Strong three – meson couplings : relativistic quark model vs QCD sum rules

Dmitri Melikhov

D. V. Skobeltsyn Institute of Nuclear Physics, M. V. Lomonosov Moscow State University

HEPHY, Austrian Acaemy of Sciences, Vienna, Austria

We compare the calculation of complicated three-meson couplings within the two theoretical frameworks: QCD sum rules and relativistic quark model formulated as spectral representation over mass variables. Arguments are given in favour of the results from the relativistic quark model which in many cases predicts much larger values of the couplings.

Based on W. Lucha, D. M., H. Sazdjian, S. Simula, Physical Review D93, 016004 (2016).

Introduction and definitions

The strong couplings of our interest, $g_{PV'V}$ and $g_{PP'V}$:

$$\langle P'(p_2)V(q)|P(p_1)\rangle = -\tfrac{1}{2}g_{PP'V}(p_1+p_2)^\mu\varepsilon^*_\mu(q),$$
$$\langle V'(p_2)V(q)|P(p_1)\rangle = -\epsilon_{\varepsilon^*(q)\varepsilon^*(p_2)p_1p_2}\,g_{PV'V},$$

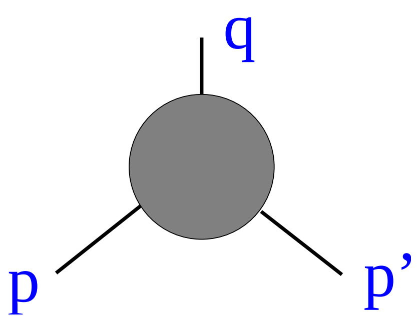
with momentum transfer $q = p_1 - p_2$. $g_{PP'V}$ is dimensionless; $g_{PV'V}$ has inverse mass dimension.

The couplings g are related to the residues of the poles in the transition form factors at time-like momentum transfer arising from contributions of intermediate meson states in the transition amplitudes.

The form factors $F_+^{P\to P'}(q^2)$, $V^{P\to V}(q^2)$, and $A_0^{P\to V}(q^2)$, related to the transition amplitudes induced by quark currents: vector $\bar q_2\gamma_\mu q_1$ and axial-vector $\bar q_2\gamma_\mu\gamma_5 q_1$:

$$\langle P'(p_2)|\bar q_2\gamma_\mu q_1|P(p_1)\rangle = F_+^{P\to P'}(q^2)(p_1+p_2)_\mu + \cdots,$$
$$\langle V(p_2)|\bar q_2\gamma_\mu q_1|P(p_1)\rangle = \frac{2V^{P\to V}(q^2)}{M_P+M_V}\epsilon_{\mu\varepsilon^*(p_2)p_1p_2},$$
$$\langle V(p_2)|\bar q_2\gamma_\mu\gamma_5 q_1|P(p_1)\rangle = \mathrm{i}q_\mu(\varepsilon^*(p_2)p_1)\frac{2M_V}{q^2}A_0^{P\to V}(q^2) + \cdots,$$

dots stand for other Lorentz structures.



The poles in the above form factors are of the form

$$
\begin{aligned}
F_+^{P\to P'}(q^2) &= \frac{g_{PP'V_R} f_{V_R}}{2M_{V_R}} \frac{1}{1-q^2/M_{V_R}^2} + \cdots, \\
V^{P\to V}(q^2) &= \frac{(M_V+M_P) g_{PVV_R} f_{V_R}}{2M_{V_R}} \frac{1}{1-q^2/M_{V_R}^2} + \cdots, \\
A_0^{P\to V}(q^2) &= \frac{g_{PP_RV} f_{P_R}}{2M_V} \frac{1}{1-q^2/M_{P_R}^2} + \cdots.
\end{aligned}
$$

In these relations, P_R and V_R label pseudoscalar and vector resonances with appropriate quantum numbers; f_P and f_V are the leptonic decay constants of the pseudoscalar and vector mesons, respectively, defined in terms of the amplitude of the meson-to-vacuum transition induced by the axial-vector or vector quark currents according to

$$
\begin{aligned}
\langle 0|\bar{q}_1\gamma_\mu\gamma_5 q_2|P(p)\rangle &= \mathrm{i} f_P p_\mu, \\
\langle 0|\bar{q}_1\gamma_\mu q_2|V(p)\rangle &= f_V M_V \varepsilon_\mu(p).
\end{aligned}
$$

Relativistic constituent quark model as dispersion approach

• Relativistic quark models treat mesons as two-particle bound states of effective objects — constituent quarks.

• Relativistic treatment of two-particle contributions to the bound-state structure may be consistently formulated using relativistic dispersion approach which takes into account only two-particle intermediate (constituent) quark-antiquark states in Feynman diagrams.

• This formulation is explicitly relativistic-invariant: hadron observables (decay constants, form factors) are given by spectral representations over the invariant masses of the quark-antiquark intermediate states.

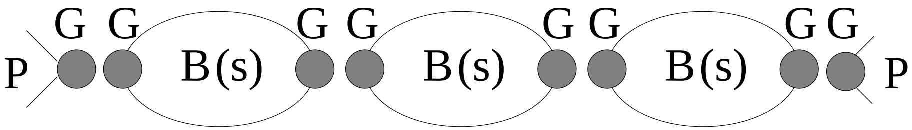
• The main benefit of the relativistic formulation is the possibility to obtain the weak form factors in the decay region by analytic continuation. Then anomalous cuts and the anomalous contributions to the form factors emerge. The anomalous contributions are given via wave functions in the physical region (i.e. via $\psi(s)$ **at** $s \geq (m_1 + m_2)^2$**)**

Two-particle unitarity of the partial-wave scattering amplitude $A_0(s)$,

$$\mathrm{Im}\,A_0(s) = \rho(s)|A_0(s)|^2$$

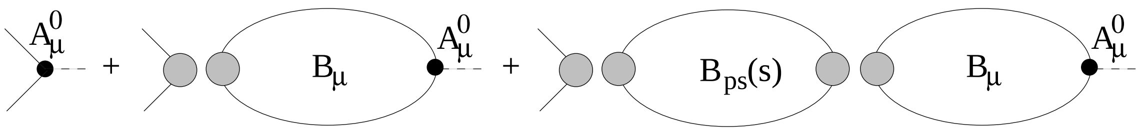
has the following solution

$$A_0(s) = G(s)\frac{1}{1-B(s)}G(s), \qquad B(p^2) = \int \frac{ds}{s-p^2}G^2(s).$$



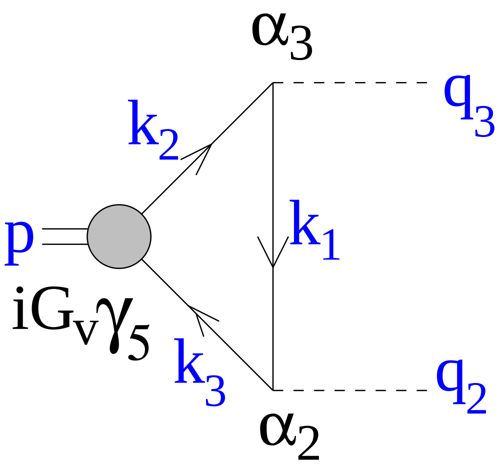
The pole in this amplitude is the bound state, and in the two-particle approximation its properties can be calculated by the following procedure:

Decay constant of the bound state:

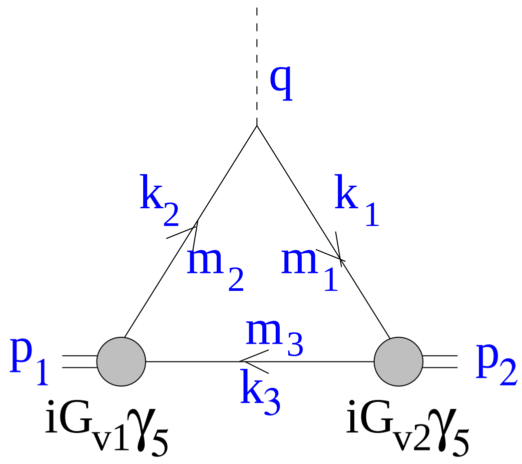


$$f = \int ds \frac{G(s)}{s-M^2}\rho(s)$$

Two-current decay of the bound state:



Elastic or transition form factor of the bound state:



$$F(q_2^2, q_3^2) = \int ds \frac{G_v(s)}{s - M^2}\rho(s, q_2^2, q_3^2), \qquad F(q^2) = \int ds_1 \frac{G_{v1}(s_1)}{s_1 - M_1^2} \int ds_2 \frac{G_{v2}(s_2)}{s_2 - M_2^2}\Delta(s_1, s_2, q^2).$$

• **The wave function of the bound state is:** $\psi(s) = \frac{G_v(s)}{s-M^2}$**.**

• **Normalization condition for** ψ**: the elastic form factor at zero momentum transfer is equal unity.**

• **Nuclei-like bound state:** $G_v(s)$ **is a regular function, pole in** $\psi(s)$**at** $s = M^2$

Confined bound state: $\psi(s)$ **is regular, no pole in** $\psi(s)$ **at** $s = M^2$

Decay constants of pseudoscalar and vector mesons:

$$f_P = \sqrt{N_c} \int\limits_{(m_1+m)^2}^{\infty} \mathrm{d}s\, \phi_P(s)\, (m_1+m) \frac{\lambda^{1/2}(s, m_1^2, m^2)}{8\pi^2 s} \frac{s-(m_1-m)^2}{s},$$

$$f_V = \sqrt{N_c} \int\limits_{(m_1+m)^2}^{\infty} \mathrm{d}s\, \phi_V(s)\, \frac{2\sqrt{s}+m_1+m}{3} \frac{\lambda^{1/2}(s, m_1^2, m^2)}{8\pi^2 s} \frac{s-(m_1-m)^2}{s},$$

The wave functions $\phi_i(s)$, $i = P, V$, can be written as

$$\phi_i(s) = \frac{\pi}{\sqrt{2}} \frac{\sqrt{s^2-(m_1^2-m^2)^2}}{\sqrt{s-(m_1-m)^2}} \frac{w_i(k^2)}{s^{3/4}}, \qquad k^2 = \frac{\lambda(s, m_1^2, m^2)}{4s},$$

with $w_i(k^2)$ normalized according to

$$\int \mathrm{d}k\, k^2 w_i^2(k^2) = 1.$$

The $M_1(p_1) \to M_2(p_2)$ transition ffs induced by the constituent-quark transition $\bar{Q}_1 \to Q_2$ reads

$$F_i(q^2) = \int \mathrm{d}s_1\, \psi_1(s_1) \int \mathrm{d}s_2\, \psi_2(s_2)\, \Delta_i(s_1, s_2, q^2).$$

At $q^2 < 0$, this form factors is equal to the form factors of the light-front relativistic constituent quark model: the double spectral representation at $q^2 < 0$ may be rewritten as the convolution of the light-cone wave functions of the initial and the final hadrons.

In the decay region $0 < q^2 \leq (m_2 - m_1)^2$ the analytic continuation in q^2 leads to the anomalous contributions. Both the normal and the anomalous contributions involve the s_1 and s_2 integrations over the corresponding two-particle cuts, i.e. for $k_1^2 > 0$ and $k_2^2 > 0$.

For ffs in this region, a Gaussian parameterization can be adopted: $w_i(k^2) \propto \exp(-k^2/2\beta_i^2)$.

• **The transition form factors from dispersion approach satisfy the rigorous constraints of non-perturbative QCD in the limit of the heavy-to-heavy (HQET) and heavy-to light quark transitions (LEET).**

• **The spectral representation is based on constituent-quark degrees of freedom and we apply it to calculate the form factors in the region $q^2 < (m_2 - m_1)^2$. We then numerically interpolate the results of our calculations and use the obtained parameterizations to study the form factors at $q^2 > (m_1 - m_2)^2$, where one expects the appearance of meson resonance at $q^2 = M_R^2$.**

QCD quark currents vs constituent quark currents

An essential feature of the constituent quark picture is the appropriate matching of the quark currents in QCD ($\bar{q}\gamma_\mu q$, $\bar{q}\gamma_\mu\gamma_5 q$, etc.) and the associated currents formulated in terms of constituent quarks ($\bar{Q}\gamma_\mu Q$, $\bar{Q}\gamma_\mu\gamma_5 Q$, etc.).

For currents containing heavy quarks, these matching conditions are simple:

$$\begin{aligned} \bar{q}_1\gamma_\mu q_2 &= g_V\bar{Q}_1\gamma_\mu Q_2 + \cdots, \\ \bar{q}_1\gamma_\mu\gamma_5 q_2 &= g_A\bar{Q}_1\gamma_\mu\gamma_5 Q_2 + \cdots, \end{aligned}$$

where dots indicate other possible Lorentz structures.

Constituent quarks Q_1 and Q_2 have masses m_1 and m_2, respectively. In general, the form factors g_V and g_A depend on the momentum transfer. Vector current conservation requires $g_V = 1$ at zero momentum transfer for the elastic current and at zero recoil for the heavy-to-heavy quark transition. The specific values of the form factors g_V and g_A and their momentum dependences belong to the parameters of the model, as well as the quark masses and the wave functions of mesons regarded as relativistic quark–antiquark bound states.

Parameters of the model

- **constituent-quark matching couplings g_A and g_V**

$$g_V = g_A = 1.$$

- **constituent quark masses**

$$m_d = m_u = 0.23\ \text{GeV}, \qquad m_s = 0.35\ \text{GeV}, \qquad m_c = 1.45\ \text{GeV}.$$

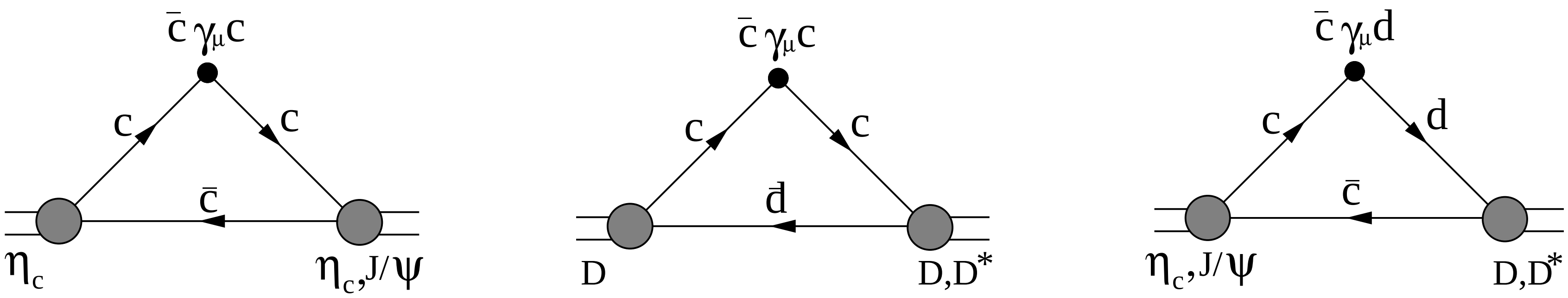
- **wave function shape and size parameter β**

For the wave functions, we make use of the simple Gaussian wave-function Ansatz which proved to provide a reliable picture of a large class of transition form factors.

With the above quark couplings and masses, and the meson wave-function parameters β collected in Table, the decay constants from our dispersion approach reproduce the best-known decay constants of pseudoscalar and vector mesons.

	D	D^*	D_s	D_s^*	η_c	J/ψ
M **(GeV)**	1.87	2.010	1.97	2.11	2.980	3.097
f **(MeV)**	206 ± 8	260 ± 10	248 ± 2.5	311 ± 9	394.7 ± 2.4	405 ± 7
β **(GeV)**	0.475	0.48	0.545	0.54	0.77	0.68

Form factors



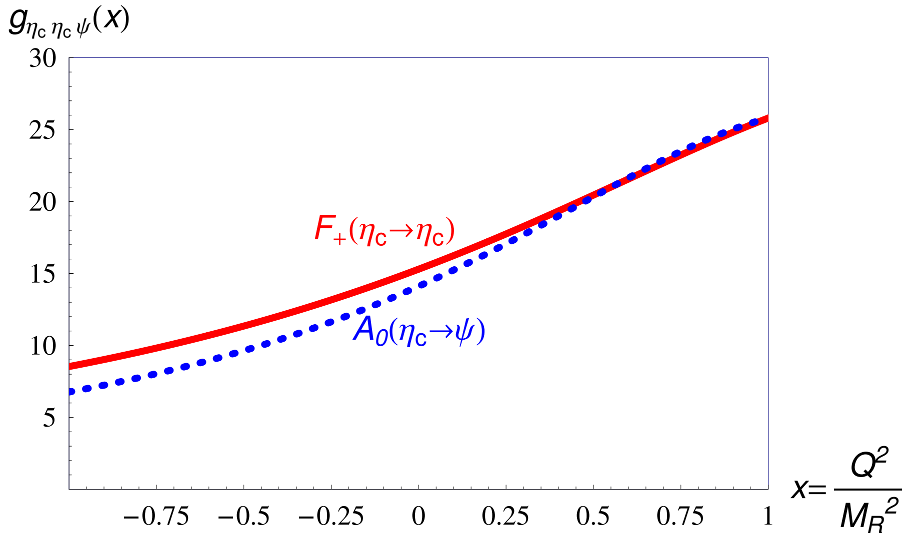
• **We calculate the ffs via double spectral representations**

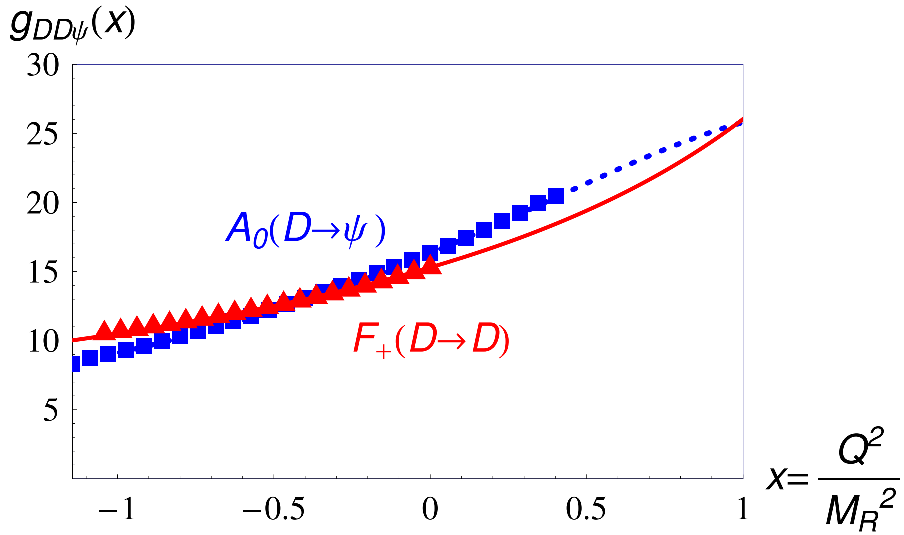
• **We interpolate by**

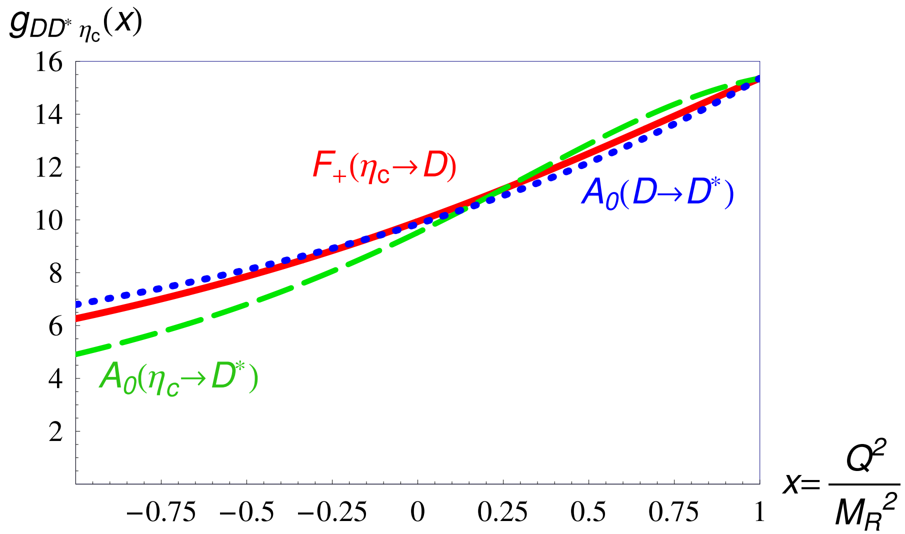
$$F(q^2) = \frac{F(0)}{\left(1 - q^2/M_R^2\right)\left(1 - \sigma_1 q^2/M_R^2 + \sigma_2 q^4/M_R^4\right)},$$

where $M_R = M_V$ for F_+ and V, and $M_R = M_P$ for A_0. The value of M_R obtained by the fit is very close to the mass of the resonance with the appropriate quantum numbers. The residue is given by products of the (known) weak and the strong couplings g to be determined.

• **In some cases, the residues of different form factors involve the same strong coupling; for such form-factor sets a constrained interpolation will be done.**







$$F_+^{P\to P'}(q^2) = \frac{g_{PP'V_R} f_{V_R}}{2M_{V_R}} \frac{1}{1-q^2/M_{V_R}^2} + \cdots \to \frac{2M_{V_R}}{f_{V_R}} F_+^{P\to P'}(q^2)(1 - q^2/M_{V_R}^2) \equiv g_{PP'V_R}(q^2)$$

$$A_0^{P\to V}(q^2) = \frac{g_{PP_R V} f_{P_R}}{2M_V} \frac{1}{1-q^2/M_{P_R}^2} + \cdots \to \frac{2M_V}{f_{P_R}} A_0^{P\to V}(q^2)(1 - q^2/M_{P_R}^2) \equiv g_{PP_R V}(q^2)$$

Our results may be summarized as follows:

- **for the couplings involving J/ψ and η_c mesons,**

$$g_{\eta_c\eta_c\psi} = 25.8 \pm 1.7, \qquad g_{\eta_c\psi\psi} = (10.6 \pm 1.5)\ \mathrm{GeV}^{-1},$$

- **for the J/ψ and η_c couplings to charmed mesons,**

$$g_{DD\psi} = 26.04 \pm 1.43, \qquad g_{DD^*\psi} = (10.7 \pm 0.4)\ \mathrm{GeV}^{-1},$$
$$g_{DD^*\eta_c} = 15.51 \pm 0.45, \qquad g_{D^*D^*\eta_c} = (9.76 \pm 0.32)\ \mathrm{GeV}^{-1},$$

- **and, for the J/ψ and η_c couplings to charmed strange mesons,**

$$g_{D_sD_s\psi} = 23.83 \pm 0.78, \qquad g_{D_sD_s^*\psi} = (9.6 \pm 0.8)\ \mathrm{GeV}^{-1},$$
$$g_{D_sD_s^*\eta_c} = 14.15 \pm 0.52, \qquad g_{D_s^*D_s^*\eta_c} = (8.27 \pm 0.37)\ \mathrm{GeV}^{-1}.$$

Comparison of the couplings predicted by the dispersion approach with the results from experiment and lattice QCD in those cases where such results are available, allows us to expect the accuracy of our predictions to be not worse than 15–20%.

Our results considerably exceed the ones from QCD sum rules.

	$g_{DD\psi}$	$g_{DD^*\psi}$ **(GeV^{-1})**	$g_{D_sD_s\psi}$	$g_{D_sD_s^*\psi}$ **(GeV^{-1})**
This work	26.04 ± 1.43	10.7 ± 0.4	23.83 ± 0.78	9.6 ± 0.8
QCD sum rules	11.6 ± 1.8	4.0 ± 0.6	11.96 ± 1.34	4.30 ± 1.53

Properties of individual resonances from OPE

• **The basic object**

T-product of a number of the interpolating currents $j(x)$:

$$\langle\Omega|j(0)|M\rangle = f_M \neq 0.$$

(E.g. $j(x) = \bar{q}_1(x) O q_2(x)$ for "normal" mesons, 4-quark currents for exotic mesons).

The simplest object—2-point function

$$\Pi(p^2) = i\int d^4x\, e^{ipx}\left\langle\Omega\left|T\left(j(x)j^\dagger(0)\right)\right|\Omega\right\rangle$$

• **Wilsonian OPE - separation of distances:**

$$T\left(j(x)j^\dagger(0)\right) = C_0(x^2,\mu)\hat{1} + \sum_n C_n(x^2,\mu) : \hat{O}_n(x=0,\mu) :$$

$$\Pi(p^2) = \Pi_{\rm pert}(p^2,\mu) + \sum_n \frac{C_n}{(p^2)^n}\langle\Omega| : \hat{O}_n(x=0,\mu) : |\Omega\rangle$$

• **Physical QCD vacuum $|\Omega\rangle$ is complicated and differs from perturbative QCD vacuum $|0\rangle$.**

Condensates – nonzero expectation values of gauge-invariant operators over physical vacuum:

$$\boxed{\langle\Omega| : \hat{O}(0,\mu) : |\Omega\rangle \neq 0}$$

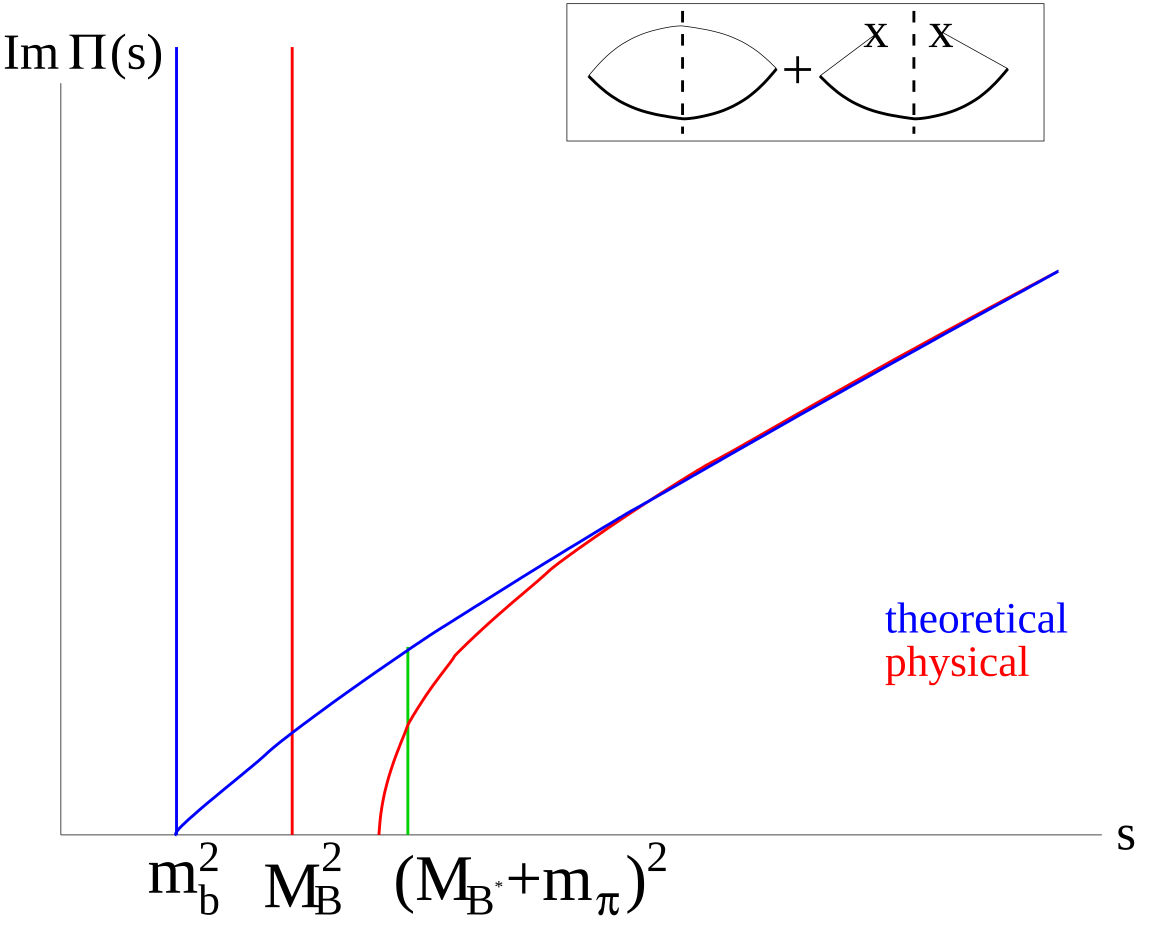
$$\langle\Omega|\bar{q}q(2\ \text{GeV}|\Omega\rangle = -(271 \pm 3\ \text{MeV})^3, \qquad \langle\Omega|\alpha_s/\pi GG|\Omega\rangle = 0.012 \pm 0.006\ \text{GeV}.$$

2-point function is analytic function of p^2

$$\Pi(p^2) = \int \frac{ds}{s - p^2}\rho(s),$$

One calculates the spectral densities using OPE and using hadron states

$$\rho_{\text{theor}}(s) = \left[\rho_{\text{pert}}(s,\mu) + \sum_n C_n \delta^{(n)}(s)\langle\Omega|O_n(\mu)|\Omega\rangle\right], \qquad \rho_{\text{hadr}}(s) = f^2\delta(s - M^2) + \rho_{\text{cont}}(s)$$



How to relate to each other truncated $\Pi_{\rm OPE}(p^2)$ and $\Pi_{\rm hadron}(p^2)$?

Borel transform $p^2 \to \tau$ [$\frac{1}{s-p^2} \to \exp(-\tau p^2)$]

$$\Pi(\tau) = \int ds\, \exp(-s\tau)\rho(s) = f^2 e^{-M_B^2\tau} + \int\limits_{s_{\rm phys}}^{\infty} ds\, e^{-s\tau}\rho_{\rm hadr}(s) = \int\limits_{(m_b+m)^2}^{\infty} ds\, e^{-s\tau}\rho_{\rm pert}(s,\mu) + \Pi_{\rm power}(\tau,\mu).$$

Here $s_{\rm phys}$ is the physical threshold, and f is the decay constant defined by

$$\langle 0|\bar{q}Ob|B\rangle = f.$$

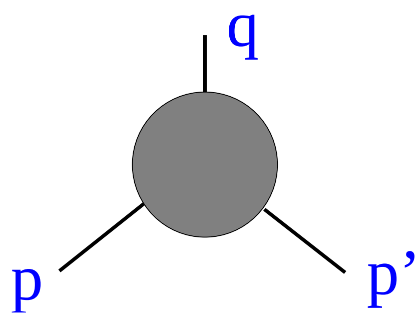
To get rid of the excited-state contributions, one adopts the *duality Ansatz*: all contributions of excited states are counterbalanced by the perturbative contribution above an *effective continuum threshold*, $s_{\rm eff}(\tau)$ which differs from the physical continuum threshold.

Applying the duality assumption yields:

$$f^2 e^{-M_B^2\tau} = \int\limits_{(m_b+m)^2}^{s_{\rm eff}(\tau)} ds\, e^{-s\tau}\rho_{\rm pert}(s,\mu) + \Pi_{\rm power}(\tau,\mu).$$

Strong decays from 3 – point vertex functions

• **The basic object:**



$$\Gamma(p, p', q) = \int \langle 0|T(J(x) j(0) j'(x')|0\rangle \exp(ipx - ip' x') dx dx'$$

This correlator contains the triple-pole in the Minkowski region: namely

$$\Gamma(p, p', q) = \frac{f f'}{(p^2 - M^2)(p'^2 - M'^2)} F(q^2) + \cdots$$

where the form factor $F(q^2)$ contains pole at $q^2 = M_q^2$:

$$F(q^2) = \frac{f_q g_{MM'M_q}}{(q^2 - M_q^2)} + \cdots$$

$g_{MM'M_q}$ **describes the $M \to M_1 M_2$ strong transition;**
f**,** f'**, and** f_{M_q} **are the decay constants of the mesons $\langle 0|j(0)|M\rangle = f_M$.**

The three-point function satisfies the double spectral representation

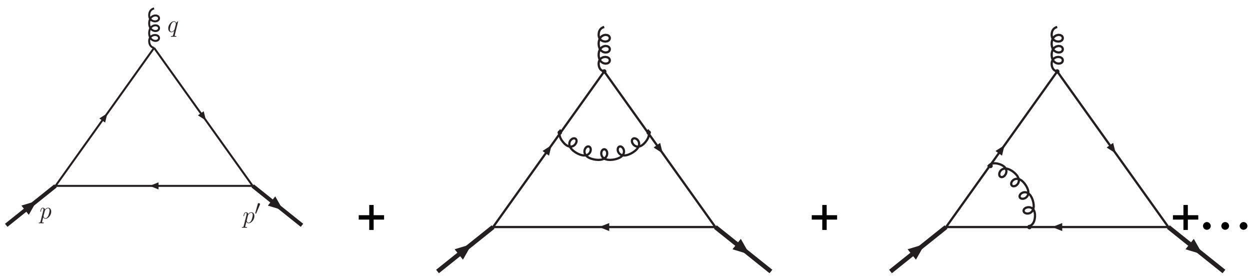
$$\Gamma(p,p',q)=\int\frac{ds}{s-p^2}\frac{ds'}{s'-p'^2}\Delta(s,s',q^2)$$

Perform double Borel transform $p^2\to\tau$, $p'^2\to\tau'$ and applying duality we obtain

$$\exp(-M^2\tau)\exp(-M'^2\tau')ff'F(q^2)=\int^{s_{\rm eff}}ds\exp(-s\tau)\int^{s'_{\rm eff}}ds'\exp(-s'\tau')\Delta_{\rm OPE}(s,s',q^2)$$

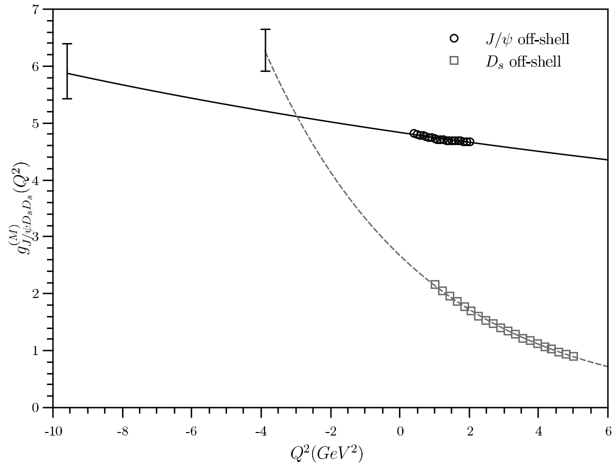
Γ has the following perturbative expansion

$$\Gamma_{\rm OPE}(p^2,p'^2,q^2)=\Gamma_0(p^2,p'^2,q^2)+\alpha_s\Gamma_1(p^2,p'^2,q^2)+\dots$$



A one-loop zero-order diagram has a nonzero double-spectral density and provides a nonzero contribution to the form factor at small and intermediate momentum transfers (and to the coupling). Radiative corrections are crucial for large q^2; at small q^2 one has to specfy the behaviour $\alpha_s(q^2)$. Assuming freezing of $\alpha_s(q^2)$, $O(1)$ and $O(\alpha_s)$ contributions to the pion elastic form factor give comparable contributions at $q^2=0$.

Radiative 2-loop corrections to the 3-point functions relevant for strong couplings are unknown.



The presently available sum-rule extractions have several shortcomings:

• Radiative corrections were not included in the correlator;

the q^2-dependence of the effective threshold has been neglected.

• Calculation in a limited q^2-range may be done

• Extrapolation over large q^2-ranges; unphysical parametrizations are used for some of the ffs

Left aside: **calculation of the coupling using light-cone sum rules**

$$\Gamma(p,q) = \int dx \exp(-iqx)\langle 0|T(j(x)j(0))|M(p)\rangle$$

and expressing the result via the light-cone wave function of the $M(p)$.

Severe and so far unsettled problems in describing the $D^*D\pi$ coupling: $g_{D^*D\pi} = 10 \pm 2$ (vs exp: 18)

Summary and conclusions

We discussed three-meson couplings using constituent quark picture and QCD sum rules.

• Dispersion approach

a. Relativistic dispersion approach based on the constituent quark picture provides form factors as double dispersion representations in terms of the nonperturbative meson wafe functions.

b. These spectral representations satisfy rigorous constraints from nonperturbative QCD in the limit of a heavy-to-heavy and heavy-to-light quark transitions.

c. Fixing a few parameters (constituent quark masses, wave-function parameters) allows one to calculate many form factors in a broad range of momentum transfers (far from hadron thresholds and resonances). Good agreement with results from lattice QCD and the data.

d. Numerical interpolation of the calculated results shows the behaviour well compatible with the presence of the poles at timelike momenta at the "right" locations (i.e. at the location of hadron resonances with the appropriate quantum numbers). This behaviour allows us to extract the strong couplings related to the residues of the form factors.

One and the same coupling may be extracted from the residues of different form factors (related to different transitions like e.g. $D \to D$ and $J\psi \to D$). The extracted values are excellently compatible with each other rendering further credit to our results.

• QCD sum rules have two shortcomings: the method allows one to calculate the form factors in a narrow q^2 range and needs the extrapolation over a wide range of q^2.

• The dispersion approach reports sizeably larger predictions for the three-meson couplings compared to the existing results from QCD sum rules.