+2\nu)\Gamma(2+j)}{\Gamma(3+j+2\nu)\Gamma(3/2+j)} H_{j+2\nu,j+1}(t)$$

- Main result of Müller, Polyakov and KS'14: MB representation for the kernel

$$K_{2\nu}(x,\eta|y) = K_{2\nu}^{J\neq 0}(x,\eta|y) - \eta^{2\nu}\, p_{2\nu-1}(x,\eta)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2+2\nu)}{2^{2\nu}\Gamma\left(\frac{3}{2}+2\nu\right)}\,\frac{1}{y} \quad ;$$

$$K_{2\nu}^{J\neq 0}(x,\eta|y) = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} dj\, \eta^{2\nu}\, \frac{p_{j+2\nu}(x,\eta)}{\sin(\pi[1+j])}\, \frac{\Gamma(3+j+2\nu)\Gamma\left(\frac{3}{2}+j\right)}{2^{2\nu}\Gamma\left(\frac{5}{2}+j+2\nu\right)\Gamma(2+j)}\, y^j\, \hat{d}_{00}^{j+1}(\eta)\,.$$

- Straightforward calculation recovers the dual parametrization result.

- Extraction of the information on GPDs from the Compton F.Fs is the problem of deconvolution.
- Consider the elementary amplitude:

$$\mathcal{H}^{(+)}(\xi,t)=\int_0^1 dx H(x,\xi,t)\left[\frac{1}{\xi-x-i0}-\frac{1}{\xi+x-i0}\right]=4\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{\substack{l=0\\ \text{even}}}^{n+1}B_{n\,l}(t)P_l\left(\frac{1}{\xi}\right);$$

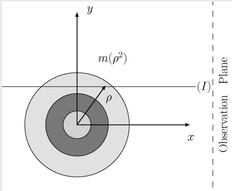
$$\mathrm{Im}\mathcal{H}^{(+)}(\xi,t)=2\int_{\frac{1-\sqrt{1-\xi^2}}{\xi}}^{1}\frac{dx}{x}N(x,t)\frac{1}{\sqrt{\frac{2x}{\xi}-x^2-1}}.$$

- Explicit expression also exists for $\mathrm{Re}\mathcal{H}^{(+)}(\xi,t)$.
- GPD quintessence:
  $N(x,t)=\sum_{\nu=0}^{\infty}x^{2\nu}Q_{2\nu}(x,t)=Q_0(x)+x^2Q_2(x)+x^4Q_4(x)+\ldots$
- The amplitude automatically satisfies the dispersion relation in $\omega=\frac{1}{\xi}$ (O. Teryaev'05) with the subtraction constant given by the D-FF:

$$D(t)=\int_0^1\frac{dx}{x}\left(\frac{1}{\sqrt{1+x^2}}-1\right)+\int_0^1\frac{dx}{x}[N(x,t)-Q_0(x,t)]\frac{1}{\sqrt{1+x^2}}$$

- GPD quintessence and D-FF is the maximal amount of info one can obtain about GPDs from the amplitude.

Abel transform tomography



The observer at ∞ looking along a line parallel to the x-axis a distance y above the origin sees the projection:

$$a(y^2) = \int_{-\infty}^{\infty} dx\, m(\rho^2) = \int_{y^2}^{\infty} d\rho^2 \frac{m(\rho^2)}{\sqrt{\rho^2 - y^2}}$$

- M. Polyakov'07: with the help of Joukowski conformal map $\frac{1}{w} = \frac{1}{2}\left(x + \frac{1}{x}\right)$ it is possible to present the relation between $\mathrm{Im}\mathcal{H}(\xi)$ and GPD quintessence $N(x)$ in the form of the Abel integral equation.
- The inverse transform for $N(x)$:

$$N(x) = \frac{1}{\pi} \frac{x(1-x^2)}{(1+x)^{\frac{3}{2}}} \int_{\frac{2x}{1+x^2}}^{1} \frac{d\xi}{\xi^{\frac{3}{2}}} \frac{1}{\sqrt{\xi - \frac{2x}{1+x^2}}} \left\{ \frac{1}{2} \mathrm{Im}\mathcal{H}^{(+)}(\xi) - \xi \frac{d}{d\xi} \mathrm{Im}\mathcal{H}^{(+)}(\xi) \right\}.$$

- $N(x,t) = \underbrace{Q_0(x,t)}_{\text{PDFs}} + x^2 \underbrace{Q_2(x,t)}_{\text{FFs of EMT tensor}} + x^4 Q_4(x,t) + \ldots$
- For massless hadrons:
  $\int_0^1 dx x^{J-1} N(x,t) = B_{J-1\,J}(t) + B_{J+1\,J}(t) + B_{J+3\,J}(t) + \ldots \equiv F_J(t).$

Gribov'61, Froissart'61

DR for the elementary amplitude analytically continued to the t-channel:

$$\mathcal{H}^{(+)}(\cos\theta_t, t) = \int_0^1 dz\, \frac{2z}{1-z^2} \Phi^{(+)}(z, \cos\theta_t, t) = \int_0^1 dx \frac{2x\cos^2\theta_t}{1-x^2\cos^2\theta_t} H^{(+)}(x, x, t) + 4D(t)$$

where $\Phi^{(+)}(z, \omega, t) = H^{(+)}\left(\frac{z}{\omega}, \eta = \frac{1}{\omega}, t\right)$.

Let us define

- SO(3) PWAs

$$a_J(t) \equiv \frac{1}{2} \int_{-1}^{1} d(\cos\theta_t)\, P_J(\cos\theta_t) \mathcal{H}^{(+)}(\cos\theta_t, t)$$

- GDAs with a definite angular momentum J

$$\Phi_J^{(+)}(z, t) = \frac{1}{2} \int_{-1}^{1} d(\cos\theta_t)\, P_J(\cos\theta_t)\, \Phi^{(+)}(z, \cos\theta_t, t)$$

Neumann's integral representation for the Legendre functions $\mathcal{Q}_J$

$$\frac{1}{2}\int_{-1}^{1} dz P_J(z) \frac{1}{z'-z} = \mathcal{Q}_J(z') \quad J \geq 0, \quad \text{integer.}$$

- For even positive J

$$a_{J>0}(t) = \int_0^1 dz \frac{2z}{1-z^2} \Phi_J^{(+)}(z,t) = 2\int_0^1 dx \frac{\mathcal{Q}_J(1/x)}{x^2} H^{(+)}(x,x,t)\,.$$

- For $J=0$ we get

$$a_{J=0}(t) = 2\int_0^1 dx \left[\frac{\mathcal{Q}_0(1/x)}{x^2} - \frac{1}{x}\right] H^{(+)}(x,x,t) + 4D(t)\,.$$

- N.B. $\frac{\mathcal{Q}_J(1/x)}{x^2} \sim x^{J-1}$ for small x.

Mellin moments of GPD quintessence ⇔ Froissart- Gribov projection

$$\int_0^1 dy\, y^{J-1} N(y,t) = \int_0^1 dx \left[\frac{1}{\sqrt{x}} \frac{d}{dx} R_J(x)\right] H^{(+)}(x,x,t)\,,$$

where the auxiliary functions

$$\frac{1}{\sqrt{x}} \frac{d}{dx} R_J(x) = (\frac{1}{2}+J)\, \frac{\mathcal{Q}_J(1/x)}{x^2}\,.$$

Froissart- Gribov projection III

- For even $J > 0$ we get

$$a_{J>0}(t) = \frac{4}{2J+1} \sum_{\substack{n=J-1 \\ \text{odd}}}^{\infty} B_{nJ}(t) = \frac{4}{2J+1} \int_0^1 dy y^{J-1} N(y,t).$$

- For $J = 0$ it reads

$$\begin{aligned} a_{J=0}(t) &= 4 \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} B_{n0}(t) = 4 \, \mathrm{Reg} \int_0^1 \frac{dy}{y} \left(N(y,t) - Q_0(y,t) \right) \\ &= 4 \int_{(0)}^1 \frac{dy}{y} \left(N(y,t) - Q_0(y,t) \right) + 4 D^{\text{f.p.}}(t). \end{aligned}$$

Non-analytic contribution into $a_{J=0}(t)$:

$$-4 \int_{(0)}^1 \frac{dy}{y} Q_0(y,t) + 4 D^{\text{f.p.}}(t) \equiv -2 \int_{(0)}^1 \frac{dx}{x} H^{(+)}(x,0,t) + 4 D^{\text{f.p.}}(t).$$

N. Khuri'63

- Consider $2 \to 2$ scattering amplitude

$$T(s,t) \doteq \frac{1}{\pi}\int_{(m+m_\pi)^2}^{\infty} \frac{ds'}{s'-s} A(s',t) + \frac{1}{\pi}\int_{(m+m_\pi)^2}^{\infty} \frac{ds'}{s'-u} A(s',t)$$

- Power series in s and u (instead of the Legendre series in Regge formulation):

$$T(s,t) = \sum_{n=0}^{\infty} \underbrace{\frac{1}{\pi}\int_{(m+m_\pi)^2}^{\infty} \frac{ds'}{s'^{n+1}} A(s',t)}_{a_n(s,t)} (s^n + u^n)$$

- Consider $a_n(t)$ as the analytic function of $z = n$: $a(z,t)$ for $n \geq N$.
- Analytic properties of $a(z,t)$ govern the asymptotic behavior of $T(s,t)$:

$$\text{Khuri pole}: \quad a(z,t) = \frac{\beta(t)}{\alpha(t) - z} \quad \Rightarrow$$

$$T(s,t) \sim \frac{\pi\beta(t)}{\sin(\pi\alpha(t))} \left\{1 + e^{-i\pi\alpha(t)}\right\} s^{\alpha(t)} \quad \text{for} \quad s \to \infty.$$

- Interpretation: $\alpha(t) = n$ corresponds to poles in the cross channel (moving poles).
- Maximal analyticity hypothesis: $a(z, t)$ can be analytically continued to the left semiplane: can fix subtraction constants (if needed) and provide finite energy sum rules.
- What can spoil analyticity in z? Kronecker δ singularity: $a'_n = a_n + c\delta_{nJ_0}$.
- Source: fixed pole singularity - exchange with an elementary particle of spin J_0 (non-Reggeized) in the cross channel or contact interaction term for $J_0 = 0$. Does not move in z plane with change of t.

Fixed pole in real Compton scattering

M.Creutz, S. Drell and E. Pashos'69: Regge-pole representation of the real forward Compton scattering amplitude:

$$f_1(\nu) = \sum_{\alpha\neq 0} \frac{\beta_\alpha \nu^\alpha}{4\pi} \frac{-1 - e^{-i\pi\alpha}}{\sin(\pi\alpha)} + C_\infty \quad \text{with} \quad \nu = \frac{s-u}{4M} .$$

Thomson limit

$$f_1(0) = \lim_{\nu\to 0} f_1(\nu) = -\frac{e_p^2}{4\pi M}.$$

- $J = 0$ fixed pole sum rule $(\mathrm{Im} f_1(\nu) = \frac{\nu}{4\pi}\sigma_T(\nu))$

$$C_\infty = f_1(0) - \frac{2}{\pi}\int_{\nu_{\rm thr}}^{(\infty)} \frac{d\nu}{\nu}\mathrm{Im} f_1(\nu),$$

First attempt to extract: C. A. Dominguez, C. Ferro Fontan and R. Suaya'70

PHYSICAL REVIEW D VOLUME 5, NUMBER 11 1 JUNE 1972

Are There Fixed Singularities in T_1?

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(Received 15 February 1972)

Nonforward scaling and the light-cone commutator are shown to require the existence of a Kronecker δ term in T_1. The present data may be consistent with the possibility that this Kronecker δ term vanishes at $t=0$ and $q^2=-\infty$. Analyticity implies a connection between scaling behavior and Regge behavior.

PHYSICAL REVIEW D VOLUME 7, NUMBER 5 1 MARCH 1973

Comment on "Are There Fixed Singularities in T_1?"*

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(Received 17 July 1972)

We show that the conclusion of Zee on the presence of fixed $J=0$ singularities in virtual Compton scattering is unfounded. Thus, there is no model-independent argument for such a behavior.

PHYSICAL REVIEW D VOLUME 5, NUMBER 6 15 MARCH 1972

Compton Scattering and Fixed Poles in Parton Field-Theoretic Models*

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(Received 12 October 1971)

We extend a class of parton models to a fully gauge-invariant theory for the full Compton amplitude. The existence of local electromagnetic interactions is shown to always give rise to a constant real part in the high-energy behavior of the amplitude $T_1(\nu, q^2)$. In the language of Reggeization this is interpreted as a fixed pole at $J=0$ in T_1 and νT_2, with residue polynomial in the photon mass squared.

PHYSICAL REVIEW D **79**, 033012 (2009)

Local two-photon couplings and the $J = 0$ fixed pole in real and virtual Compton scattering

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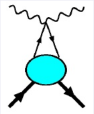
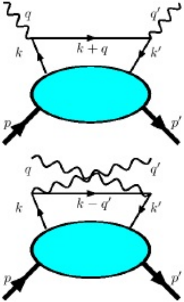
The local coupling of two photons to the fundamental quark currents of a hadron gives an energy-independent contribution to the Compton amplitude proportional to the charge squared of the struck quark, a contribution which has no analog in hadron scattering reactions. We show that this local contribution has a real phase and is universal, giving the same contribution for real or virtual Compton scattering for any photon virtuality and skewness at fixed momentum transfer squared t. The t dependence of this $J = 0$ fixed Regge pole is parameterized by a yet unmeasured even charge-conjugation form factor of the target nucleon. The $t = 0$ limit gives an important constraint on the dependence of the nucleon mass on the quark mass through the Weisberger relation. We discuss how this $1/x$ form factor can be extracted from high-energy deeply virtual Compton scattering and examine predictions given by models of the H generalized parton distribution.

$J = 0$ fixed pole manifestation in DVCS

- S. Brodsky et al.'08: local coupling of two photons to a quark in the high energy limit.





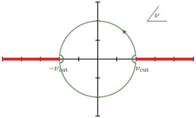
$$\frac{\hat{k}+\hat{q}+m}{(k+q)^2-m^2+i\epsilon} \to \frac{\gamma^+}{2p^+}\left(\frac{1}{x}+\frac{\xi}{x}\frac{1}{x-\xi++i\epsilon}\right)$$

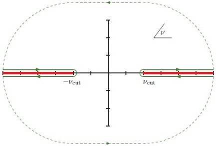
$$\frac{\hat{k}-\hat{q}'+m}{(k-q')^2-m^2+i\epsilon} \to \frac{\gamma^+}{2p^+}\left(\frac{1}{x}-\frac{\xi}{x}\frac{1}{x+\xi++i\epsilon}\right)$$

Universal $J = 0$ fixed pole contribution into Compton amplitude

$$\mathcal{H}_\infty(t|Q_1^2, Q_2^2) = -\frac{1}{\pi}\int_{\nu_{\rm cut}}^{(\infty)} d\nu \frac{2}{\nu}\, {\rm Im}\mathcal{H}(\nu, t|Q_1^2 = Q_2^2) \quad \textbf{(Conjecture)}.$$

$$\gamma^{(*)}(q_1) + N(p_1) \to \gamma^{(*)}(q_2) + N(p_2) \quad \nu = \frac{s-u}{4M}$$





- Unsubtracted DR:

$$\mathcal{H}(\nu, t|Q_1^2, Q_2^2) = \mathcal{H}_\infty(t|Q_1^2, Q_2^2) + \frac{1}{\pi}\int_{\nu_{\rm cut}}^{(\infty)} d\nu' \, \frac{2\nu'}{\nu'^2 - \nu^2 - i\epsilon} \mathrm{Im}\,\mathcal{H}(\nu', t|Q_1^2, Q_2^2)\,.$$

- Usual subtracted DR:

$$\mathcal{H}(\nu, t|Q_1^2, Q_2^2) \overset{\rm d.v.}{=} \mathcal{H}_0(t|Q_1^2, Q_2^2) + \frac{1}{\pi}\int_{\nu_{\rm cut}}^{\infty} \frac{d\nu'}{\nu'} \, \frac{2\nu^2}{\nu'^2 - \nu^2 - i\epsilon} \mathrm{Im}\,\mathcal{H}(\nu', t|Q_1^2, Q_2^2)\,.$$

Relation between the two constants

$$\mathcal{H}_\infty(t|Q_1^2, Q_2^2) = \mathcal{H}_0(t|Q_1^2, Q_2^2) - \frac{2}{\pi}\int_{\nu_{\rm cut}}^{(\infty)} \frac{d\nu}{\nu} \, \mathrm{Im}\mathcal{H}(\nu, t|Q_1^2, Q_2^2)$$

$$\gamma^{(*)}(q_1) + N(p_1) \to \gamma^{(*)}(q_2) + N(p_2)$$

Scaling variables

$$\xi = \frac{Q^2}{P\cdot q} = \frac{Q^2}{2M\nu}; \qquad \eta = -\frac{\Delta\cdot q}{P\cdot q} = -\frac{\Delta\cdot q}{2M\nu}, \quad \text{where} \quad Q^2 = -q^2 \equiv -\frac{(q_1+q_2)^2}{4}.$$

Useful variable: $\vartheta \equiv \eta/\xi = \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2} + \mathcal{O}(t/Q^2)$

- For $t = 0$, the case $\vartheta = 0$ corresponds to the usual DIS kinematics.
- The case $\vartheta = 1$ corresponds to the DVCS kinematics.

LO Compton FF

$$\mathcal{H}(\xi, t|\vartheta) \overset{\text{LO}}{=} \int_0^1 dx \, \frac{2x}{\xi^2 - x^2 - i\epsilon} H^{(+)}(x, \eta = \vartheta\xi, t).$$

Dispersive approach in the scaling regime II

DRs within scaling variables

$$\mathcal{H}(\xi,t|\vartheta) = \frac{1}{\pi}\int_{(0)}^{1} \frac{d\xi'}{\xi'} \frac{2\xi^2}{\xi^2-\xi'^2-i\epsilon}\mathrm{Im}\mathcal{H}(\xi',t|\vartheta) + \mathcal{H}_\infty(t|\vartheta)\,,$$

$$\mathcal{H}(\xi,t|\vartheta) = \frac{1}{\pi}\int_{0}^{1} d\xi' \frac{2\xi'}{\xi^2-\xi'^2-i\epsilon}\mathrm{Im}\mathcal{H}(\xi',t|\vartheta) + \underbrace{\mathcal{H}_0(t|\vartheta)}_{4D(t|\vartheta)}\,.$$

Relation between subtraction constants

$$\mathcal{H}_\infty(t|\vartheta) = 4D(t|\vartheta) - \frac{2}{\pi}\int_{(0)}^{1}\frac{d\xi}{\xi}\,\mathrm{Im}\mathcal{H}(\xi,t|\vartheta)\,.$$

GPD sum rule O. Teryaev'05

$$4D(t|\vartheta) \stackrel{\mathrm{LO}}{=} \int_0^1 dx\,\frac{2x}{x^2-\xi^2}\left[H^{(+)}(x,\vartheta x,t) - H^{(+)}(x,\vartheta\xi,t)\right].$$

Dispersive approach in the scaling regime III

- Caution! 'High energy' limit $\xi \to 0$ requires attention.
- Taken naively will miss $D^{\text{f.p.}}(\eta|, t)$

$$H^{(+)}(x, \eta, t) = \mathbb{H}^{(+)}(x, \eta, t) + \theta(|\eta| - |x|) d^{\text{f.p.}}(x/|\eta|, t)\,.$$

- Split $x \in [0, \vartheta\xi]$, and $x \in [\vartheta\xi, 1]$. Then

$$4D^{\text{f.p.}}(t|\vartheta) \stackrel{\text{LO}}{=} \lim_{\xi \to 0} \int_0^{\vartheta\xi} dx\, \frac{2x}{\xi^2 - x^2}\, d^{\text{f.p.}}\left(\frac{x}{\vartheta\xi}, t\right) = \int_0^1 dx\, \frac{2x\vartheta^2}{1 - \vartheta^2 x^2}\, d^{\text{f.p.}}(x, t)\,.$$

- No proof for Brodsky et al. conjecture! Back to the discussion of the D-term as inherent part of GPD (GPD holographic property) and presence/absence of $j = -1$ fixed poles.
- Counterexamples with auxiliary D-term exist. Such situation occurs in certain dynamical models. *E.g.* pion GPD in nonlocal chiral quark model. See K.S.'08

$J = 0$ fixed pole contribution for DVCS

How this applies for the dual parametrization:

$$4D = 4\int_0^1 \frac{dx}{x} N(x)\left(\frac{1}{\sqrt{1+x^2}} - 1\right) + 4\int_{(0)}^1 \frac{dx}{x}\left[N(x) - Q_0(x)\right]$$

$$4\int_0^1 \frac{dx}{x} N(x)\left(\frac{1}{\sqrt{1+x^2}} - 1\right) + 4\int_{(0)}^1 \frac{dx}{x}\left[N(x)\right] - 2\int_{(0)}^1 \frac{dx}{x} q(x)$$

- $J = 0$ fixed pole contribution is a part of D form factor.
- It cancels in the Compton FF. No access to inverse PDF moment.

Conclusions

1. The dual parametrization approach is equivalent to the Mellin-Barnes type integral based techniques for GPDs.
2. Froissar-Gribov projection provides explanation for the properties of GPD quintessence function and Abel transform tomography.
3. There exists no proof for $J = 0$ pole universality for Compton scattering conjectured by S. Brodsky.
4. $J = 0$ pole universality property is equivalent to GPD holographic property.
5. However, this is an additional “external principle”. Hard to prove (or disprove).

Evolution of GPDs

- GPDs depend on the renormalization scale μ^2 of operators in their definition (scale at which partons are resolved)
- Generalization of DGLAP equation. Splitting functions are much more complicated: include the pieces different in different kinematical regions.
- Evolution of C-odd GPDs

$$\mu^2 \frac{d}{d\mu^2} H^{q(-)}(x,\xi,t) = \frac{1}{|\xi|} \int_{-1}^{1} dx' V_{\rm NS}\left(\frac{x}{\xi}, \frac{x'}{\xi}\right) H^{q(-)}(x',\xi,t)$$

- C-even quark GPDs mixing with gluon GPDs
- LO kernels V.Gribov et.al'83, D.Müller et.al'94
- As in the case of PDFs evolution of GPDs can be treated in terms of renormalization of the local operators corresponding to their x moments
- Leading twist operators mix under renormalization with operators having additional overall derivatives:

$$\mu^2 \frac{d}{d\mu^2} [\bar{\psi}\gamma^+ (\overleftrightarrow{D}^+)^n \psi] = \sum_{m=0}^{n} \Gamma_{nm} [(\partial^+)^{n-m} \bar{\psi}\gamma^+ (\overleftrightarrow{D}^+)^m \psi]$$

- Only for $\Delta^+ = 0$ Mellin moments have multiplicative renormalization

A note on conformal symmetry

- Let us consider non-local gauge invariant operators

$$\hat{O}(z_1 n, z_2 n) = \bar{\psi}(z_1 n)\psi(z_2 n)\,;$$

- RG evolution is driven to leading log. accuracy with tree level counterterms which have symmetries of bare $\mathcal{L}_{\rm QCD}$ (and in particular the conformal symmetry).
- Operators, belonging to different representations of conformal group $\mathrm{SL}(2,\,\mathbb{R})$ do not mix under renormalization.
- Conformal spin: $j = \frac{1}{2}(\ell + s)$ ($\ell = \frac{3}{2}$, $s = \pm\frac{1}{2}$ for quarks). Characterizes the behavior of the field under collinear conformal transformation.

$$z \to z' = \frac{az+b}{cz+d} \quad (ad - bc = 1)\,; \phi(zn) \to \phi'(zn) = (cz+d)^{-2j}\phi\left(\frac{az+b}{cz+d}n\right).$$

- In the meson sector the operators with definite conformal spin $(2j+n)$:

$$\hat{O}_{nl} \sim (i\partial_{z_1} + i\partial_{z_2})^l C_n^\nu\left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}}\right)\hat{O}(z_1 n, z_2 n), \quad n \le l \ (\nu = 2j - \frac{1}{2})\,.$$

'To pursue it with forks and hope'