Orbifolds of 4d $\mathcal{N} = 2$

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arXiv:1702.03330 *"Surface defects and instanton-vortex interaction"*
with A. Gorsky, N. Sopenko (Moscow), A. Milekhin (Princeton)

arXiv:1708.04631 *"AGT/$\mathbb{Z}_2$"*
with G. J. Turiaci (Princeton)

AGT correspondence

AGT with boundaries and cross-caps

Instantons and surface defects

Nekrasov partition functions

4d $\mathcal{N} = 2$ gauge theories

- Vector multiplet $(A_\mu, \Phi, \lambda, \tilde{\lambda})$ in adjoint representation of $\mathfrak{g}$. Coupling $\tau = 4\pi/g^2 + \theta/(2\pi)$.
- Hypermultiplet $(q, \tilde{q}^\dagger, \psi, \tilde{\psi}^\dagger) \in R$ representation of $\mathfrak{g}$.

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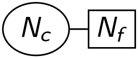
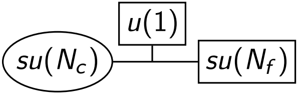
- $R =$ fundamental$^{\oplus N_f}$ $\implies$ 4d $\mathcal{N} = 2$ SQCD (conformal for $N_f = 2N_c$)

Quiver: 

or equivalently 

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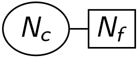
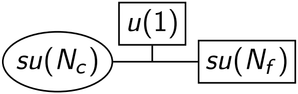
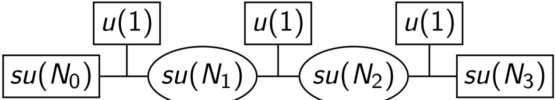
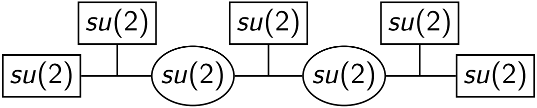
Quiver: 

or equivalently 

- 

Other example quiver gauge theory $G = su(N_1) \times su(N_2)$ and $R = N_0\overline{N_1} \oplus N_1\overline{N_2} \oplus N_2\overline{N_3}$ and $G_f = su(N_0) \times u(1)^3 \times su(N_3)$

$SU(2)$ AGT dictionary



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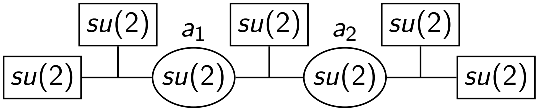
Coulomb branch: $a_1, a_2 \in \mathbb{C}/\mathbb{Z}_2$



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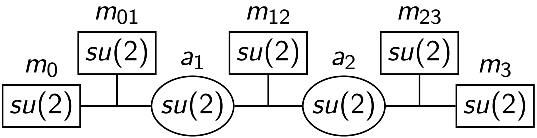

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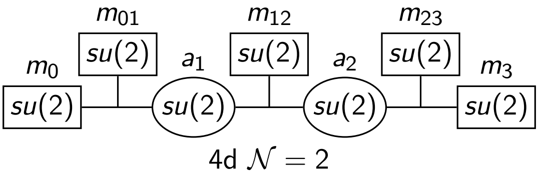
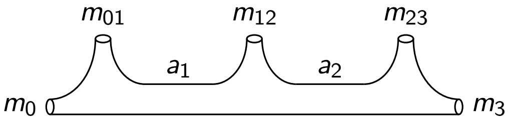


$$\Longleftrightarrow$$

6d $\mathcal{N} = (2, 0)$ A_1 SCFT twisted-compactified on Σ



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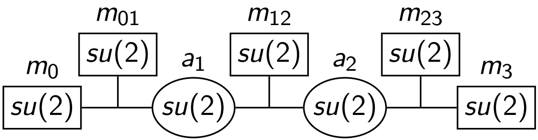

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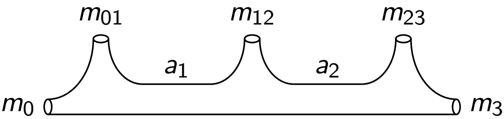

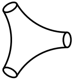 = matter,  = vector multiplet

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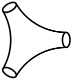
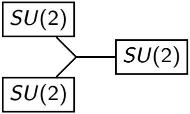
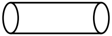
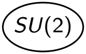
 $=$ 8 hypers $=$ 

,  $=$ 

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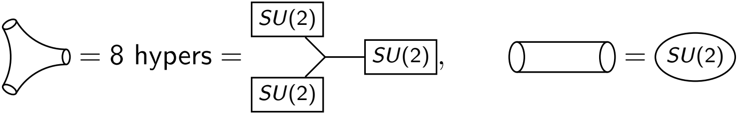
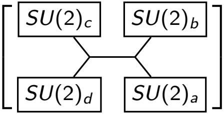


From **6d $\mathcal{N} = (2,0)$ on S^4** $\times$ **Σ** get

Z_{S^4}



$$= \left\langle V_{m_a}(0) V_{m_b}(z,\bar{z}) V_{m_c}(1) V_{m_d}(\infty) \right\rangle_{\substack{\text{Liouville} \\ \text{2d CFT}}}$$

$$= \int \mathrm{d}a\, Z_{\text{1-loop}}(m,a) \left| Z_{\text{cl,inst}}(m,a;z) \right|^2 = \int \mathrm{d}a\, C(m,a) \left| \mathcal{F}(m,a;z) \right|^2$$

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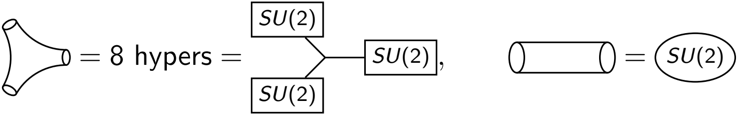

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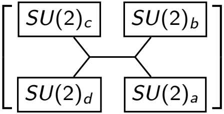

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$$= \int \mathrm{d}a\, Z_{S^3}(m,a) \left| Z_{HS^4}^{\text{Dirichlet}}(m,a;z) \right|^2$$

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2d CFT boundaries

6d $\mathcal{N} = (2,0)$ theory has no supersymmetric boundary, but

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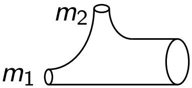

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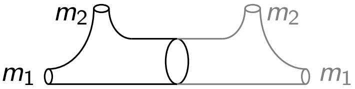

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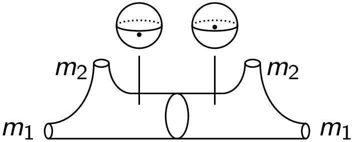

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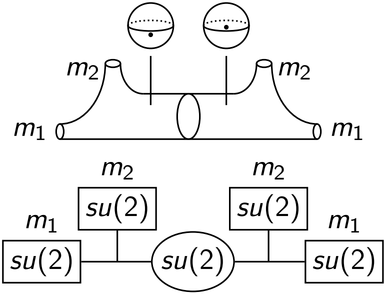

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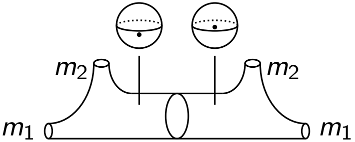
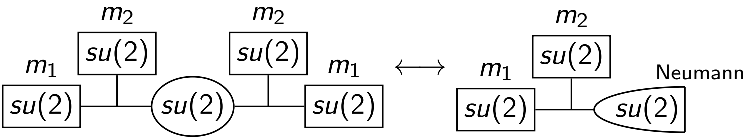
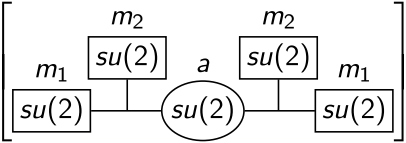




$$Z^{\text{Neumann}}_{HS^4}\left[\right.$$



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$$= \int \mathrm{d}a\, Z_{S^3}(a)\, Z^{\text{Dirichlet}}_{HS^4}(a)$$

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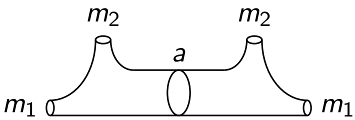

$$= \langle V_{m_1} V_{m_2} \rangle_{\mathcal{B}}$$

Disk 2-point function with boundary state

$$\mathcal{B} = \int \mathrm{d}a\, \underbrace{Z_{S^3}(a)}_{=\ \text{Wavefunction of identity brane}} |a\rangle\rangle$$

Other branes

The 6d $\mathcal{N} = (2,0)$ theory has codimension 2 operators:

- along $S^4 \times \text{pt}$: change 4d theory, add puncture to Σ
- along $S^3 \times \partial\Sigma$: 3d domain walls, change boundary state
- along $S^2 \times \Sigma$: Gukov–Witten defects, change 2d theory

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CFT	Gauge theory	Labels
Ishibashi state	Dirichlet boundary condition (scaled)	Momentum α
ZZ brane (id)	Neumann boundary condition	None
ZZ brane (all)	Neumann + Wilson line at equator	(R_1, R_2)
FZZT brane	Symmetry-breaking boundary condition + FI parameters m + Wilson line	$(H \subset G, m, R_1^H, R_2^H)$

Variants of the $\mathbb{Z}_2$ action

$\widehat{\Sigma}/\mathbb{Z}_2 \longrightarrow$ boundary or cross-cap
$S^4/\mathbb{Z}_2 \longrightarrow HS^4$ or $\mathbb{RP}^4$

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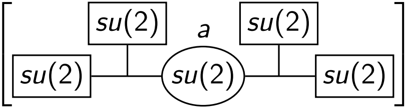

$$= \int \mathrm{d}a \, Z_{S^3/\mathbb{Z}_2}(a) \, Z^{\text{Dirichlet}}_{HS^4}(a)$$

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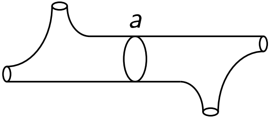

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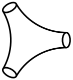 = matter on S^4

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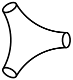 = matter on S^4

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Identity brane = vector multiplet on HS^4 with Neumann boundary

Cross-cap = vector multiplet on $\mathbb{RP}^4$

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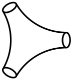 $=$ matter on S^4

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Future work:

- The 6d construction suggests
  Unusual brane $\int \mathrm{d}a\, \psi_{\otimes}(a) |a\rangle\rangle$
  Unusual cross-cap $\int \mathrm{d}a\, \psi_{\mathrm{id}}(a) |a\rangle\rangle_{\otimes}$
- Boundary-changing operators in the 2d CFT
- Verlinde loop operators $\Leftrightarrow$ Wilson–'t Hooft loops

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Instantons (codimension 4)

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For $k \geq 0$ note $\mathrm{Tr}(F \wedge \star F) = -\,\mathrm{Tr}(F \wedge F) + \frac{1}{2}\,\mathrm{Tr}(F + \star F)^2$
$\implies$ BPS bound: action $\geq k$

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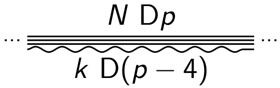
Saturated by *instantons*: $\boxed{F = - \star F}$

($F = \star F$ *anti-instantons* for $k \leq 0$)

- Important in nonperturbative effects
- ADHM construction $\mathcal{M}^{\text{instanton}}_{SU(N)} = \bigcup_{k \geq 0} \mathbb{H}^{k^2+Nk} /// U(k)$
- Same as instantons in SYM (*supersymmetric* Yang–Mills)
  $\implies$ ADHM can be understood in string theory

ADHM construction of k-instantons for $G = SU(N)$

Maximal supersymmetry
e.g. 4d $\mathcal{N} = 4$ SYM



0d $\mathcal{N} = (4,4)$
$\Downarrow$
0d $\mathcal{N} = (2,2)$
$\rightarrow$ is a chiral
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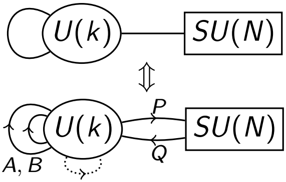

ADHM construction of k-instantons for $G = SU(N)$

Maximal supersymmetry	Half-maximal supersymmetry
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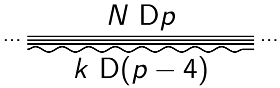

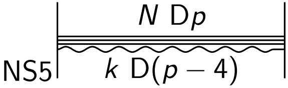

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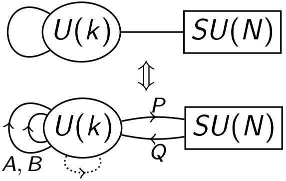
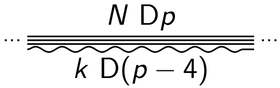
→ is a chiral
⇢ is a chiral

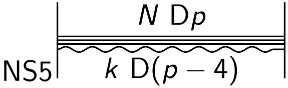


0d $\mathcal{N} = (0,4)$
$\Downarrow$
0d $\mathcal{N} = (0,2)$
→ is a chiral
⇢ is a Fermi

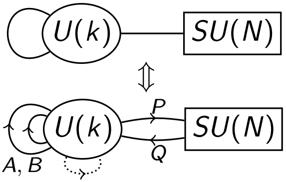
ADHM construction of k-instantons for $G = SU(N)$

Maximal supersymmetry	Half-maximal supersymmetry
e.g. 4d $\mathcal{N} = 4$ SYM	e.g. 4d $\mathcal{N} = 2$ SYM





0d $\mathcal{N} = (4,4)$
⇓
0d $\mathcal{N} = (2,2)$
→ is a chiral
⋯→⋯ is a chiral



0d $\mathcal{N} = (0,4)$
⇓
0d $\mathcal{N} = (0,2)$
→ is a chiral
⋯→⋯ is a Fermi

Classical Higgs branch of ADHM quiver:

$$\mathcal{M}^{k \text{ inst}}_{SU(N)} = \left\{ \begin{array}{l} [A,B] + PQ = 0, \\ [A,A^\dagger] + [B,B^\dagger] + QQ^\dagger - P^\dagger P = \mathbf{t} \end{array} \right\} \Big/ \ U(k)$$

$\longrightarrow$ correct hyper-Kähler metric

Gukov–Witten defect (codimension 2)

Operator at $z_1 = x^1 + ix^2 = 0$, spanning $x^3, \ldots$

- Improperly-quantized Dirac string $\begin{cases} \mathrm{Pexp} \oint_{z_1=0} A = \exp 2\pi\alpha \\ F = 2\pi(\alpha \bmod \Lambda)\delta^{(2)} \end{cases}$

  for $\alpha \in \mathfrak{t}$ generic

  (also singular sgluino)

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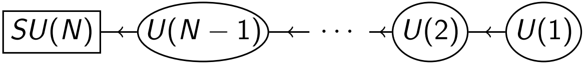
- Or couple a 2d theory by gauging its $SU(N)$ flavour symmetry



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  (also singular sgluino)

- Or couple a 2d theory by gauging its $SU(N)$ flavour symmetry

$$\boxed{SU(N)} \leftarrow U(N-1) \leftarrow \cdots \leftarrow U(2) \leftarrow U(1)$$

- Or $\mathbb{Z}_N$ orbifold by $z_1 \to \omega z_1$ and gauge transformation by $\operatorname{diag}(1, \omega, \dots, \omega^{N-1})$ with $\omega = \exp(2\pi i/N)$

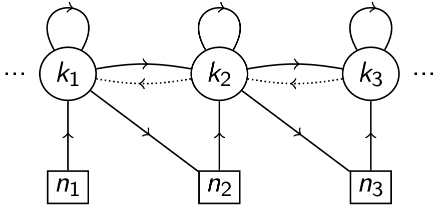
  (only true for complex structure)

Generalization that breaks $SU(N)$ to $\big(U(n_1) \times \cdots \times U(n_M)\big)/U(1)$ at the defect instead of $U(1)^{N-1}$

ADHM for $(\mathbb{C}/\mathbb{Z}_M) \times \mathbb{C}$

$$\mathcal{M}^{(k_1,\ldots,k_M)}_{(\mathbb{C}/\mathbb{Z}_M)\times\mathbb{C}} = \{A_{I+1}B_I - B_I A_I + P_{I+1}Q_I = 0\} \;/\; \prod GL(k_I,\mathbb{C})$$

Instantons of 4d $\mathcal{N}=4$ SYM on $(\mathbb{C}/\mathbb{Z}_M)\times\mathbb{C}$ have 0d $\mathcal{N}=(2,2)$
4d $\mathcal{N}=2$ 0d $\mathcal{N}=(0,2)$



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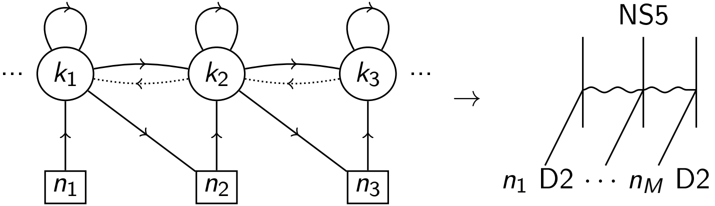

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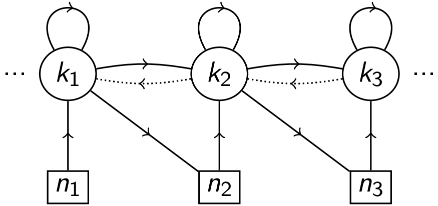

$\rightarrow$

ADHM for $(\mathbb{C}/\mathbb{Z}_M) \times \mathbb{C}$

$$\mathcal{M}^{(k_1,\ldots,k_M)}_{(\mathbb{C}/\mathbb{Z}_M)\times\mathbb{C}} = \{A_{I+1}B_I - B_I A_I + P_{I+1}Q_I = 0\} \;/\; \prod GL(k_I, \mathbb{C})$$

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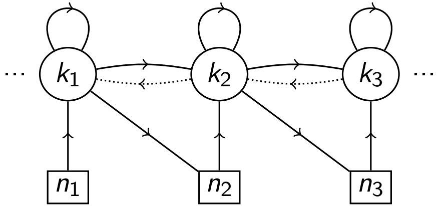
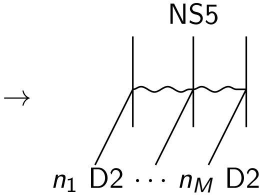
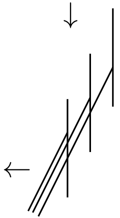
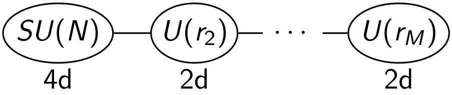






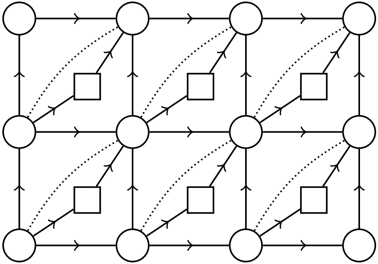
instanton-vortices of



$r_I = n_I + \cdots + n_M$

ADHM for $(\mathbb{C}/\mathbb{Z}_L) \times (\mathbb{C}/\mathbb{Z}_M)$

4d $\mathcal{N} = 2$
SYM



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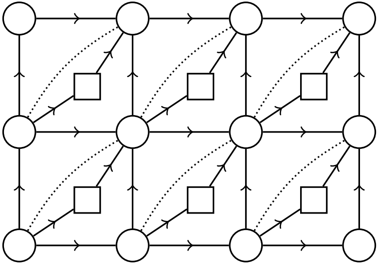

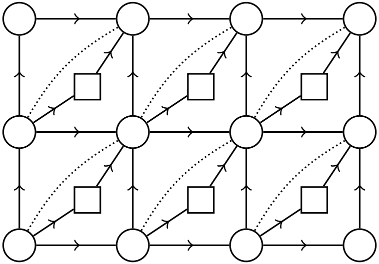

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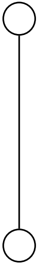

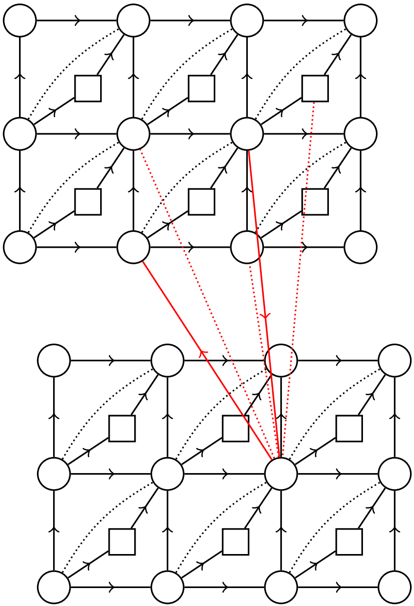

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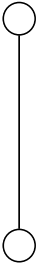
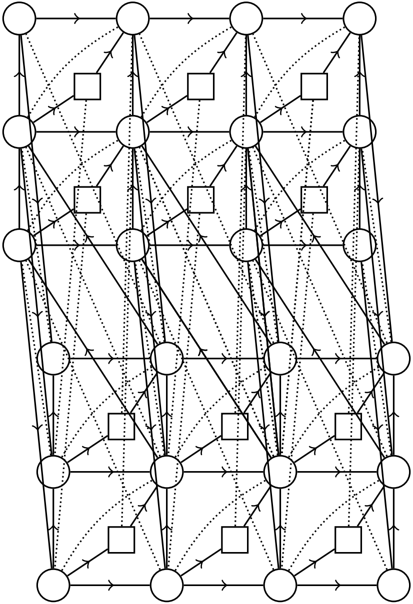




AGT correspondence

AGT with boundaries and cross-caps

Instantons and surface defects

Nekrasov partition functions

Nekrasov partition function

5d $\mathcal{N} = 1$ on $\mathbb{R}^4 \times S^1_\beta$, with $\phi(z_1, z_2, x^5 + \beta) = \phi(e^{\epsilon_1} z_1, e^{\epsilon_2} z_2, x^5)$

$\implies$ 4d $\mathcal{N} = 2$ on *Omega background* $\mathbb{R}^4_{\epsilon_1, \epsilon_2}$

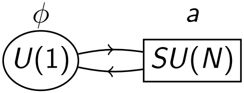
Nekrasov partition function

5d $\mathcal{N} = 1$ on $\mathbb{R}^4 \times S^1_\beta$, with $\phi(z_1, z_2, x^5 + \beta) = \phi(e^{\epsilon_1} z_1, e^{\epsilon_2} z_2, x^5)$
$\implies$ 4d $\mathcal{N} = 2$ on *Omega background* $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$

$Z_{\mathbb{R}^4_{\epsilon_1,\epsilon_2}} = \sum_{k \geq 0} q^{|k|} \operatorname{vol}\left(\mathcal{M}^{k\text{ inst}}_{SU(N)}\right)$ (equivariant). Then

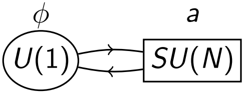
- equivariant localization to fixed points under $\mathbf{T} = U(1)^2 \times U(1)^{N-1} \subset SO(4) \times SU(N)$
- or write ADHM Coulomb branch integral and pick up poles

Nekrasov partition function (4d $\mathcal{N} = 2$ SYM)

1 instanton ADHM  0d $\mathcal{N} = (0,2)$

$$Z^{\text{4d}}_{\text{instanton}} = \int \mathrm{d}\phi \frac{1}{\prod_{A=1}^{N}(\phi - a_A + \epsilon_1 + \epsilon_2)(a_A - \phi)}$$

Nekrasov partition function (4d $\mathcal{N} = 2$ SYM)

1 instanton ADHM 

0d $\mathcal{N} = (0, 2)$

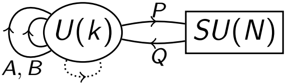
$$Z^{4d}_{\text{instanton}} = \int \mathrm{d}\phi \frac{1}{\prod_{A=1}^{N} (\phi - a_A + \epsilon_1 + \epsilon_2)(a_A - \phi)}$$

Close contours and pick up residues. Two choices:

- pick poles at $\phi = a_A$,
- pick poles at $\phi = a_A - \epsilon_1 - \epsilon_2$

Each pole $\Leftrightarrow$ fixed point of $\mathbf{T}$ on $\mathcal{M}^{1\text{ inst}}_{SU(N)} = \mathbb{C}^2 \times T^*(\mathbb{CP}^{N-1})$

Nekrasov partition function (4d $\mathcal{N} = 2$ SYM)

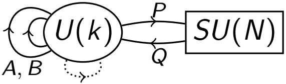


A, B, P, Q give poles, $\cdots\!\rightarrow\!\cdots$ gives zeros

$$
Z^{\text{4d}}_{\text{instanton}} = \int \frac{\mathrm{d}^k\phi}{k!} \frac{\prod_{I\neq J}(\phi_I - \phi_J)\prod_{I,J=1}^{k}(\phi_I - \phi_J + \epsilon_1 + \epsilon_2)}{\prod_{I,J=1}^{k}(\phi_I - \phi_J + \epsilon_1)(\phi_I - \phi_J + \epsilon_2)}
$$

$$
\times \frac{1}{\prod_{J=1}^{k}\prod_{A=1}^{N}(\phi_J - a_A + \epsilon_1 + \epsilon_2)(a_A - \phi_J)}
$$

Nekrasov partition function (4d $\mathcal{N} = 2$ SYM)



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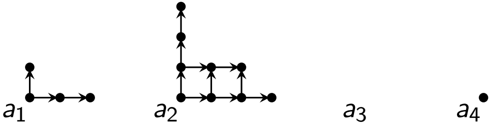
$$\times \frac{1}{\prod_{J=1}^k \prod_{A=1}^N(\phi_J - a_A + \epsilon_1 + \epsilon_2)(a_A - \phi_J)}$$

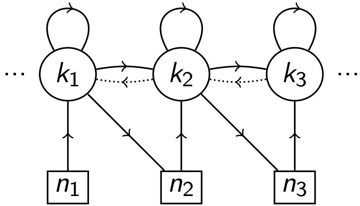
Jeffrey–Kirwan residue [...]

$\Longrightarrow$ (after ordering) Each ϕ_J is a_A or $\phi_I + \epsilon_{1,2}$ for $I < J$



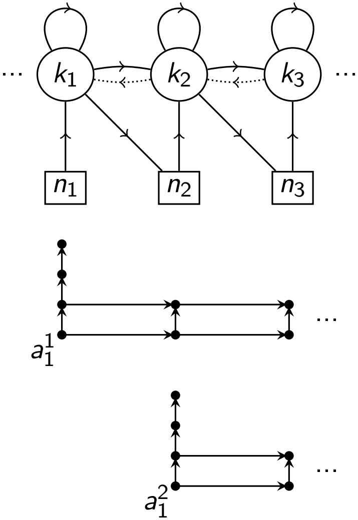
Poles at $\phi = \{a_A + (r-1)\epsilon_1 + (s-1)\epsilon_2 \mid (r,s) \in \lambda_A\}$

Nekrasov on $(\mathbb{C}/\mathbb{Z}_M) \times \mathbb{C}$



a's split into a^I_A, $A \leq n_I$

Nekrasov on $(\mathbb{C}/\mathbb{Z}_M) \times \mathbb{C}$

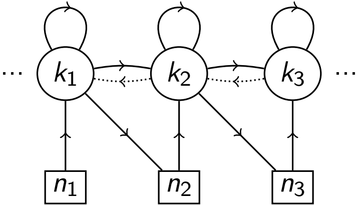


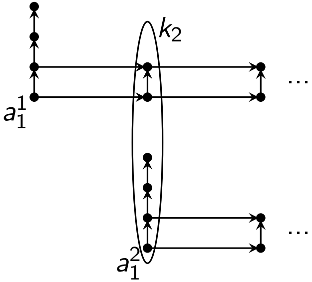
a's split into a_A^I, $A \leq n_I$

Poles still labeled by N Young diagrams λ_A^I

$$\phi = \{a_A^I + (r-1)\tfrac{\epsilon_1}{M} + (s-1)\epsilon_2 \mid (r,s) \in \lambda_A^I\}$$

Nekrasov on $(\mathbb{C}/\mathbb{Z}_M) \times \mathbb{C}$





a's split into a^I_A, $A \leq n_I$

Poles still labeled by N Young diagrams λ^I_A

$$\phi = \{a^I_A + (r-1)\frac{\epsilon_1}{M} + (s-1)\epsilon_2 \mid (r,s) \in \lambda^I_A\}$$

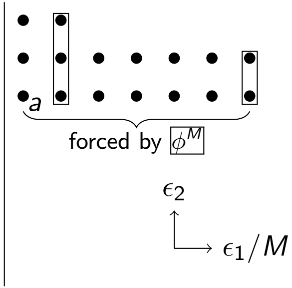
k_J counts boxes $(r,s) \in \lambda^I_A$ with $r + I \equiv J \mod M$

Change Jeffrey–Kirwan residue prescription

Flip contours for $\phi^1, \ldots, \phi^{M-1}$ but not ϕ^M.

$\implies$ ϕ^M given by (every M-th column of) the λ^I_A

$\phi^1, \ldots, \phi^{M-1}$ given by $\begin{cases} \text{what's needed for } \phi^M \\ \end{cases}$



Poles labeled by N Young diagrams and

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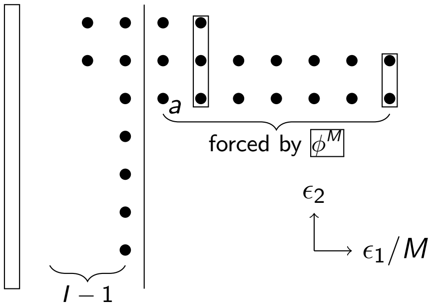
$\implies$ ϕ^M given by (every M-th column of) the λ^I_A

$$\phi^1, \ldots, \phi^{M-1} \text{ given by } \begin{cases} \text{what's needed for } \phi^M \\ \text{extra components} \end{cases}$$



Poles labeled by N Young diagrams and $\sum_{I=1}^{M}(I-1)n_I$ vorticities

4d-2d instanton-vortex partition function

$$Z^{\mathbb{Z}_M}_{\text{instanton}} = \sum_{\lambda_1,\ldots,\lambda_N} q^{|\vec{\lambda}|} \underbrace{\sum_{k^2_1,\ldots,k^2_{r_2}} z_2^{|k^2|} \cdots \sum_{k^M_1,\ldots,k^M_{r_M}} z_M^{|k^M|} (\text{residue at } \lambda, k)}_{Z^{\text{4d}}_{\text{instanton}}(\vec{\lambda}) \times \langle Z^{\text{2d}}_{\text{vortex}} \rangle_{\vec{\lambda}}}$$

4d-2d instanton-vortex partition function

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$$Z\left[\underset{\text{4d}}{\boxed{SU(N)}} \Big/ \mathbb{Z}_M \right] = Z\left[\underset{\text{4d}}{\boxed{SU(N)}} \leftarrow \underset{\text{2d}}{\boxed{U(r_2)}} \leftarrow \cdots \leftarrow \underset{\text{2d}}{\boxed{U(r_M)}} \right]$$

Here, $Z^{\text{2d}}_{\text{vortex}} = \displaystyle\int \mathrm{d}\sigma_{\text{2d}}\, z^{\sigma_{\text{2d}}} \frac{1}{\Gamma(1 + (\sigma_{\text{2d}} + m)/\epsilon_1)} \dots$ known

$$\Big\downarrow \; \langle \quad \rangle_{\vec{\lambda}}$$

$$\frac{1}{\Gamma(1 + (\sigma_{\text{2d}} + a_{\text{4d}})/\epsilon_1)} \prod_{i=1}^{k} \frac{\phi_i + \sigma_{\text{2d}} + \epsilon_1 + \epsilon_2}{\phi_i + \sigma_{\text{2d}} + \epsilon_1}$$

How to derive this prescription

Local operators give

$$\langle \mathrm{Tr}\, f_1(\Phi) \cdots \mathrm{Tr}\, f_n(\Phi) \rangle_\lambda = \langle \mathrm{Tr}\, f_1(\Phi) \rangle_\lambda \cdots \langle \mathrm{Tr}\, f_n(\Phi) \rangle_\lambda$$

$$\langle \mathrm{Tr}\, f(\Phi) \rangle_\lambda = \left(\sum_{A=1}^{N} f(a_A) \right) - \sum_{i=1}^{k} \big(f(\phi_i) + f(\phi_i + \epsilon_1 + \epsilon_2) - f(\phi_i + \epsilon_1) - f(\phi_i + \epsilon_2) \big)$$

How to derive this prescription

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4d matter gives $\left\langle \cdots \exp \int_0^\infty \frac{\mathrm{d}s}{s} \frac{\mathrm{Tr}_R\, e^{-s(\Phi - m)}}{(e^{-s\epsilon_1} - 1)(e^{-s\epsilon_2} - 1)} \right\rangle_\lambda$

2d matter (on $\mathbb{R}^2_{\epsilon_1}$) gives $\left\langle \cdots \exp \int_0^\infty \frac{\mathrm{d}s}{s} \frac{\mathrm{Tr}_R\, e^{-s(\Phi + i\sigma)}}{(e^{-s\epsilon_1} - 1)} \right\rangle_\lambda$

$$\boxed{Z^{\text{4d-2d}}_{\text{instanton-vortex}} = \left\langle Z^{\text{2d}}_{\text{vortex}}(\Phi) \right\rangle_{\text{4d}}}$$

Conclusions

- Couple fields living on *different quotients* of the same space
- Localization on non-orientable space $\mathbb{RP}^4$
- 2d CFT: unusual branes and cross-caps
- Nonlocal operators have many constructions
  Instanton-vortex partition functions match

Future work:

- Node-hopping duality
- Class S_k theories
- 2d $\mathcal{N} = (2, 2)$ dualities