QCD Wigner distribution at small-x

Yoshitaka Hatta

(Yukawa institute, Kyoto U.)

1. Probing 5D phase space distributions in DIS

   YH, Bowen Xiao, Feng Yuan, Phys.Rev.Lett. 116 (2016) 202301,

2. Computing 5D distributions in Color Glass Condensate

   Yoshikazu Hagiwara, YH, Takahiro Ueda, Phys.Rev. D94 (2016) 094036

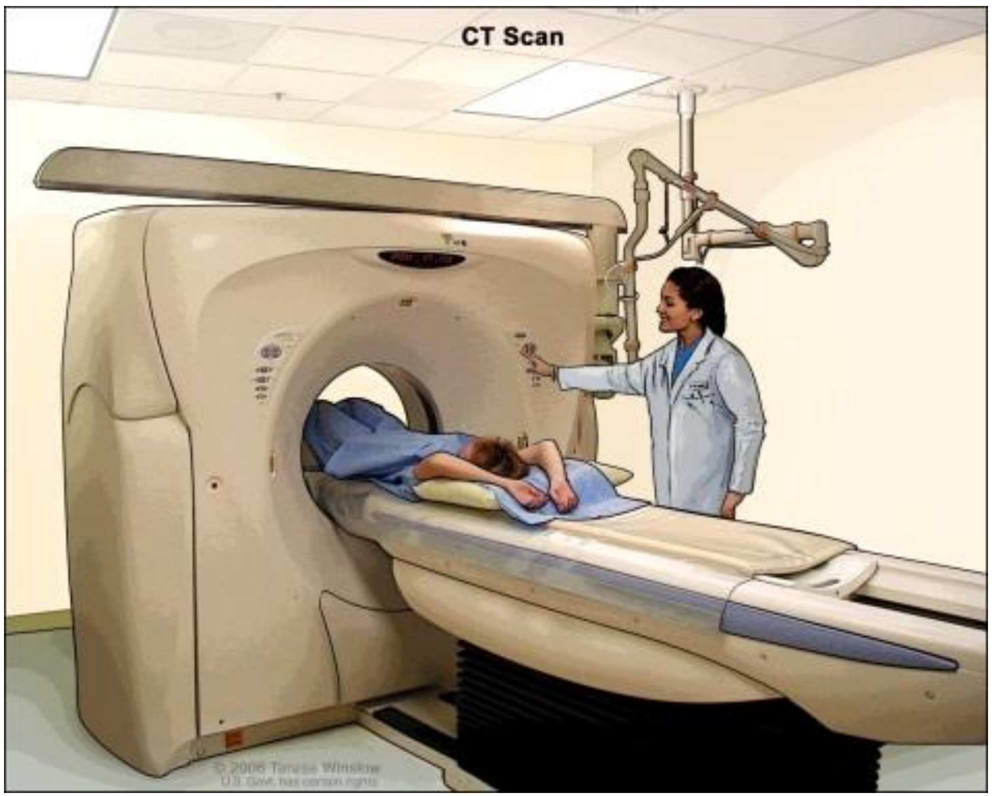
3. How to measure the gluon orbital angular momentum

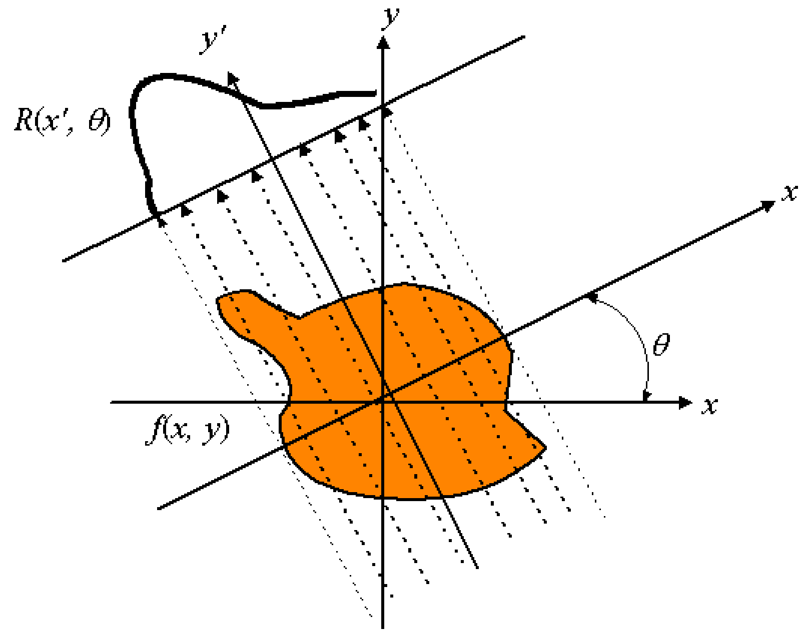
   YH, Yuya Nakagawa, Feng Yuan, Yong Zhao, in preparation

Tomography

CT = Computed Tomography

See inside an object without cutting

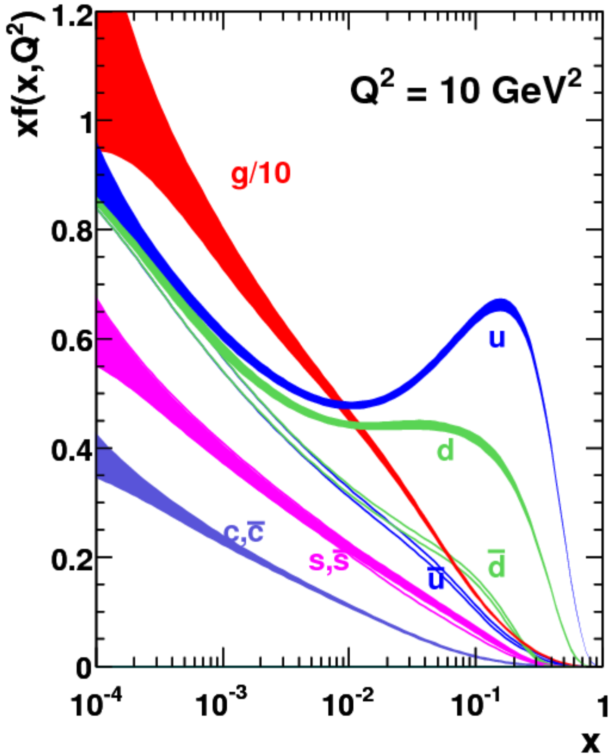




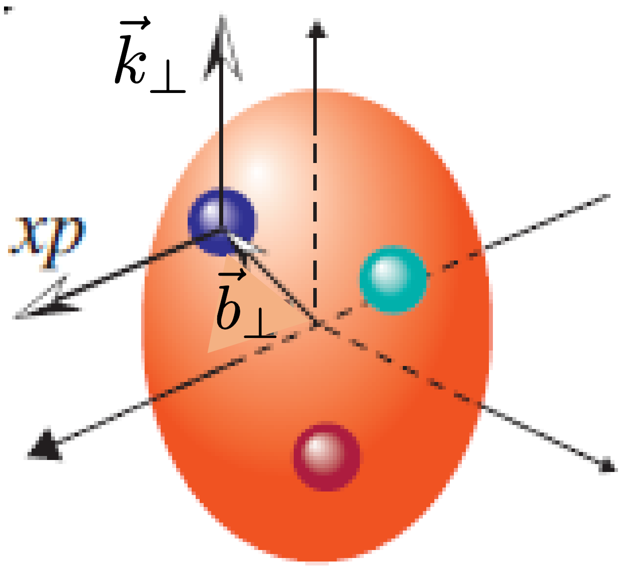
Nucleon tomography

1D tomography: Parton distribution function (PDF)

$$f(x) = \int \frac{dz^-}{4\pi} e^{ixP^+z^-} \langle P|\bar{q}(-\frac{z^-}{2})\gamma^+ q(\frac{z^-}{2})|P\rangle$$



Probability distribution of quarks and gluons with longitudinal momentum fraction $x = \frac{p^+_{parton}}{P^+_{proton}}$

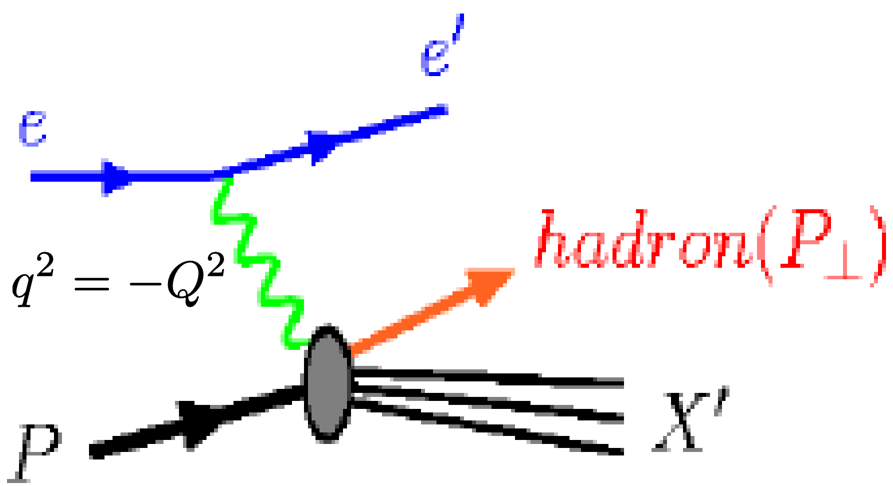


The nucleon is much more complicated!
Partons also have transverse momentum $\vec{k}_\perp$
and are spread in impact parameter space $\vec{b}_\perp$

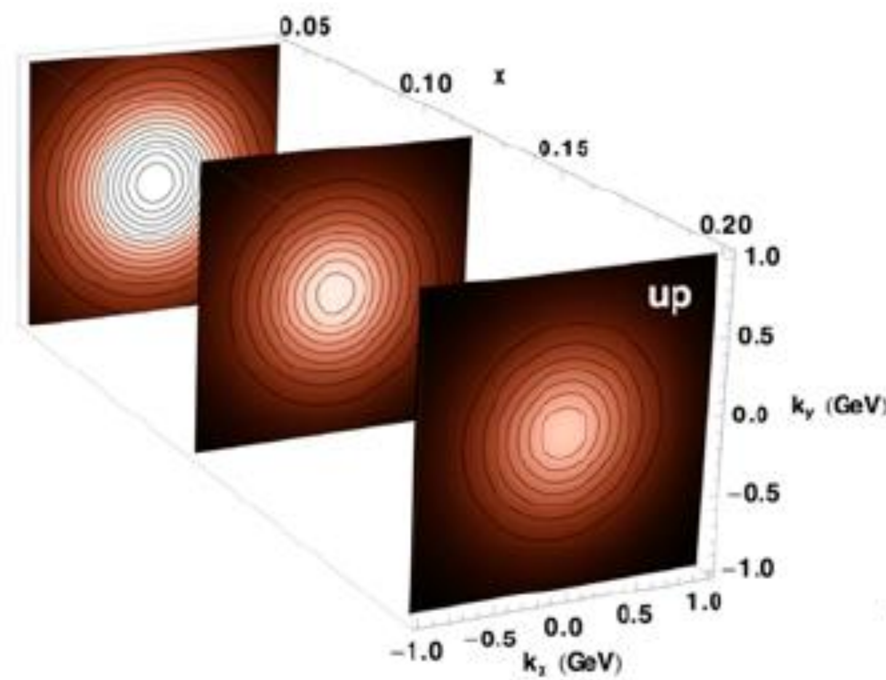
3D tomography:
Transverse momentum dependent distributions (TMD)

$$f(x, \vec{k}_\perp) = \int \frac{dz^- d^2 z_\perp}{16\pi^3} e^{ixP^+ z^- - i\vec{k}_\perp \cdot \vec{z}_\perp} \langle P | \bar{q}(-z/2)\gamma^+ W q(z/2) | P \rangle$$

Relevant in semi-inclusive DIS (SIDIS), etc.



$$Q \gg P_\perp \gtrsim \Lambda_{QCD}$$



3D tomography:
Generalized parton distributions (GPD)

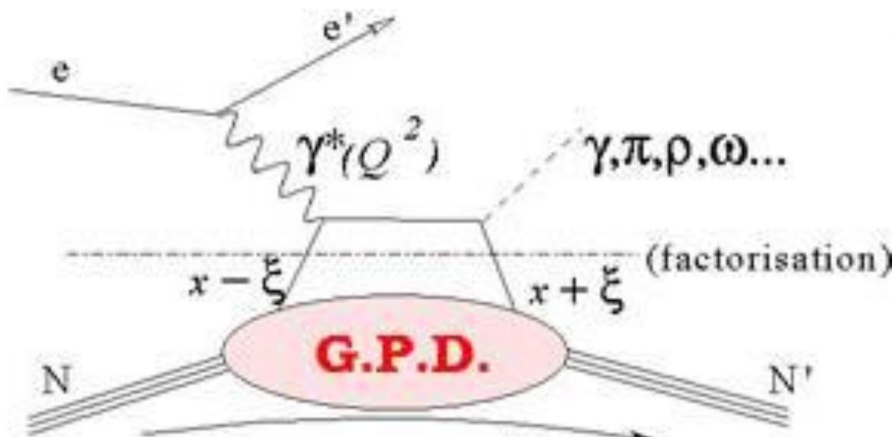
$$f(x,\vec{\Delta}_\perp) \sim \int \frac{dz^-}{4\pi} e^{ixP^+z^-} \langle P - \frac{\Delta}{2} | \bar{q}(-z/2)\gamma^+ q(z/2) | P + \frac{\Delta}{2} \rangle$$

$\Longleftrightarrow$ $f(x, \vec{b}_\perp)$ distribution of partons in impact parameter space

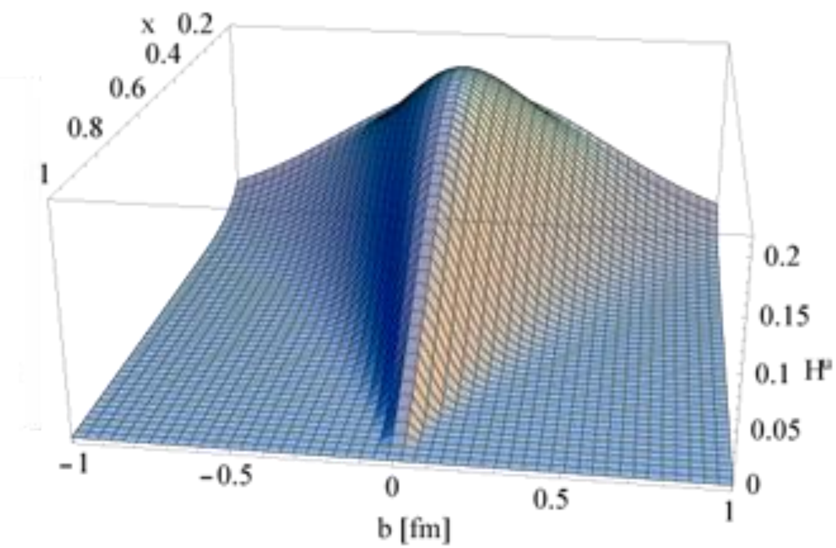
Fourier transform

$\vec{\Delta}_\perp \leftrightarrow \vec{b}_\perp$

Deeply Virtual Compton Scattering (DVCS)



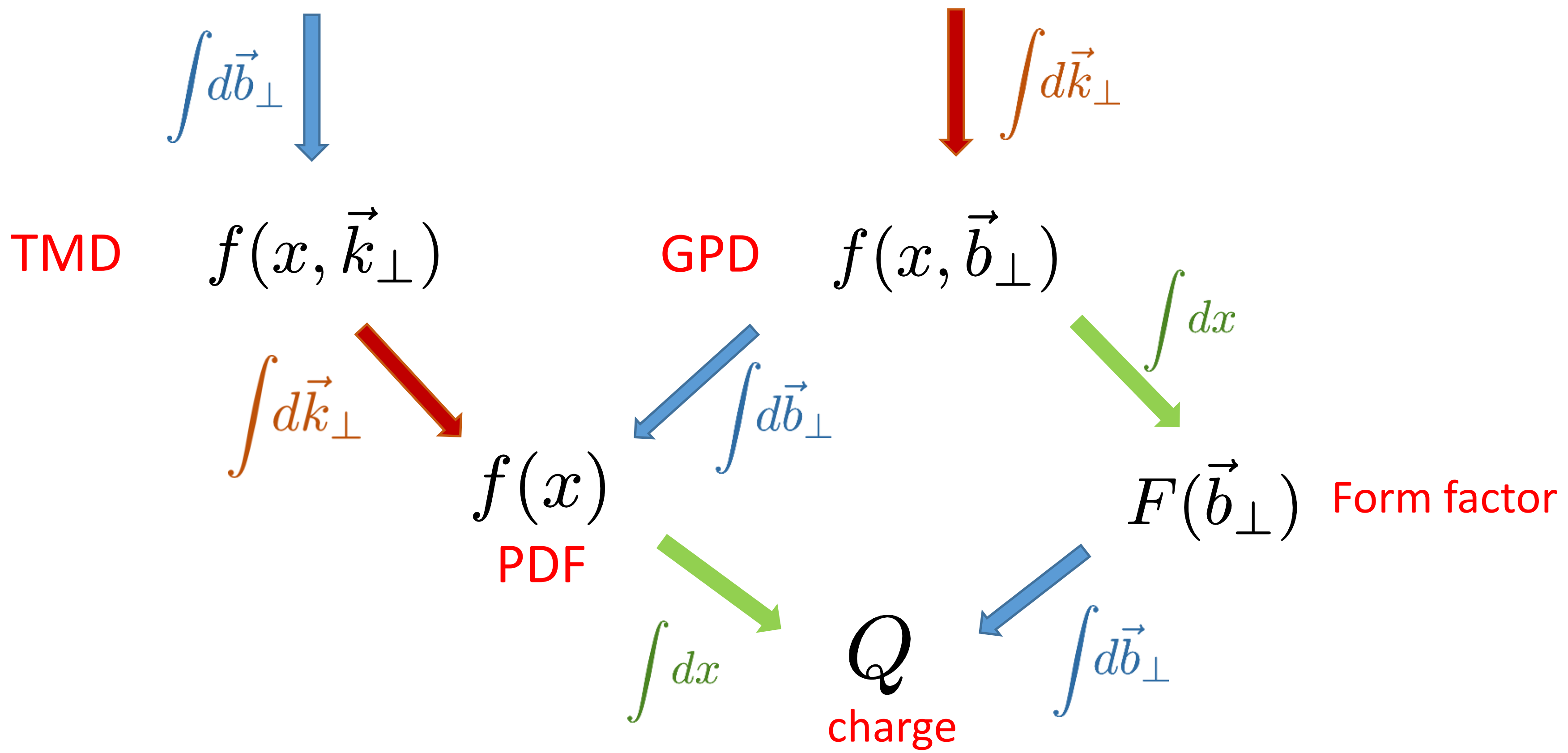
$$t = -\vec{\Delta}_\perp^2$$



5D tomography:
Wigner distribution— the "mother distribution"

Belitsky, Ji, Yuan (2003);
Lorce, Pasquini (2011)

$$W(x,\vec{k}_\perp,\vec{b}_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{i\vec{b}_\perp\cdot\vec{\Delta}_\perp} \int \frac{dz^- d^2 z_\perp}{16\pi^3} e^{ixP^+z^- - i\vec{k}_\perp\cdot\vec{z}_\perp} \langle P - \frac{\Delta}{2} | \bar{q}(-z/2)\gamma^+ q(z/2) | P + \frac{\Delta}{2} \rangle$$



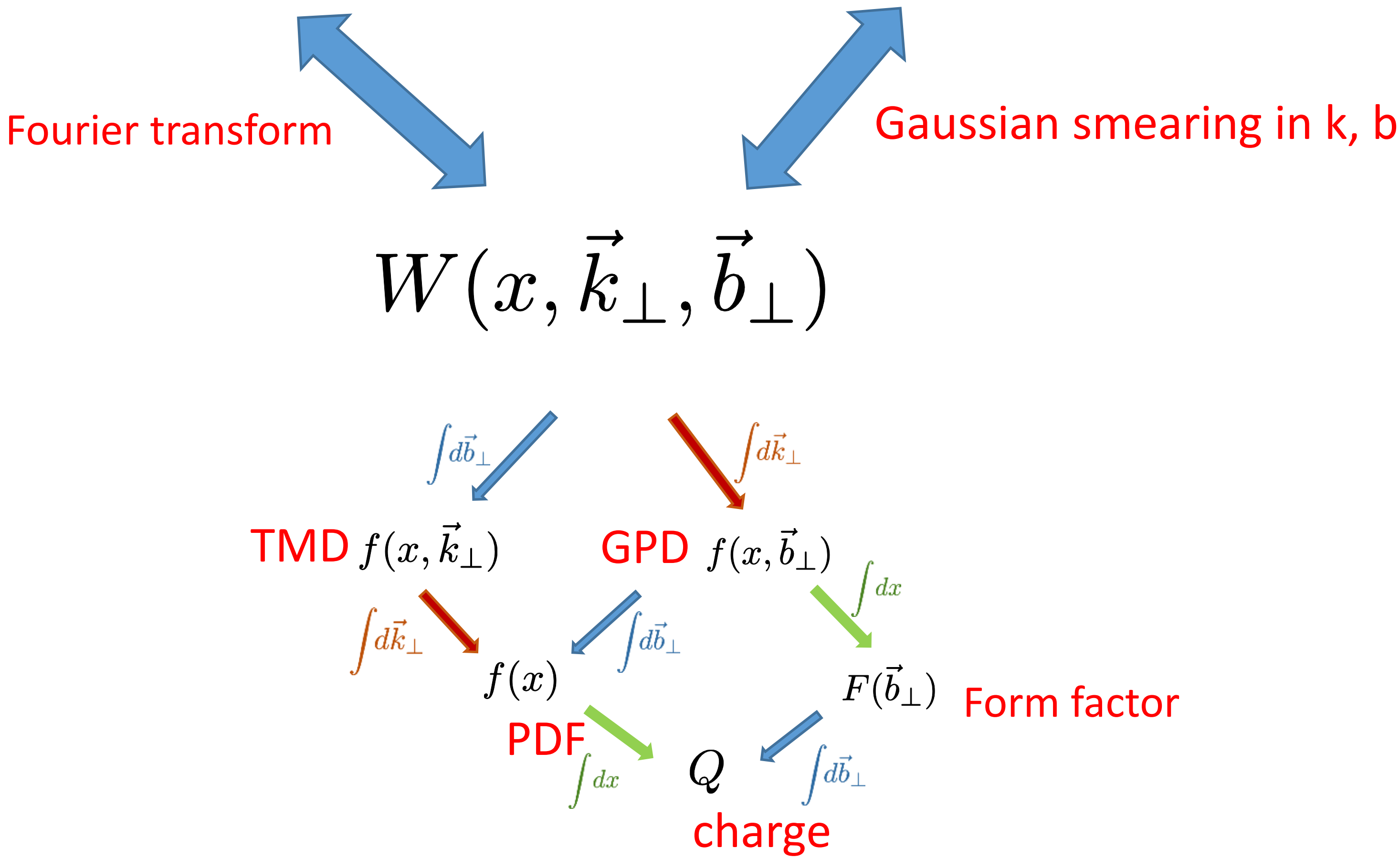
Generalized TMD and Husimi

GTMD Meissner, Metz, Schlegel (2009)

$$F(x, \vec{k}_\perp, \vec{\Delta}_\perp)$$

Husimi Hagiwara, YH (2015)

$$H(x, \vec{k}_\perp, \vec{b}_\perp)$$



Husimi distribution

Kodi Husimi (1909—2008)

Gaussian smearing of the Wigner distribution

$$H(q,p,t) = \frac{1}{\pi\hbar}\int dq'dp' e^{-m\omega(q'-q)^2/\hbar-(p'-p)^2/m\omega\hbar}W(q',p',t)$$

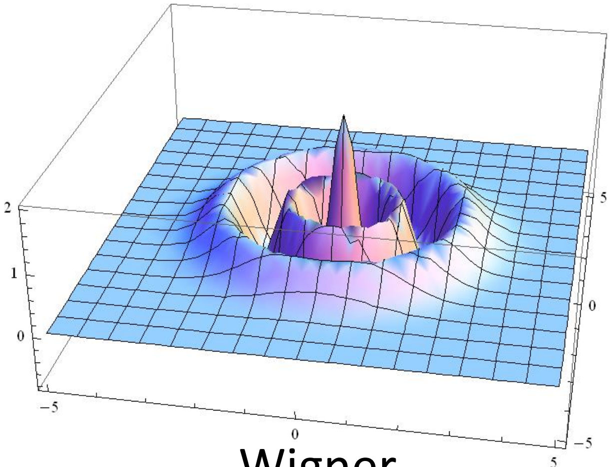
$\delta q \delta p = \hbar/2$ minimal uncertainty relation

coherent state

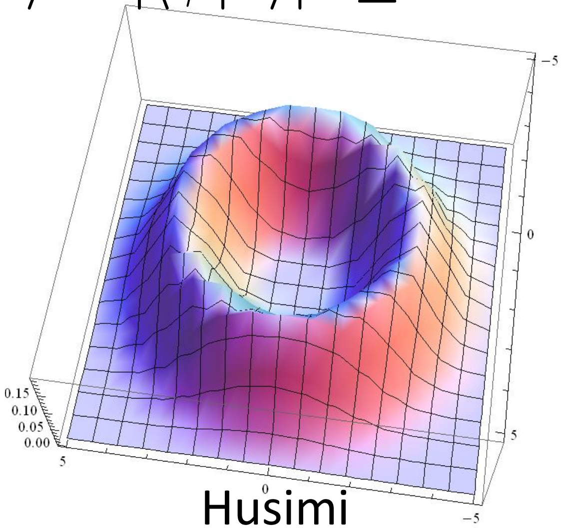
Husimi distribution is positive!

$$H(q,p,t) = \langle\lambda|\hat{\rho}|\lambda\rangle = |\langle\psi|\lambda\rangle|^2 \geq 0$$

1D harmonic oscillator, 4th excited state

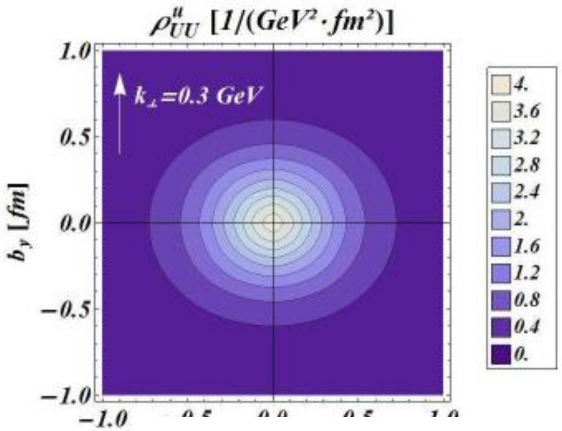


Wigner



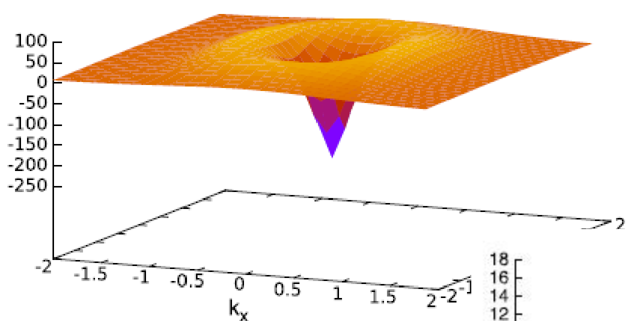
Husimi

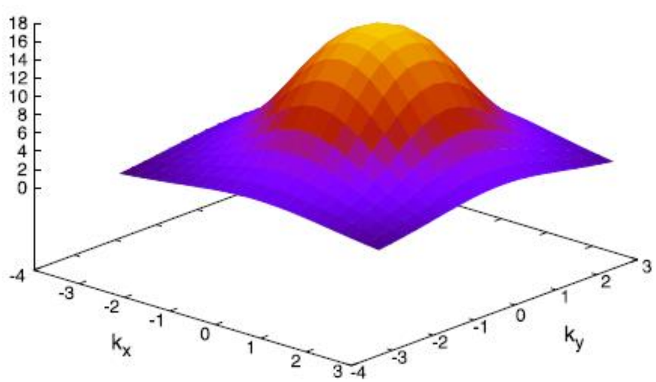
Wigner/Husimi/GTMD
model calculations, 1-loop calculations, evolution...



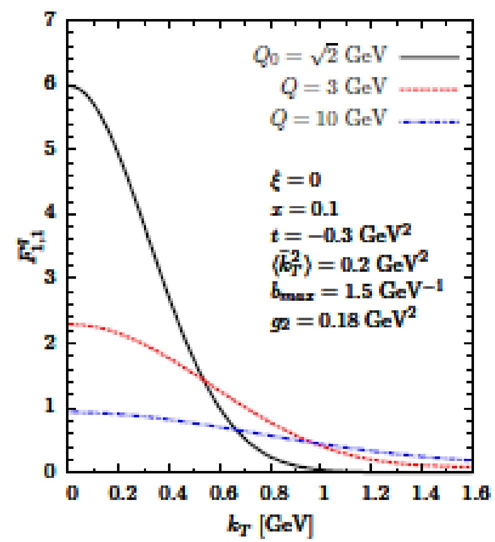
Wigner

Lorce, Pasquini
Liu, Ma
Mukherjee, Nair, Ojha
Courtoy, Goldstein, Hernandez, Liuti, Rajan
Hagiwara, YH
Echevarria, Idilbi, Kanazawa, Lorce,
Metz, Pasquini, Schlegel
Gutsche, Lyubovitskij, Schmidt



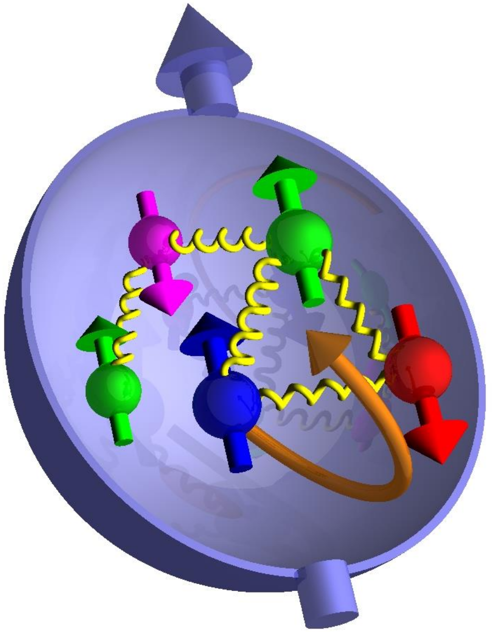


Husimi



GTMD

Parton orbital angular momentum



Nucleon spin decomposition

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q + L^g$$

Quarks' helicity

Gluons' helicity

Canonical Orbital angular momentum

$$L^{q,g} = \int dx \int d^2b_\perp d^2k_\perp (\vec{b}_\perp \times \vec{k}_\perp)_z \begin{cases} W^{q,g}(x, \vec{b}_\perp, \vec{k}_\perp) \\ H^{q,g}(x, \vec{b}_\perp, \vec{k}_\perp) \end{cases}$$

Lorce, Pasquini, (2011);
YH (2011)

Wigner distribution: Is it measurable?

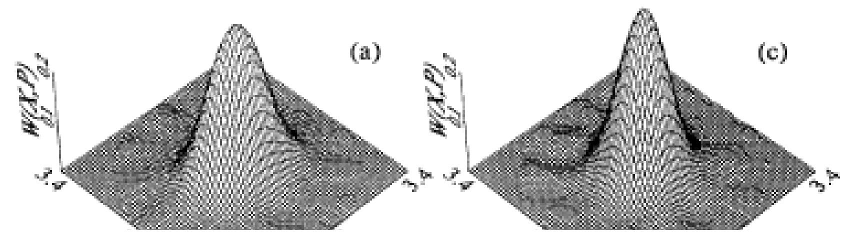
In quantum optics, yes!

VOLUME 70, NUMBER 9 PHYSICAL REVIEW LETTERS 1 MARCH 1993

Measurement of the Wigner Distribution and the Density Matrix of a Light Mode Using Optical Homodyne Tomography: Application to Squeezed States and the Vacuum

D. T. Smithey, M. Beck, and M. G. Raymer
Department of Physics and Chemical Physics Institute, U

A. Faridani
Department of Mathematics, Oregon State Uni
(Received 16 Novembe



What about in QCD? Go to small-x!

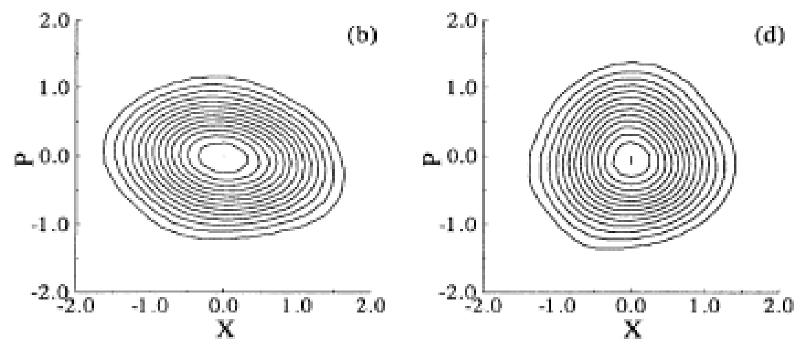


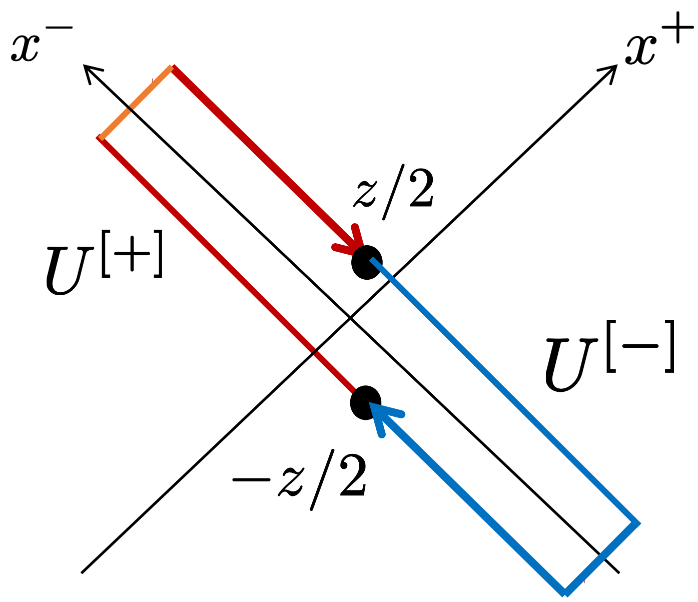
FIG. 1. Measured Wigner distributions for (a),(b) a squeezed state and (c),(d) a vacuum state, viewed in 3D and as contour plots, with equal numbers of constant-height contours. Squeezing of the noise distribution is clearly seen in (b).

Gluon Wigner distribution

$$xW(x,\vec{k}_\perp,\vec{b}_\perp)$$
$$= \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{i\vec{b}_\perp\cdot\vec{\Delta}_\perp} \int \frac{dz^- d^2 z_\perp}{16\pi^3} e^{ixP^+z^- - i\vec{k}_\perp\cdot\vec{z}_\perp} \langle P-\Delta/2|F^{+i}(-z/2)F^+_i(z/2)|P+\Delta/2\rangle$$

There are two ways to make it gauge invariant

Dominguez, Marquet, Xiao, Yuan (2011)



Dipole distribution

$$\mathrm{Tr}[F(-z/2)U^{[+]}F(z/2)U^{[-]}]$$

Weizsacker-Williams distribution

$$\mathrm{Tr}[F(-z/2)U^{[+]}F(z/2)U^{[+]}]$$

Dipole gluon Wigner distribution at small-x

YH, Xiao, Yuan (2016)

$$xW(x,\vec{k}_\perp,\vec{b}_\perp)$$
$$= \int_{\Delta_\perp} e^{i\vec{b}_\perp\cdot\vec{\Delta}_\perp} \int_{z^-,z_\perp} e^{ixP^+z^- - i\vec{k}_\perp\cdot\vec{z}_\perp} \langle P-\Delta/2|\mathrm{Tr}F^{+i}(-z/2)U^{[+]}F^+_i(z/2)U^{[-]}|P+\Delta/2\rangle$$

Approximate $e^{ixP^+z^-} \approx 1$

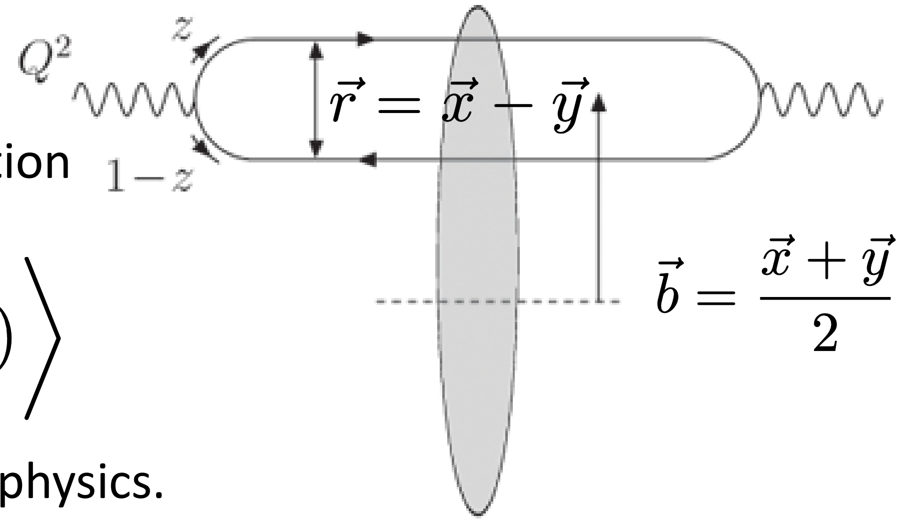
Use the identity $$\partial_z^i U_{\infty,-\infty}(\vec{z}_\perp) = -ig\int dz^- U_{\infty,z^-}\underbrace{\partial^i A^+(z^-,\vec{z}_\perp)}_{-F^{+i}+D^+A^i} U_{z^-,-\infty}$$

$$xW(x,\vec{k}_\perp,\vec{b}_\perp) \approx \frac{2N_c}{\alpha_s}\int \frac{d^2\vec{r}_\perp}{(2\pi)^2} e^{i\vec{k}_\perp\cdot\vec{r}_\perp}\left(\frac{1}{4}\vec{\nabla}_b^2 - \vec{\nabla}_r^2\right)$$
$$\times\left\langle \frac{1}{N_c}\mathrm{Tr}\, U\left(\vec{b}_\perp - \frac{\vec{r}_\perp}{2}\right)U^\dagger\left(\vec{b}_\perp + \frac{\vec{r}_\perp}{2}\right)\right\rangle_x$$

``Dipole S-matrix”

Dipole S-matrix

Forward S-matrix of a dipole ($q\bar{q}$ pair)
off a dense target in the eikonal approximation



$$S(\vec{x},\vec{y}) = \left\langle \frac{1}{N_c} \mathrm{Tr}\, U(\vec{x})\, U^\dagger(\vec{y}) \right\rangle$$

A fundamental object in small-x/saturation physics.

Satisfies the Balitsky-Kovchegov (BK) equation

$$\partial_Y S(\vec{x},\vec{y}) = \bar{\alpha}_s \int \frac{d^2\vec{z}}{2\pi} \frac{(\vec{x}-\vec{y})^2}{(\vec{x}-\vec{z})^2(\vec{z}-\vec{y})^2} \left(S(\vec{x},\vec{z})S(\vec{z},\vec{y}) - S(\vec{x},\vec{y})\right)$$

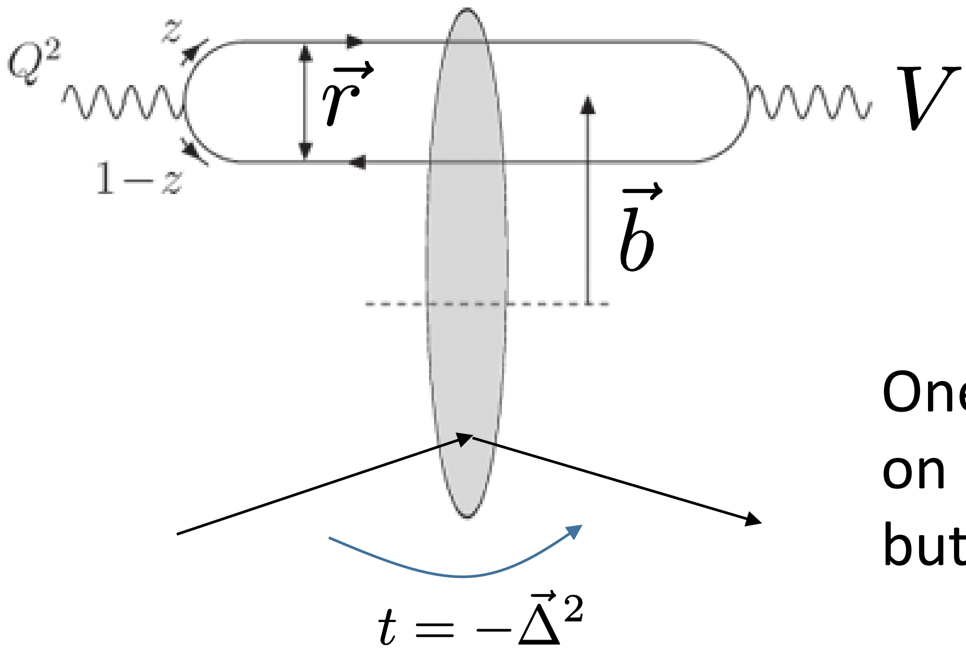
$Y = \ln 1/x$

Find a process which is sensitive to both $\vec{b}$ and $\vec{r}$.

Diffractive vector meson production

$$\frac{d\sigma}{dt} = \frac{1}{4\pi} \left| \int d^2b \, e^{-i\vec{\Delta}\cdot\vec{b}} A(\vec{b}) \right|^2$$

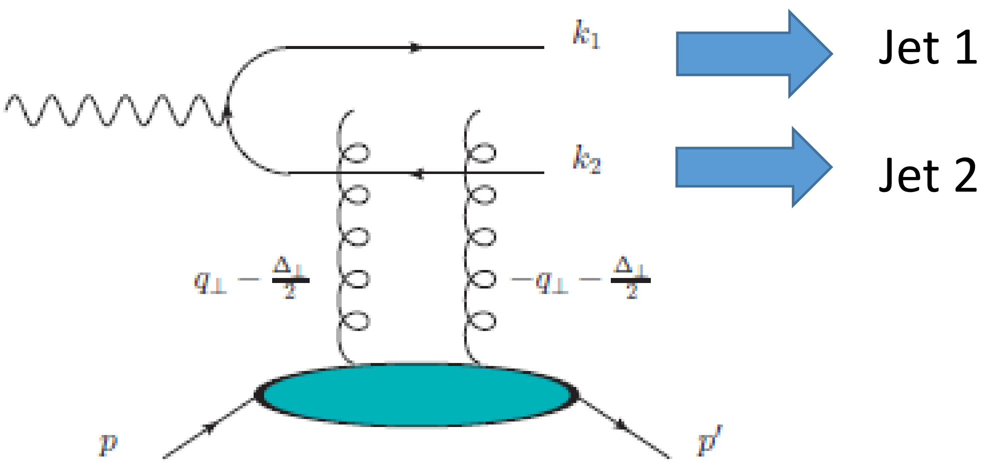
$$A(\vec{b}) = i \int d^2\vec{r} \int dz \Psi^{\gamma^*}(Q, z, \vec{r})(1 - S(\vec{r}, \vec{b}))\Psi^V(\vec{r}, z)$$



One can study the dependence on $\vec{b}$ $(\leftrightarrow \vec{\Delta})$ (Munier, Stasto, Mueller, 2001) but not on $\vec{k}$ $(\leftrightarrow \vec{r})$

Diffractive dijet production in DIS

YH, Xiao, Yuan (2016) see also, Altinoluk, Armesto, Beuf, Rezaeian (2015)



$$\vec{\Delta}_\perp = -(\vec{k}_{1\perp} + \vec{k}_{2\perp})$$

Proton recoil momentum

$$\vec{P}_\perp = \frac{1}{2}(\vec{k}_{2\perp} - \vec{k}_{1\perp})$$

Dijet relative momentum

Fourier transform of $S(\vec{r}_\perp, \vec{b}_\perp)$

$$\frac{d\sigma^{\gamma_T^* A \to q\bar{q}X}}{dy_1 d^2k_{1\perp} dy_2 d^2 k_{2\perp}} \propto z(1-z)[z^2 + (1-z)^2] \int d^2q_\perp d^2q'_\perp \mathcal{F}(q_\perp, \Delta_\perp)\mathcal{F}(q'_\perp, \Delta_\perp)$$

$$\times \left[\frac{\vec{P}_\perp}{P_\perp^2 + \epsilon^2} - \frac{\vec{P}_\perp - \vec{q}_\perp}{(P_\perp - q_\perp)^2 + \epsilon^2}\right] \cdot \left[\frac{\vec{P}_\perp}{P_\perp^2 + \epsilon^2} - \frac{\vec{P}_\perp - \vec{q}'_\perp}{(P_\perp - q'_\perp)^2 + \epsilon^2}\right]$$

$$\sim \left(\mathcal{F}(\vec{P}_\perp, \vec{\Delta}_\perp)\right)^2$$ (for $\epsilon \sim Q$ small)

`Elliptic' Wigner distribution

$\cos 2\phi$ correlation between $\vec{P}_\perp$ and $\vec{\Delta}_\perp$ expected at small-x

$$W(x,\vec{k}_\perp,\vec{b}_\perp) = W_0(x,k_\perp,b_\perp) + 2\cos 2(\phi_k - \phi_b)W_1(x,k_\perp,b_\perp) + \cdots$$

$$\frac{d\sigma}{d^2k_{1\perp}d^2k_{2\perp}} \sim d\sigma_0 + 2\cos 2(\phi_P - \phi_\Delta)d\tilde{\sigma}$$

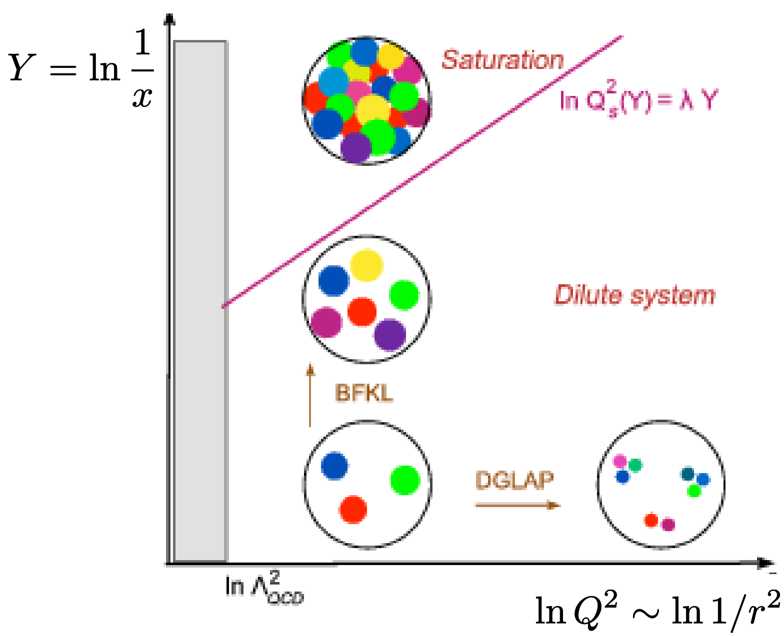
Measurable at EIC

Wigner/Husimi/GTMD in the Color Glass Condensate

Hagiwara, YH, Ueda (2016)

Small-x evolution of dipole S-matrix including gluon saturation
→ Balitsky-Kovchegov (BK) equation

$$\partial_Y S(\vec{x},\vec{y}) = \int \frac{d^2\vec{z}}{2\pi} \frac{(\vec{x}-\vec{y})^2}{(\vec{x}-\vec{z})^2(\vec{z}-\vec{y})^2} \left(S(\vec{x},\vec{z})S(\vec{z},\vec{y}) - S(\vec{x},\vec{y})\right)$$



$$S(\vec{x},\vec{y}) = S(\vec{r},\vec{b}) \qquad \vec{r} = \vec{x}-\vec{y}$$

$$\vec{b} = \frac{\vec{x}+\vec{y}}{2}$$

Solve BK keeping the b-dependence

Golec-Biernat, Stasto (2003)
Ikeda, McLerran (2004)
Marquet, Soyez (2005)
Berger, Stasto (2010)

Hidden SO(3) symmetry

BK equation conformally symmetric.

Assume that a SO(3) subgroup of conformal group survives — Gubser (2011)

$$S(\vec{x}, \vec{y}) = S(d^2(\vec{x}, \vec{y}))$$

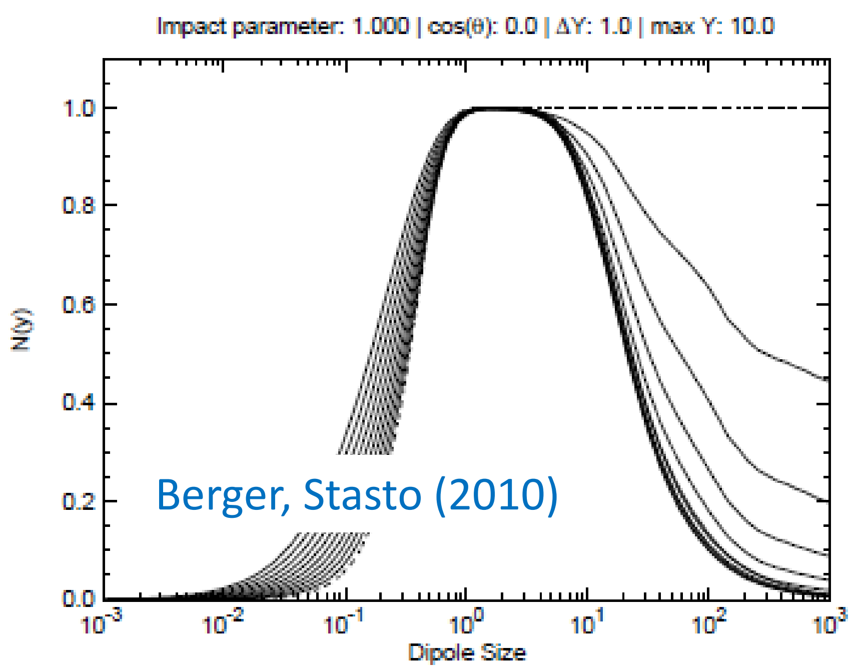
$$d^2(\vec{x}, \vec{y}) = \frac{R^2(\vec{x} - \vec{y})^2}{(R^2 + x^2)(R^2 + y^2)}$$

Realistic initial conditions break this symmetry…

HOWEVER, the symmetry is dynamically restored by the equation!

"What is interesting is the fact that the amplitude has a maximum for the dipole size which is twice its impact parameter r=2b"

Stasto, Golec-Biernat (2003)



$d^2(\vec{r},\vec{b})$ exactly invariant under $r \leftrightarrow \dfrac{4(b^2+R^2)}{r}$

Turning point: $r = 2\sqrt{b^2+R^2}$

We just need to solve

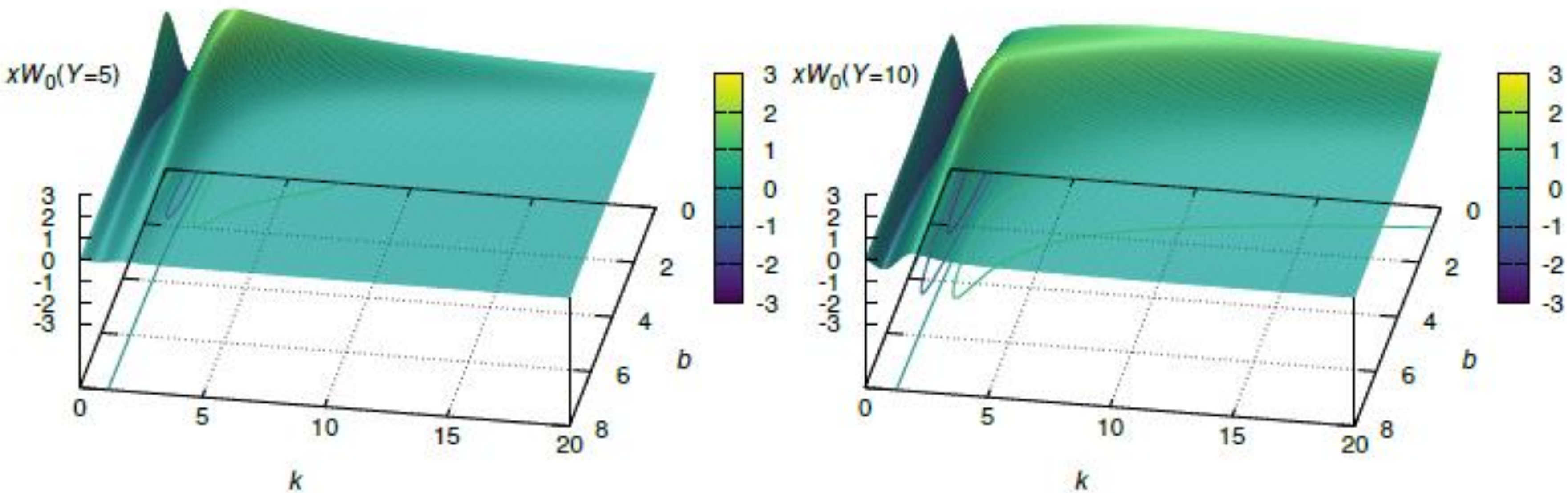
$$\partial_Y g(x) = \bar{\alpha}_s \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{\infty} \frac{dz}{z} \frac{x^2}{(x^2+z^2-2xz\cos\phi)} \times \left\{ g_Y\left(\sqrt{\frac{R^4(x^2+z^2-2xz\cos\phi)}{R^4+x^2z^2+2R^2xz\cos\phi}}\right) g_Y(z) - g_Y(x) \right\}$$

$$S_Y(\boldsymbol{x},\boldsymbol{y}) = S_Y(d^2(\boldsymbol{x},\boldsymbol{y})) = g_Y\left(\sqrt{\frac{R^2 d^2(\boldsymbol{x},\boldsymbol{y})}{1-d^2(\boldsymbol{x},\boldsymbol{y})}}\right)$$

Wigner

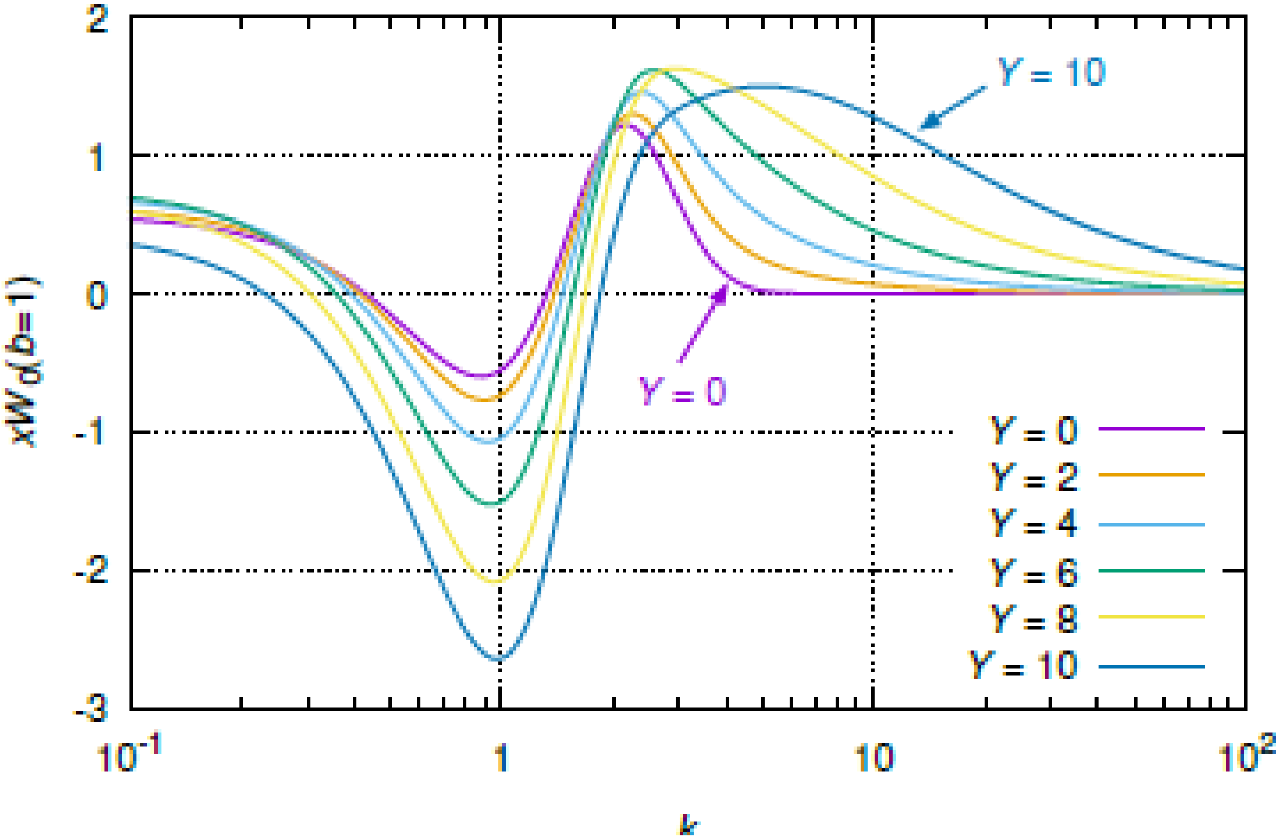
$$xW(x,\vec{k}_\perp,\vec{b}_\perp) \approx \frac{2N_c}{\alpha_s}\int \frac{d^2\vec{r}_\perp}{(2\pi)^2} e^{i\vec{k}_\perp\cdot\vec{r}_\perp} e^{-\epsilon r^2} \left(\frac{1}{4}\vec{\nabla}_b^2 - \vec{\nabla}_r^2\right) S(\vec{b}_\perp,\vec{r}_\perp)$$

insert a Gaussian factor
(mimic confinement)



Peak at the saturation momentum $k = Q_s(Y,b)$

Rapidity evolution

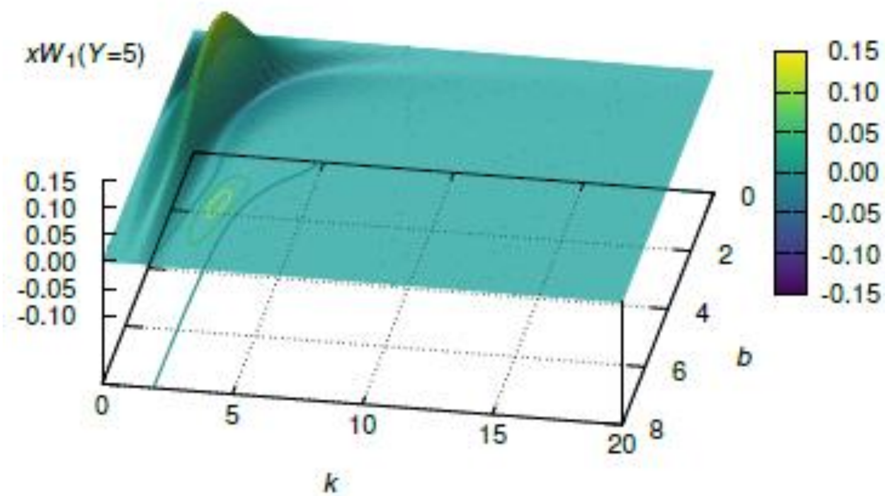


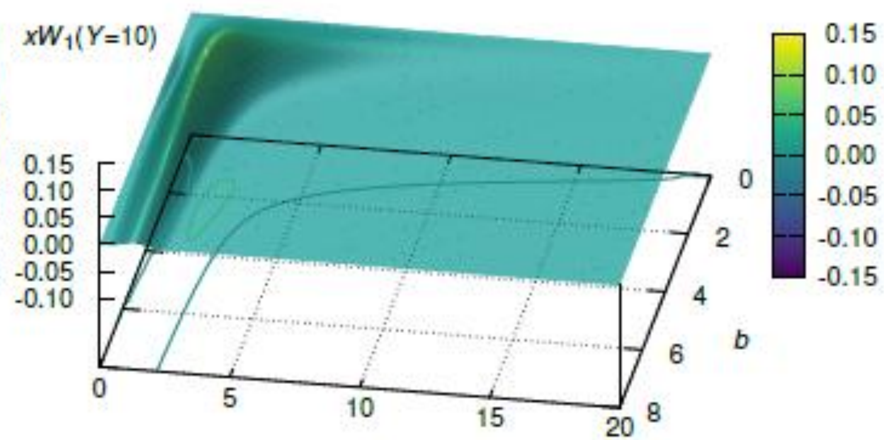
$$k = Q_s(Y, b) \sim e^{\#Y}$$

increasing function of Y,
decreasing function of b

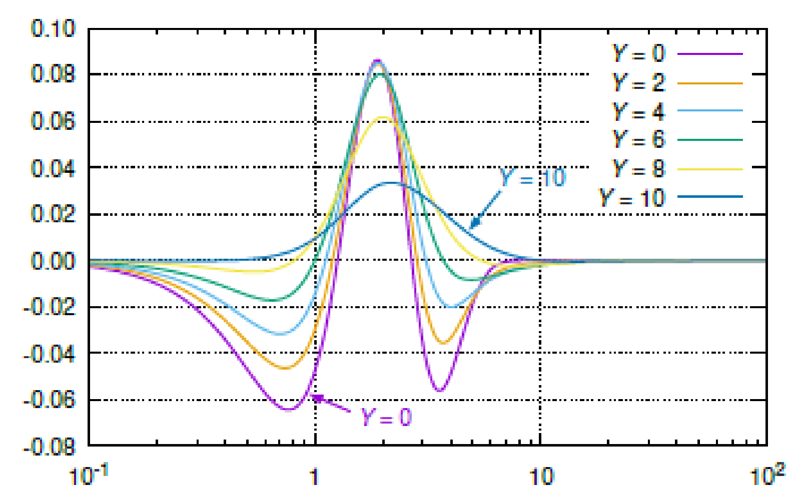
Elliptic Wigner

$$W(x, \vec{k}_\perp, \vec{b}_\perp) = W_0(x, k_\perp, b_\perp) + 2\cos 2(\phi_k - \phi_b) W_1(x, k_\perp, b_\perp) + \cdots$$





A few percent effect.
Peak does not move with Y



No geometric scaling in the elliptic part

SO(3)-symmetric geometric scaling

$$S_Y(r_\perp, b_\perp) \sim f(Q_s^2(Y) d^2(b_\perp, r_\perp))$$

$$d^2 \approx \frac{r^2}{R^2}\left(1 + \frac{b^2 r^2}{2R^4}\cos 2\phi_{br}\right)$$

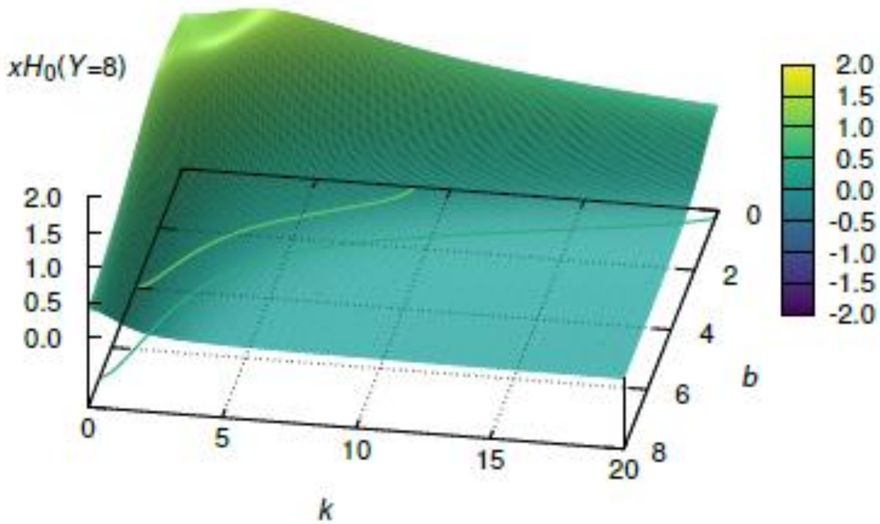
$$k_{peak} \sim \sqrt{Q_s(Y)}$$

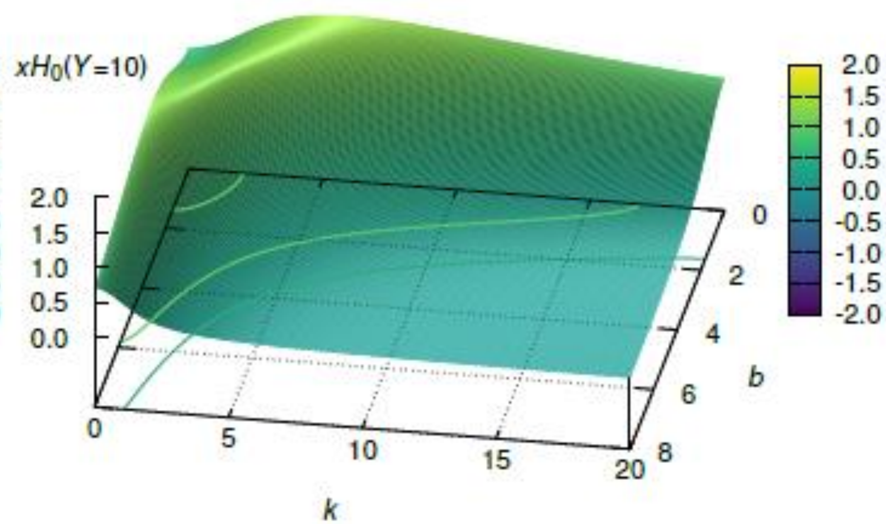
Husimi

$$xH(x, b_\perp, k_\perp) := \frac{1}{\pi^2} \int d^2 b'_\perp d^2 k'_\perp e^{-\frac{1}{l^2}(b_\perp - b'_\perp)^2 - l^2 (k_\perp - k'_\perp)^2} xW(x, k'_\perp, b'_\perp)$$

Two Gaussian widths inversely related such that $\delta k \delta b = \frac{\hbar}{2}$

Large-r region naturally suppressed.



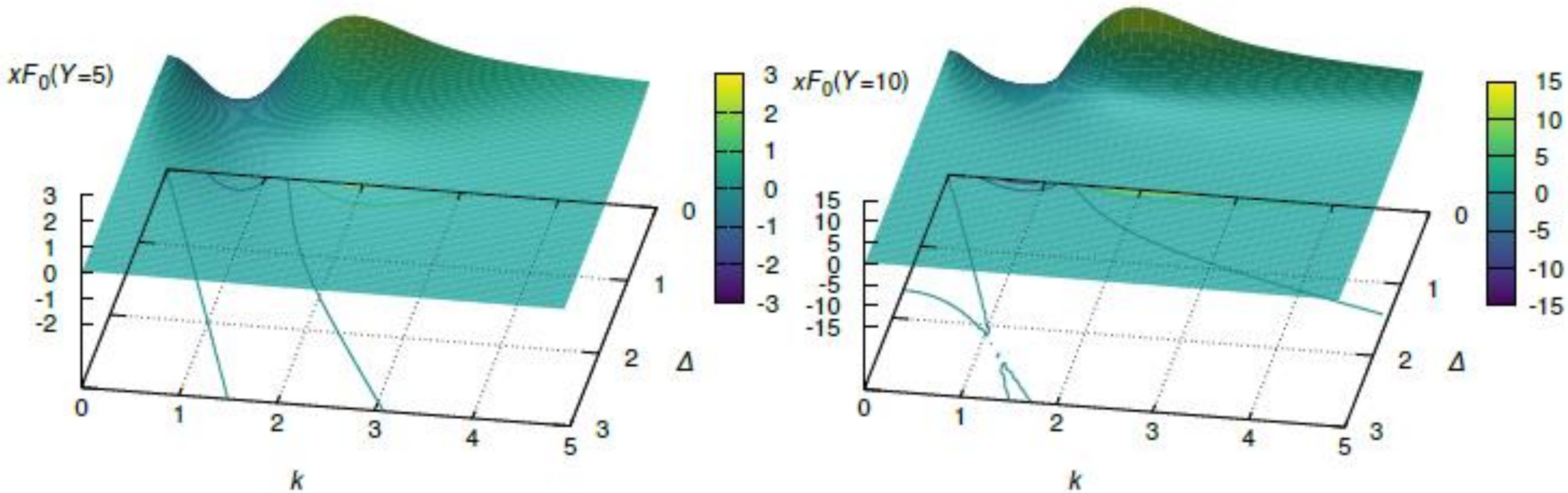


Positive everywhere → probability distribution of gluons in CGC

GTMD

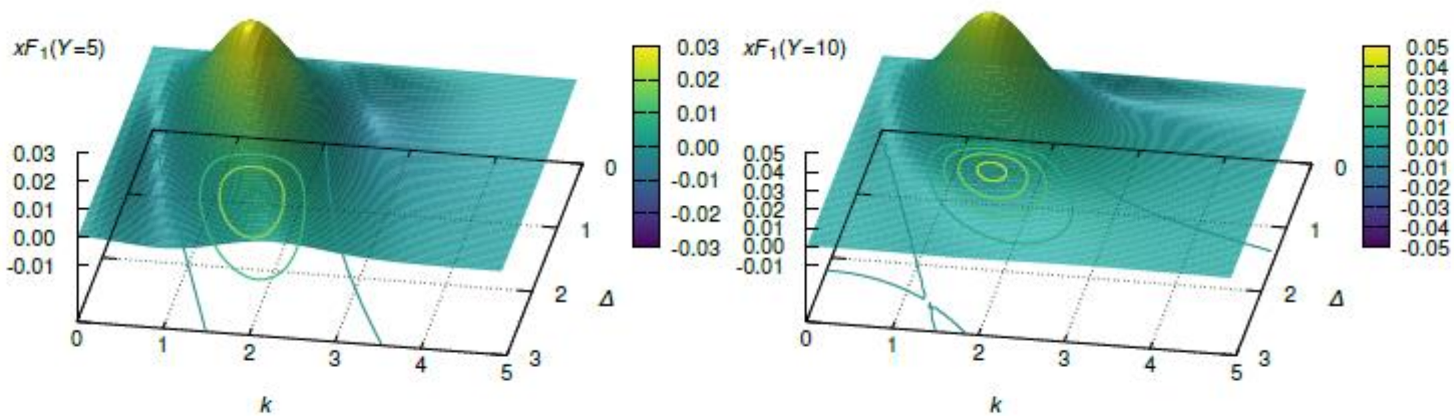
$$xF(x, k_\perp, \Delta_\perp) \equiv \int \frac{d^2b}{(2\pi)^2}\, e^{ib_\perp \cdot \Delta_\perp} xW(x, k_\perp, b_\perp)$$

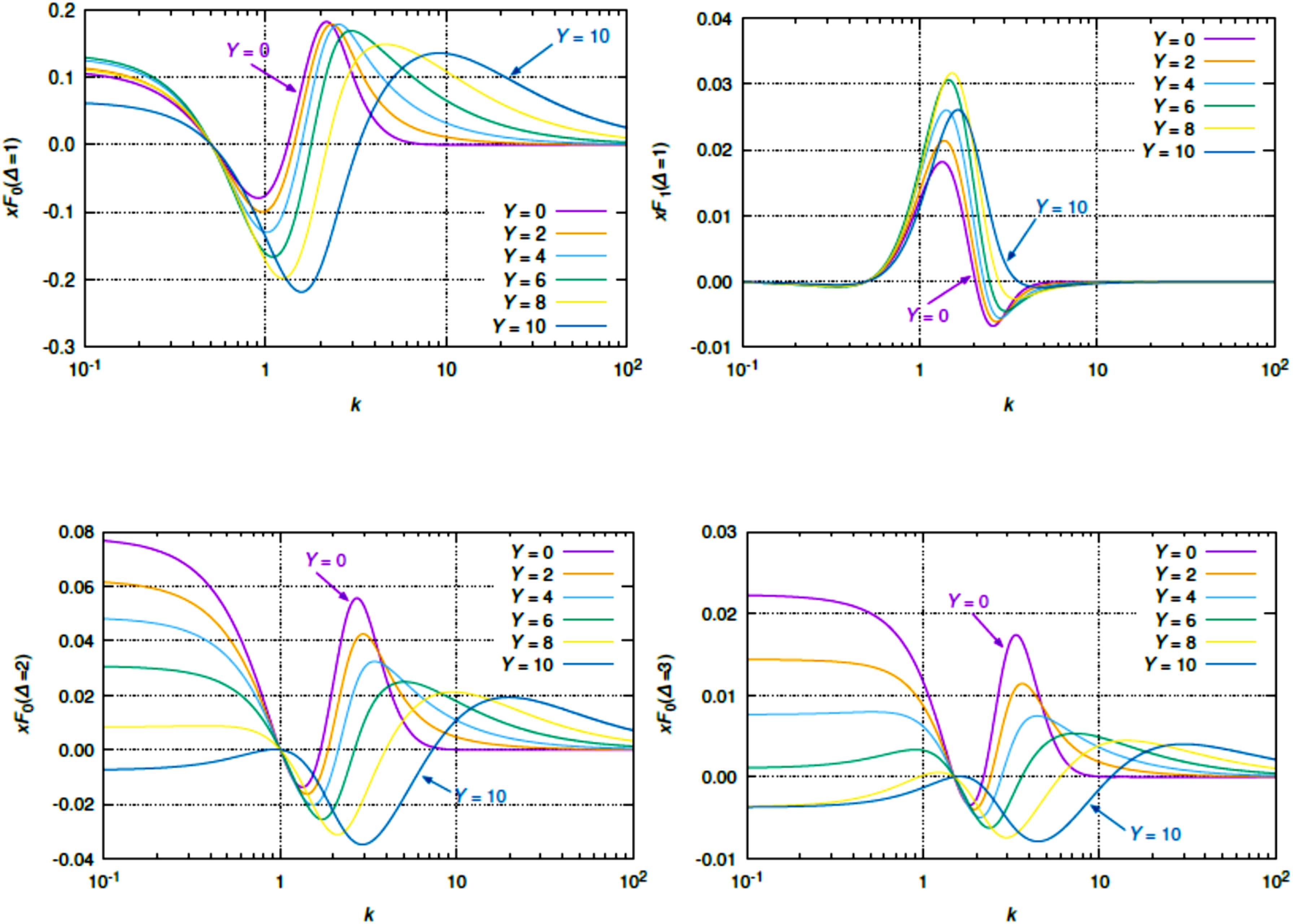
Directly connected to dijet cross section



$k = Q_s(Y, \Delta)$ increasing function of Δ

Marquet, Soyez (2005)





0.2
0.1
0.0
-0.1
-0.2
-0.3
xF0(Δ=1)
Y = 0
Y = 10
Y = 0
Y = 2
Y = 4
Y = 6
Y = 8
Y = 10
10-1
1
10
102
k
0.04
0.03
0.02
0.01
0.00
-0.01
xF1(Δ=1)
Y = 0
Y = 2
Y = 4
Y = 6
Y = 8
Y = 10
Y = 10
Y = 0
10-1
1
10
102
k
0.08
0.06
0.04
0.02
0.00
-0.02
-0.04
xF0(Δ=2)
Y = 0
Y = 0
Y = 2
Y = 4
Y = 6
Y = 8
Y = 10
Y = 10
10-1
1
10
102
k
0.03
0.02
0.01
0.00
-0.01
xF0(Δ=3)
Y = 0
Y = 0
Y = 2
Y = 4
Y = 6
Y = 8
Y = 10
Y = 10
10-1
1
10
102
k

How to measure the gluon OAM

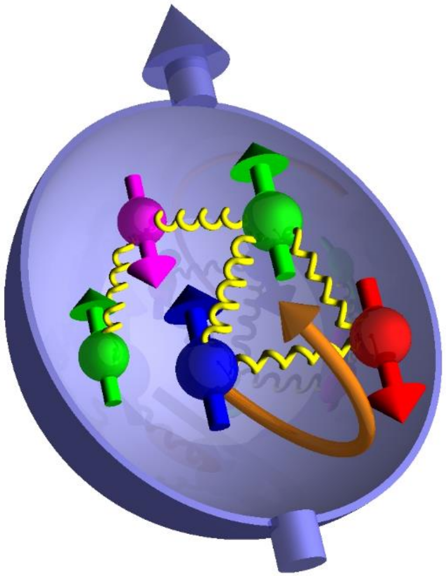
YH, Yuya Nakagawa, Feng Yuan, Yong Zhao, in preparation

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q + L^g$$

Quarks' helicity

Gluons' helicity

Canonical Orbital angular momentum



Missing pieces of the Jaffe-Manohar decomposition
→ OAM is the future of spin physics!

Even the proper gauge invariant definition not available until recently.

Nobody knows how to measure them.

OAM parton distribution

One can define the partonic density of OAM

YH, Yoshida (2012)

$$L^{q,g}(x) = \int d^2b_\perp d^2k_\perp (\vec{b}_\perp \times \vec{k}_\perp) W^{q,g}(x, \vec{b}_\perp, \vec{k}_\perp)$$

$$\begin{aligned} L^g_{can}(x) = & \frac{x}{2}\int_x^{\epsilon(x)} \frac{dx'}{x'^2}(H_g(x') + E_g(x')) - x\int_x^{\epsilon(x)} \frac{dx'}{x'^2}\Delta G(x') \\ & + 2x \int_x^{\epsilon(x)} \frac{dx'}{x'^3}\int dX \Phi_F(X, x') + 2x\int_x^{\epsilon(x)} dx_1 \int_{-1}^{1} dx_2 \tilde{M}_F(x_1, x_2)\mathcal{P}\frac{1}{x_1^3(x_1 - x_2)} \\ & + 2x \int_x^{\epsilon(x)} dx_1 \int_{-1}^{1} dx_2 M_F(x_1, x_2)\mathcal{P}\frac{2x_1 - x_2}{x_1^3(x_1 - x_2)^2} \end{aligned}$$

Look at the component $\quad W^{q,g} = i\frac{S^+}{P^+}\epsilon^{ij} q_\perp^i \Delta_\perp^j f(x, q_\perp) + \cdots$

Then $\quad L_{q,g}(x) = \int d^2q_\perp q_\perp^2 f(x, q_\perp)$

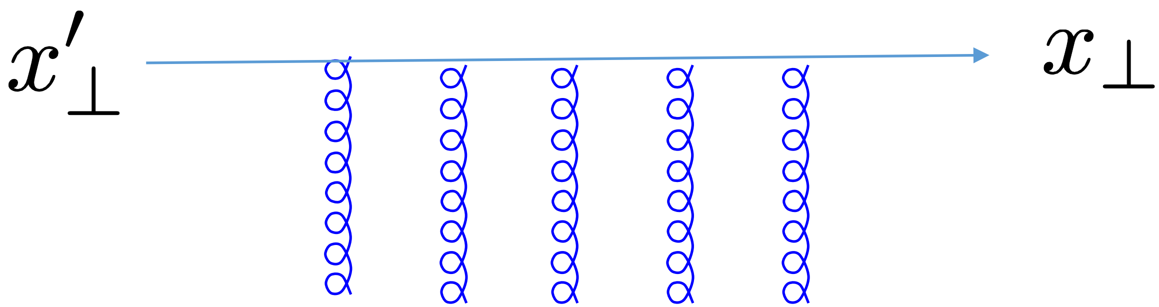
Lorce, Pasquini (2011)
YH (2011)

OAM as a next-to-eikonal effect

$$e^{ixP^+z^-} \approx 1 + ixP^+z^-$$

Strict eikonal limit,
no information on spin!

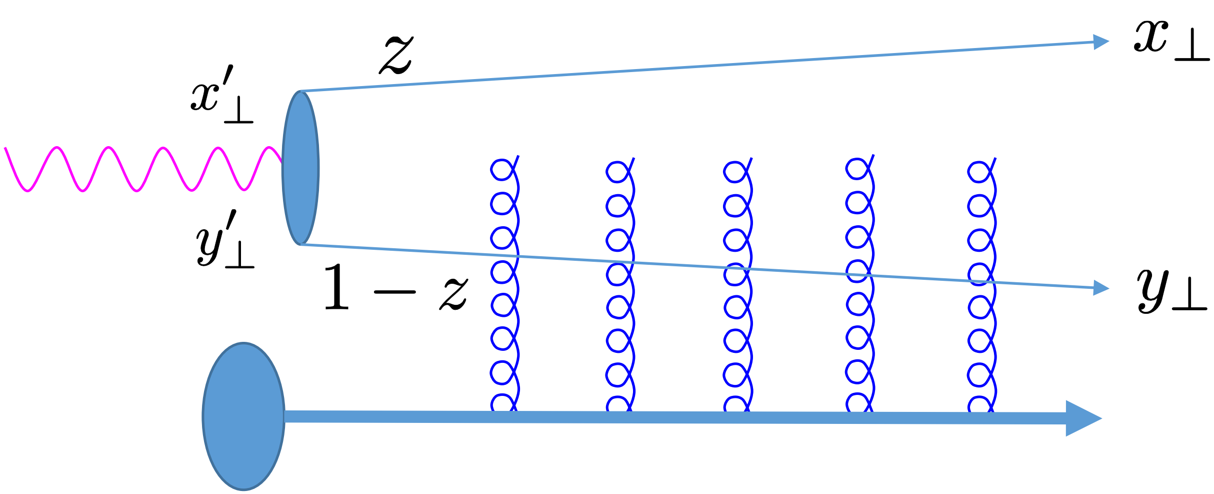
Next-to-eikonal corrections.
This is where OAM sits



$$U(x_\perp, x'_\perp) \equiv U(x_\perp)\delta^{(2)}(x_\perp - x'_\perp) + \frac{i}{2k^-}\int_{-\infty}^{\infty} dz^- U_{\infty z^-}(x_\perp) D^2_{x_\perp}\delta^{(2)}(x_\perp - x'_\perp) U_{z^- -\infty}(x'_\perp)$$

cf. Altinoluk, Armesto, Beuf, Martinez, Salgado (2014)

Observable: Longitudinal single spin asymmetry in diffractive dijet production



$$\vec{\Delta}_\perp = -(\vec{k}_{1\perp} + \vec{k}_{2\perp})$$

$$\vec{P}_\perp = \frac{1}{2}(\vec{k}_{2\perp} - \vec{k}_{1\perp})$$

$$\frac{d\Delta\sigma}{dy_1 d^2k_{1\perp} dy_2 d^2k_{2\perp}} \overset{\text{large-}Q^2}{\sim} (1-2z)\sin\phi_{P\Delta} O(x) L_g(x) + \mathcal{O}\left(\frac{\sin 2\phi_{P\Delta}}{Q^2}\right)$$

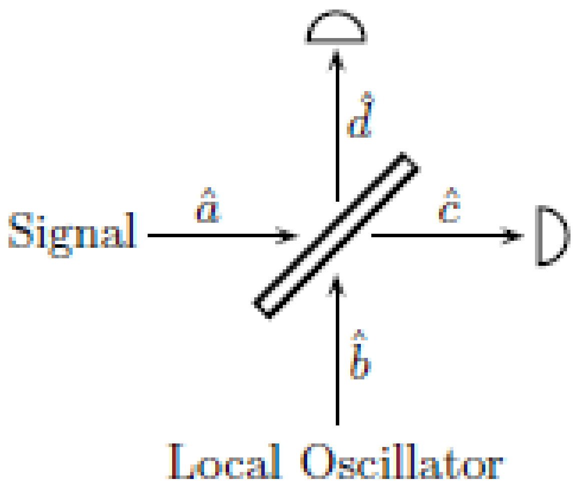
odderon! OAM!

Conclusions

- Let’s get 5 dimensional. Even richer physics than TMD+GPD combined, still largely unexplored.
- 5D distributions experimentally accessible in DIS
- 5D distribution calculated in the saturation regime.
  Wigner, GTMD: subject to uncertainties related to confinement, a nonperturbative factor needed (like in TMD).
- The elliptic angular correlation—a few percent effect.
  Peak position doesn’t move with Y.
- First-ever proposal of the direct measurement of OAM

Homodyne Detection

Two electromagnetic fields are incident on a beam splitter



(coherent state with phase ϕ)

Photon count difference at detectors C and D

$\propto P(x_\phi)$: Probability distribution of

$$\frac{\hat{a}^\dagger e^{i\phi} + \hat{a} e^{-i\phi}}{\sqrt{2}} = \hat{x}\cos\phi + \hat{p}\sin\phi \equiv \hat{x}_\phi$$

c.f. $\int dx W(x,p) = P(p) \quad \int dp W(x,p) = P(x)$

Reconstruct $W(x,p)$ via the inverse Radon transformation