Angular Momentum Sum Rules in the Front Form

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based on
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with
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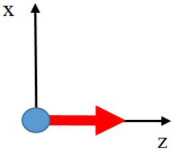
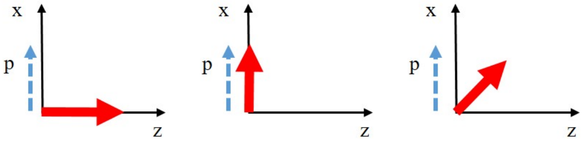
Spin of a relativistic particle

For a massive particle with spin aligned with the z-axis in its rest frame,



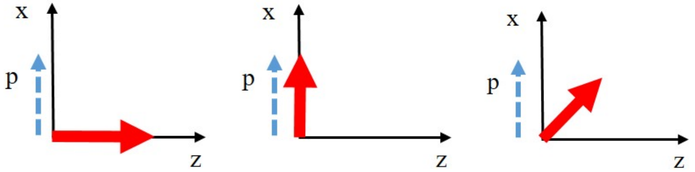
What is the the particle's spin vector in a Lorentz frame in which the particle's motion is in the x-direction?



(a) (b) (c)

Spin of a relativistic particle

It depends.
The Lorentz transformation from the rest frame to a frame in which the particle moves with momentum p is not unique.



(a) Canonical spin $s_c^\mu(p) = (0, 0, 0, m)$ (b) Wick-helicity spin $s_h^\mu(p) = (p, E, 0, 0)$ (c) Light-front spin $s_L^\mu(p) = (\frac{p^2}{E}, p, 0, \frac{m^2}{E})$

Which spin definition should we use?

Angular momentum conservation law

Non-relativistic quantum mechanics

- For a n-particle system, the angular momentum conservation law can be expressed in terms of the angular quantum numbers (j^3, s^3, l^3) :

$$j^3 = \sum_{a=1}^{n} s_a^3 + \sum_{a=1}^{n-1} l_a^3.$$

- In non-relativistic physics, spin and momentum are decoupled in the Hamiltonian, eg. Heisenberg model

Relativistic QFT

- How does spin change under Lorentz transformations?
- Angular momentum conservation is rarely used in high energy physics – particles are usually labeled by the helicity, which is not conserved
- Similar angular momentum conservation law?

Overview

1. Construct spin representations for relativistic particles
   - Start with a standard reference frame in which the spin is unambiguously defined
   - Apply Lorentz transformations to obtain the spin in any arbitrary Lorentz frame
     $\Rightarrow$ Different choices of relativistic spin (canonical, Wick-helicity, light-front, etc.)
2. Why is light-front spin advantageous?
   - z-projection of the light-front spin is Poincaré invariant
   - Frame-independent angular momentum conservation law
3. Selection rule for the orbital angular momentum in the front form
   - Eliminates certain interaction vertices in QED and QCD
   - Constrains the change of orbital angular momentum:
     $|\Delta l^3| \leq n$ in the n-th order perturbative expansion of a renormalizable theory
4. Discussion

Relativistic spin of a massive particle

- For a spin-s massive particle, start with the rest frame and polarize spin in the $\hat{z} = \hat{x}^3$ direction; denote as $|\mathring{p}; \lambda\rangle$
- Spin is labeled by s^3 or helicity λ, the $(2s+1)$ eigenvalues of the spin generator S^3:

$$S^3 |\mathring{p}; \lambda\rangle = \lambda |\mathring{p}; \lambda\rangle = s^3 |\mathring{p}; \lambda\rangle \quad \text{for } \lambda = -s, -s+1, ..., 0, ..., s-1, s$$

- Apply Lorentz transformations to $|\mathring{p}; \lambda\rangle$ to obtain a state $|p; \lambda\rangle$ with some momentum p:

$$|p; \lambda\rangle \equiv \Lambda(\mathring{p} \to p) |\mathring{p}; \lambda\rangle$$

- $\Lambda(\mathring{p} \to p)$ is not unique $\Rightarrow$ different definitions of relativistic spin
  - Canonical spin (standard) : obtained via a *pure boost*
  - Wick-helicity spin : spin is along the direction of motion
  - Light-front spin : light-front version of helicity spin

Canonical spin (massive particles)

- The canonical spin state $|p;\lambda\rangle_c$ is related to the rest frame $|\mathring{p};\lambda\rangle$ via a *rotationless pure boost* along the direction of the 3-momentum $\mathbf{p}$
- $$\begin{aligned} |p;\lambda\rangle_c &\equiv \Lambda_c(\mathring{p}\to p)\,|\mathring{p};\lambda\rangle \\ &= R(\hat{z}\to\hat{\mathbf{p}})\,B(\mathring{p}\to p^3=|\mathbf{p}|)\,R^{-1}(\hat{z}\to\hat{\mathbf{p}})\,|\mathring{p};\lambda\rangle\,, \end{aligned}$$

  where

  $$R(\hat{z}\to\hat{\mathbf{p}}) = e^{-iM^{12}\phi}e^{-iM^{31}\theta},\ \ \phi = tan^{-1}\frac{p^1}{p^2},\ \theta = tan^{-1}\frac{\sqrt{(p^1)^2+(p^2)^2}}{p^3}$$

  and

  $$B(\mathring{p}\to p^3=|\mathbf{p}|) = e^{-iM^{03}\rho},\ \ \rho = tanh^{-1}\frac{|\mathbf{p}|}{p^0}$$

Canonical spin (massive particles)

- 4-vector representation of Λ_c:

$$(\Lambda_c)^{\mu}{}_{\nu}(\mathring{p} \to p) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{p^0}{m} & \frac{p^1}{m} & \frac{p^2}{m} & \frac{p^3}{m} \\ \frac{p^1}{m} & 1+\frac{p^1p^1}{m(p^0+m)} & \frac{p^1p^2}{m(p^0+m)} & \frac{p^1p^3}{m(p^0+m)} \\ \frac{p^2}{m} & \frac{p^2p^1}{m(p^0+m)} & 1+\frac{p^2p^2}{m(p^0+m)} & \frac{p^2p^3}{m(p^0+m)} \\ \frac{p^3}{m} & \frac{p^3p^1}{m(p^0+m)} & \frac{p^3p^2}{m(p^0+m)} & 1+\frac{p^3p^3}{m(p^0+m)} \end{bmatrix} \end{array}$$

- For a particle polarized along the z-direction in its rest frame with the covariant spin 4-vector $s^\mu = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & m \end{bmatrix} \end{array}$, after performing Λ_c, the spatial components of s^μ in general will not be aligned with the 3-momentum $\mathbf{p}$

Wick-helicity spin (massive particles)

- Wick-helicity state $|p;\lambda\rangle_h$ is defined such that the spin of the moving particle is parallel or anti-parallel to the direction of the 3-momentum $\mathbf{p}$.
-

$$\begin{aligned} |p;\lambda\rangle_h &\equiv \Lambda_h(\mathring{p} \to p)\, |\mathring{p};\lambda\rangle \\ &= R(\hat{z} \to \hat{\mathbf{p}})\, B(\mathring{p} \to p^3 = |\mathbf{p}|)\, |\mathring{p};\lambda\rangle \end{aligned}$$

- Check :
  for a covariant spin vector pointed along the z-axis in the rest frame, after performing $|p;\lambda\rangle_h$, the spin will be aligned with the 3-momentum $\mathbf{p}$

Wick-helicity spin (massive particles)

- 4-vector representation of Λ_h:

$$(\Lambda_h)^{\mu}{}_{\nu}(\mathring{p} \to p) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{p^0}{m} & 0 & 0 & \frac{|\mathbf{p}|}{m} \\ \frac{p^1}{m} & \frac{p^1 p^3}{|\mathbf{p}|\left|p^{\perp}\right|} & \frac{-p^2}{\left|p^{\perp}\right|} & \frac{p^0 p^1}{m|\mathbf{p}|} \\ \frac{p^2}{m} & \frac{p^2 p^3}{|\mathbf{p}|\left|p^{\perp}\right|} & \frac{p^1}{\left|p^{\perp}\right|} & \frac{p^0 p^2}{m|\mathbf{p}|} \\ \frac{p^3}{m} & \frac{-\left|p^{\perp}\right|}{|\mathbf{p}|} & 0 & \frac{p^0 p^3}{m|\mathbf{p}|} \end{bmatrix} \end{array}$$

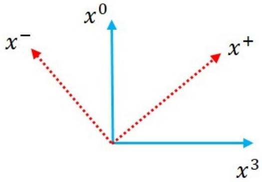
Light-front spin – What is light-front?

- Light-front coordinates are defined by

  $x^+ = x^0 + x^3$: light-front time

  $x^- = x^0 - x^3$: longitudinal spacelike coordinate

  $x^\perp$: light-front transverse coordinates, $\perp = 1, 2$



- $p \cdot x = \frac{p^-}{2} x^+ + \frac{p^+}{2} x^- - p^1 x^1 - p^2 x^2 \Rightarrow$

  p^- : light-front Hamiltonian

  p^+ : light-front longitudinal momentum

  $p^\perp$: light-front transverse momentum

Light-front spin – Kinematical generators of Poincaré group

Kinematical vs dynamical generators

- Kinematical : leaves the quantization plane at one instant of "time" invariant
- Dynamical : not kinematical –involves evolution in "time"

Instant form – boosts are dynamical

- Kinematical : translation P^i; rotation M^{ij}
- Dynamical : boost M^{0i}; time translation P^0

Front form – boosts are kinematical

- Kinematical : translation P^+, $P^\perp$; rotation M^{12}; transverse boost $M^{+\perp}$; longitudinal boost M^{+-}
- Dynamical : transverse rotation $M^{-\perp}$, LF time translation P^-

Light-front spin – Kinematical generators of Poincaré group

- Instant form boosts are dynamical – impossible to boost bound state wavefunctions
- Front form boosts are kinematical – independent of interactions and conserves particle number
- Light-front direction $x^+ = x^0 + x^3 = t + z$ is chosen to coincide with the z-axis used to label spin in the particle's rest frame
  $\Rightarrow$ z-component of spin is special under LF Lorentz transformations
- Use light-front *kinematical* boost generators, $M^{+\perp} = M^{0\perp} + M^{3\perp}$ and $M^{+-} = -2M^{03}$ to construct light-front spin state $|p; \lambda\rangle_L$

Light-front spin (massive particles)

- Light-front spin state $|p;\lambda\rangle_L$ is defined such that the spin of the moving particle will be parallel or anti-parallel to the light-front 3-momentum $(p^+, p^\perp)$
-

$$|p;\lambda\rangle_L \equiv \Lambda_L(\mathring{p} \to p)\,|\mathring{p};\lambda\rangle$$
$$= e^{-iM^{+\perp}\theta^\perp} e^{-i\frac{M^{+-}\omega}{2}}\,|\mathring{p};\lambda\rangle$$

where

$$\theta^\perp = \frac{p^\perp}{p^+}, \quad \perp = 1, 2$$
$$e^\omega = \frac{m}{p^+}$$

- Parameters $(\theta^\perp, \omega)$ have simple relations with momentum p – light-front boosts are *kinematical*

Light-front spin (massive particles)

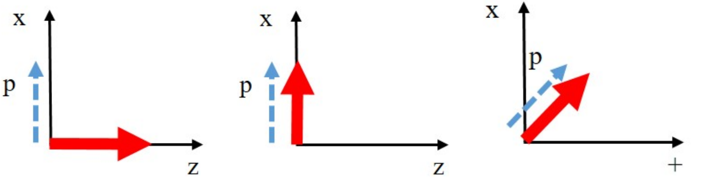
- 4-vector representation of Λ_L:

$$(\Lambda_L)^{\mu}{}_{\nu}\,(\mathring{p} \to p) = \begin{array}{c} \\ + \\ - \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} + & - & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{p^+}{m} & 0 & 0 & 0 \\ \frac{|p^\perp|^2}{mp^+} & \frac{m}{p^+} & \frac{2p^1}{p^+} & \frac{2p^2}{p^+} \\ \frac{p^1}{m} & 0 & 1 & 0 \\ \frac{p^2}{m} & 0 & 0 & 1 \end{bmatrix} \end{array}$$

- For a particle polarized in the z-axis in the rest frame with covariant spin 4-vector $s^\mu = \begin{array}{c} \begin{array}{cccc} + & - & 1 & 2 \end{array} \\ \begin{bmatrix} m & m & 0 & 0 \end{bmatrix} \end{array}$, after performing Λ_L, the spin will be aligned with the light-front 3-momentum $(p^+, p^\perp)$

Spin of a massive relativistic particle

For a massive particle with spin aligned with the z-axis in its rest frame, when it is boosted to a frame in which the momentum $p^\mu = (E, p, 0, 0)$, the spin vectors can be....



(a) Canonical spin $s_c^\mu(p) = (0, 0, 0, m)$ (b) Wick-helicity spin $s_h^\mu(p) = (p, E, 0, 0)$ (c) Light-front spin $s_L^\mu(p) = (\frac{p^2}{E}, p, 0, \frac{m^2}{E})$

Relativistic spin of a massless particle

- For a spin-s massless particle, start with a frame in which the particle moves along the z-direction with $\bar{p}^\mu \equiv \begin{matrix} 0 & 1 & 2 & 3 \\ [\ \bar{p} & 0 & 0 & \bar{p}\ ] \end{matrix}$, and polarize spin in the z-direction; denote as $|\bar{p};\lambda\rangle$
- Spin is labeled by λ, the 2 eigenvalues of the spin generator S^3 :

$$S^3\,|\bar{p};\lambda\rangle = \lambda\,|\bar{p};\lambda\rangle = s^3\,|\bar{p};\lambda\rangle \quad \text{for } \lambda = \pm s$$

- Apply Lorentz transformations to $|\bar{p};\lambda\rangle$ to obtain a state $|p;\lambda\rangle$ with any arbitrary momentum p:

$$|p;\lambda\rangle \equiv \Lambda(\mathring{p} \to p)\,|\bar{p};\lambda\rangle$$

- $\Lambda(\bar{p} \to p)$ is not unique $\Rightarrow$ different definitions of relativistic spin
  - ~~Canonical spin~~ : Not suitable as it requires a rest frame
  - Wick-helicity spin : spin is along the direction of motion
  - Light-front spin : light-front version of helicity spin

Wick-helicity spin (massless particles)

- Wick-helicity state $|p;\lambda\rangle_h$ is defined such that the spin of the moving particle is aligned with the direction of the 3-momentum $\mathbf{p}$.
-

$$
\begin{aligned}
|p;\lambda\rangle_h &\equiv \Lambda_h(\bar{p}\to p)\,|\bar{p};\lambda\rangle \\
&= R(\hat{z}\to\hat{\mathbf{p}})\;B(\bar{p}\to p^3=|\mathbf{p}|)\;\;|\bar{p};\lambda\rangle\,,
\end{aligned}
$$

where

$$
R(\hat{z}\to\hat{\mathbf{p}}) = e^{-iM^{12}\phi}e^{-iM^{31}\theta},\;\; \phi = tan^{-1}\frac{p^1}{p^2},\; \theta = tan^{-1}\frac{|p^{\perp}|}{p^3}
$$

and

$$
B(\bar{p}\to p^3=|\mathbf{p}|) = e^{-iM^{03}\rho},\;\; e^{\rho} = \frac{|\mathbf{p}|}{\bar{p}}
$$

Wick-helicity spin (massless particles)

- 4-vector representation of Λ_h:

$$
(\Lambda_h)^{\mu}{}_{\nu}(\bar{p} \to p) =
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
\frac{|\mathbf{p}|^2+(\bar{p})^2}{|\mathbf{p}|\bar{p}} & 0 & 0 & \frac{|\mathbf{p}|^2-(\bar{p})^2}{|\mathbf{p}|\bar{p}} \\
\frac{p^1\left(|\mathbf{p}|^2-(\bar{p})^2\right)}{|\mathbf{p}|^2\bar{p}} & \frac{p^1 p^3}{|\mathbf{p}|\left|p^\perp\right|} & \frac{-p^2}{\left|p^\perp\right|} & \frac{p^1\left(|\mathbf{p}|^2+(\bar{p})^2\right)}{|\mathbf{p}|^2\bar{p}} \\
\frac{p^2\left(|\mathbf{p}|^2-(\bar{p})^2\right)}{|\mathbf{p}|^2\bar{p}} & \frac{p^2 p^3}{|\mathbf{p}|\left|p^\perp\right|} & \frac{p^1}{\left|p^\perp\right|} & \frac{p^2\left(|\mathbf{p}|^2+(\bar{p})^2\right)}{|\mathbf{p}|^2\bar{p}} \\
\frac{p^3\left(|\mathbf{p}|^2-(\bar{p})^2\right)}{|\mathbf{p}|^2\bar{p}} & \frac{-\left|p^\perp\right|}{|\mathbf{p}|} & 0 & \frac{p^3\left(|\mathbf{p}|^2+(\bar{p})^2\right)}{|\mathbf{p}|^2\bar{p}}
\end{bmatrix}
\end{array}
$$

- Check :
  for a covariant spin vector pointed along the z-axis in $|\bar{p};\lambda\rangle$, after performing Λ_h, the spin will be aligned with the 3-momentum $\mathbf{p}$

Light-front spin (massless particles)

- Light-front spin state $|p;\lambda\rangle_L$ is defined such that the spin of the moving particle will be aligned with the light-front 3-momentum $(p^+, p^\perp)$
-

$$|p;\lambda\rangle_L \equiv \Lambda_L(\bar{p} \to p)\,|\bar{p};\lambda\rangle$$

$$= e^{-iM^{+\perp}\theta^\perp} e^{-i\frac{M^{+-}\omega}{2}}\,|\bar{p};\lambda\rangle$$

where

$$\theta^\perp = \frac{p^\perp}{p^+}, \quad \perp = 1, 2$$

$$e^\omega = \frac{2\bar{p}}{p^+}$$

- Expression is identical to Λ_L for massive case, except $m \to 2\bar{p}$

Light-front spin (massless particles)

- 4-vector representation of Λ_L:

$$(\Lambda_L)^{\mu}{}_{\nu}(\bar{p} \to p) = \begin{array}{c} \\ + \\ - \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} + & - & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{p^+}{2\bar{p}} & 0 & 0 & 0 \\ \frac{|p^\perp|^2}{2\bar{p}p^+} & \frac{2\bar{p}}{p^+} & \frac{2p^1}{p^+} & \frac{2p^2}{p^+} \\ \frac{p^1}{2\bar{p}} & 0 & 1 & 0 \\ \frac{p^2}{2\bar{p}} & 0 & 0 & 1 \end{bmatrix} \end{array}$$

- Check :
  for a covariant spin vector pointed along the z-axis in $|\bar{p};\lambda\rangle$, after performing Λ_L, the spin will be aligned with the light-front 3-momentum $(p^+, p^\perp)$

LF spin representation for spin-$\frac{1}{2}$ particles

- LF spinors: ($p^{\mathrm{R}} = p^1 + ip^2$ and $p^{\mathrm{L}} = p^1 - ip^2$)

$$u_\uparrow(p) = \sqrt{\frac{1}{2p^+}} \begin{bmatrix} p^+ + m \\ p^{\mathrm{R}} \\ p^+ - m \\ p^{\mathrm{R}} \end{bmatrix} \qquad u_\downarrow(p) = \sqrt{\frac{1}{2p^+}} \begin{bmatrix} -p^{\mathrm{L}} \\ p^+ + m \\ p^{\mathrm{L}} \\ -p^+ + m \end{bmatrix}$$

$$v_\uparrow(p) = \sqrt{\frac{1}{2p^+}} \begin{bmatrix} -p^{\mathrm{L}} \\ p^+ - m \\ p^{\mathrm{L}} \\ -p^+ - m \end{bmatrix} \qquad v_\downarrow(p) = \sqrt{\frac{1}{2p^+}} \begin{bmatrix} p^+ - m \\ p^{\mathrm{R}} \\ p^+ + m \\ p^{\mathrm{R}} \end{bmatrix}$$

- Widely used in amplitude calculations

LF spin representation for massless spin-1 particles

- LF polarization vectors

$$\varepsilon^\mu_+(p) = \begin{matrix} + \\ - \\ 1 \\ 2 \end{matrix} \begin{bmatrix} 0 \\ \frac{-\sqrt{2}\,p^{\mathrm{R}}}{p^+} \\ \frac{-1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} \qquad \varepsilon^\mu_-(p) = \begin{matrix} + \\ - \\ 1 \\ 2 \end{matrix} \begin{bmatrix} 0 \\ \frac{\sqrt{2}\,p^{\mathrm{L}}}{p^+} \\ \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix}$$

- Light-front gauge condition $A^+ = 0$ is preserved under light-front Lorentz transformation, in contrast to other choices of Lorentz transformations where gauge conditions are generally not preserved
- There are only 2 *physical* transverse polarizations for photons and gluons

Spin of particles in motion – Why is light-front spin special?

- For a massive particle with spin polarized in the z-direction in the rest frame, the spin expectation values along the x, y, z axes are

$$\langle S^i \rangle(\mathring{p}) = \frac{\langle \mathring{p}; \lambda = s^3 | \, S^i \, | \mathring{p}; \lambda = s^3 \rangle}{\langle \mathring{p}; \lambda | \mathring{p}; \lambda \rangle} = \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad i = 1, 2, 3$$

- What is the spin expectation value $\langle S^i \rangle(p)$ when the particle is moving ?

$$\langle S^i \rangle(p) = \frac{\langle p; \lambda | \, S^i \, | p; \lambda \rangle}{\langle p; \lambda | p; \lambda \rangle}$$

- Can be computed using the Lorentz algebra

$\Rightarrow \langle S^i \rangle(p)$ depends on the definition of spin!

Spin of particles in motion – Why is light-front spin special?

Use spin-1 representation and obtain...

Canonical spin

- $$\langle S^i \rangle_c(p) = \frac{{}_c\langle p;\lambda| S^i |p;\lambda\rangle_c}{{}_c\langle p;\lambda|p;\lambda\rangle_c} = \frac{\lambda(p^0-m)}{2p^0m(p^0+m)} \begin{bmatrix} -p^1p^3 \\ -p^2p^3 \\ \frac{2p^0m(p^0+m)}{p^0-m} + \left|p^\perp\right|^2 \end{bmatrix}$$
- Canonical spin is generally not aligned along the direction of motion
- In the non-relativistic limit,

$$\langle S^i \rangle_c(p) \underset{|\mathbf{p}|^2 \ll m^2}{\rightarrow} \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \langle S^i \rangle(\mathring{p})$$

Spin of particles in motion – Why is light-front spin special?

Wick helicity spin

- $$\langle S^i \rangle_h(p) = \frac{{}_h\langle p;\lambda| S^i |p;\lambda\rangle_h}{{}_h\langle p;\lambda|p;\lambda\rangle_h} = \frac{\lambda}{|\mathbf{p}|}\begin{bmatrix} p^1 \\ p^2 \\ p^3 \end{bmatrix}$$
- Helicity spin points along the 3-momentum direction
- $\langle S^{i=3} \rangle_h(p) = \lambda \frac{p^3}{|\mathbf{p}|} \neq \lambda!!$
- In a reference frame where the observer moves with infinite momentum(IMF) in the negative z-direction,

$$\langle S^i \rangle_h(p) \underset{p^3 \approx |\mathbf{p}|}{\longrightarrow} \lambda \begin{bmatrix} \frac{p^1}{p^3} \\ \frac{p^2}{p^3} \\ 1 \end{bmatrix}$$

Spin of particles in motion – Why is light-front spin special?

Light-front spin

- $$\langle S^i \rangle_L(p) = \frac{{}_L\langle p;\lambda|S^i\,|p;\lambda\rangle_L}{{}_L\langle p;\lambda|p;\lambda\rangle_L} = \lambda \begin{bmatrix} \frac{p^1}{p^+} \\ \frac{p^2}{p^+} \\ 1 \end{bmatrix}$$
- $\langle S^{i=3} \rangle_L(p) = \lambda$
  z-projection of LF spin state is invariant under LF boosts!
- In the non-relativistic limit, $\langle S^i \rangle_c(p) = \langle S^i \rangle_L(p)$
  In the IMF limit, $\langle S^i \rangle_h(p) = \langle S^i \rangle_L(p)$ if p^3 is identified with p^+
  LF spin is applicable to both non-relativistic and relativistic regimes

Invariance of LF spin for elementary particles

Spin of an elementary particle is preserved in the z-direction under Lorentz transformations generated by the LF boost generators :

$$\langle J^3 \rangle_L(p) = \frac{{}_L\langle p; \lambda | J^3 | p; \lambda \rangle_L}{{}_L\langle p; \lambda | p; \lambda \rangle_L} = s^3$$

Proof:

Define a momentum dependent operator $S_L^3(p)$ such that

$$S_L^3(p)\, |p; \lambda\rangle_L = s^3\, |p; \lambda\rangle_L$$

Use

$$J^3\, |\mathring{p}; \lambda\rangle = s^3\, |\mathring{p}; \lambda\rangle \text{ in the rest frame}$$

$$\text{and} \quad |\mathring{p}; \lambda\rangle = \Lambda_L^{-1}(\mathring{p} \to p)\, |p; \lambda\rangle_L$$

$$\Rightarrow S_L^3(p) = \Lambda_L(\mathring{p} \to p)\, J^3\, \Lambda_L^{-1}(\mathring{p} \to p)$$

Invariance of LF spin for elementary particles

In terms of the Poincaré generators:

$$S_L^3(p) = J^3 - \frac{P^1}{P^+}S^{+2} + \frac{P^2}{P^+}S^{+1} - L_L^3(p),$$

with

$$L_L^3(p) = \frac{P^1}{P^+}L^{+2} - \frac{P^2}{P^+}L^{+1}$$

Rewrite

$$\begin{aligned}\langle J^3\rangle_L(p) &= \langle S_L^3(p)\rangle + \frac{{}_L\langle p;\lambda|\frac{P^1}{P^+}S^{+2} - \frac{P^2}{P^+}S^{+1}|p;\lambda\rangle_L}{{}_L\langle p;\lambda|p;\lambda\rangle_L} + \frac{{}_L\langle p;\lambda|\,L_L^3(p)\,|p;\lambda\rangle_L}{{}_L\langle p;\lambda|p;\lambda\rangle_L}\\ &= s^3\ ?\end{aligned}$$

Invariance of LF spin for elementary particles

$$
\begin{aligned}
\langle p;\lambda|\, S^{+\perp}\, |p;\lambda\rangle_L &= \langle \mathring{p};\lambda|\, \Lambda_L^{-1}(\mathring{p}\to p)\; S^{+\perp}\; \Lambda_L(\mathring{p}\to p)\, |\mathring{p};\lambda\rangle \\
&= \langle \mathring{p};\lambda|\, e^{i\frac{S^{+-}\omega}{2}}\; S^{+\perp}\; e^{-i\frac{S^{+-}\omega}{2}}\, |\mathring{p};\lambda\rangle\,,\ \text{with } e^{\omega} = \frac{m}{p^+} \\
&= e^{-\omega}\, \langle \mathring{p};\lambda|\, S^{+\perp}\, |\mathring{p};\lambda\rangle \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
{}_L\langle p;\lambda| L_L^3(p)\, |p;\lambda\rangle_L &= {}_L\langle p;\lambda| \frac{P^1}{P^+} L^{+2} - \frac{P^2}{P^+} L^{+1}\, |p;\lambda\rangle_L \\
&= {}_L\langle p;\lambda| i(p^1 \frac{\partial}{\partial p_2} - p^2 \frac{\partial}{\partial p_1})\, |p;\lambda\rangle_L \\
&= {}_L\langle p;\lambda| L^3\, |p;\lambda\rangle_L
\end{aligned}
$$

Invariance of LF spin for elementary particles

$$\langle J^3\rangle_L(p) = s^3 + {}_L\langle p;\lambda| L^3 |p;\lambda\rangle_L$$

For an elementary particle with *static* momentum, there is no OAM around a fixed point (OAM term becomes essential for composite particles):

$$\boxed{\langle J^3\rangle_L(p) = s^3}$$

For elementary particles, z-projection of LF spin is invariant under Lorentz transformations generated by the LF boost generators

Angular momentum conservation law in the front form

Internal angular momentum of a composite system is conserved in the z-direction under LF Lorentz transformations Λ_L:

$$\langle J^3 \rangle_L(p) = j^3 = \sum_{a=1}^{n} s_a^3 + \sum_{a=1}^{n-1} l_a^3$$

s_a^3: z-projection of spin of each constituent
l_a^3: internal OAM of the composite system

Angular momentum conservation law in the front form

Proof:

A LF bound state has the following free Fock state decomposition :

$$|p;j^3\rangle_L = \sum_n \int [\mathrm{d}x][\mathrm{d}^2k^\perp]\ \psi_n(x_a, k_a^\perp, s_a^3)\,|n;p_a;s_a^3\rangle_L\,, \quad \forall\ s_a^3$$

with Lorentz invariant measure

$$[\mathrm{d}x] = \prod_{a=1}^{n} \frac{\mathrm{d}x_a}{\sqrt{x_a}}\delta(1-\sum_{a=1}^{n} x_a) \quad \left[\mathrm{d}^2k^\perp\right] = 16\pi^3 \prod_{a=1}^{n} \frac{\mathrm{d}^2k_a^\perp}{16\pi^3}\ \delta^2(\sum_{a=1}^{n} k_a^\perp)$$

- Physics of the bound state is described by the *frame-independent* $\psi_n(x_a, k_a^\perp, s_a^3)$
- For each constituent a, the light-front 3-momentum $(p_a^+, p_a^\perp)$ is parametrized by $(p^+, p^\perp)$ and the internal Lorentz-invariant variables $(x_a, k_a^\perp)$:

$$p_a^+ = x_a p^+ \quad p_a^\perp = x_a p^\perp + k_a^\perp$$

Angular momentum conservation law in the front form

$$\langle J^3\rangle_L(p) = \frac{{}_L\langle n; p_a; s_a^3|\ J^3\ \ |n; p_a; s_a^3\rangle_L}{{}_L\langle n; p_a; s_a^3|n; p_a; s_a^3\rangle_L}$$

$$= \sum_{a=1}^{n} \frac{{}_L\langle n; p_a; s_a^3|\ J_a^3\ \ |n; p_a; s_a^3\rangle_L}{{}_L\langle n; p_a; s_a^3|n; p_a; s_a^3\rangle_L}$$

$$= \sum_{a=1}^{n}\ \ s_a^3 + \frac{{}_L\langle n; p_a; s_a^3|\ L_a^3\ \ |n; p_a; s_a^3\rangle_L}{{}_L\langle n; p_a; s_a^3|n; p_a; s_a^3\rangle_L},$$

where

$$\sum_{a=1}^{n} {}_L\langle n; p_a; s_a^3|\ L_a^3\ \ |n; p_a; s_a^3\rangle_L$$

$$= \sum_{a=1}^{n} {}_L\langle n; p_a; s_a^3|\ i(p_a^2 \frac{\partial}{\partial p_a^1} - p_a^1 \frac{\partial}{\partial p_a^2})\ \ |n; p_a; s_a^3\rangle_L$$

Angular momentum conservation law in the front form

OAM can be decomposed (external) + (internal) OAM :

$$\sum_{a=1}^{n} {}_L\langle n; p_a; s_a^3 |\ L_a^3\ | n; p_a; s_a^3 \rangle_L$$

$$= \ {}_L\langle n; p_a; s_a^3 |\ i(p^2 \frac{\partial}{\partial p^1} - p^1 \frac{\partial}{\partial p^2})\ | n; p_a; s_a^3 \rangle_L$$

$$+ \sum_{a=1}^{n-1} {}_L\langle n; p_a; s_a^3 |\ i(k_a^2 \frac{\partial}{\partial k_a^1} - k_a^1 \frac{\partial}{\partial k_a^2})\ | n; p_a; s_a^3 \rangle_L$$

- External OAM is neglected as it has no relevance to the internal structure
- Internal OAM is Lorentz invariant:

$$\sum_{a=1}^{n-1} {}_L\langle n; p_a; s_a^3 |\ i(k_a^2 \frac{\partial}{\partial k_a^1} - k_a^1 \frac{\partial}{\partial k_a^2}) | n; p_a; s_a^3 \rangle_L = \sum_{a=1}^{n-1} l_a^3\ {}_L\langle n; p_a; s_a^3 | n; p_a; s_a^3 \rangle_L$$

Angular momentum conservation law in the front form

$$\boxed{\langle J^3 \rangle_L(p) = j^3 = \sum_{a=1}^{n} s_a^3 + \sum_{a=1}^{n-1} l_a^3}$$

Since z-projection of constituent spin and the internal OAM are Lorentz invariant $\Rightarrow$

- Angular momentum conservation law in the front form is frame independent!
- j^3 is conserved in all scattering amplitudes! (Not true for helicity)

Selection rule for the orbital angular momentum in LF

In the n-th order perturbative expansion of a renormalizable theory, the change of orbital angular momentum between the initial and final states in the front form is constrained by $|\Delta\, l^3| \leq n$

Proof:
In non-relativistic QM :

$$P^{\mathrm{R}}\,|l^3\rangle \propto |l^3+1\rangle \qquad\qquad P^{\mathrm{L}}\,|l^3\rangle \propto |l^3-1\rangle\,,$$

with $P^{\mathrm{R}} \equiv P^1 + iP^2$ and $P^{\mathrm{L}} \equiv P^1 - iP^2$
Can be shown by

$$[P^{\mathrm{R}}, L^3] = -P^{\mathrm{R}} \qquad\qquad [P^{\mathrm{L}}, L^3] = P^{\mathrm{L}}$$

More generally,

$$(P^{\mathrm{R}})^n\ \,|l^3\rangle \propto |l^3+n\rangle \qquad\qquad (P^{\mathrm{L}})^n\ \,|l^3\rangle \propto |l^3-n\rangle$$

Selection rule for the orbital angular momentum in LF

Non-relativistic QM

In non-relativistic QM, for an interaction $H_I \propto (P^{\perp})^n$,

$$\langle p', l' | \, H_I \, | p, l \rangle \neq 0, \quad \text{only if} \quad |\Delta \, l^3| \leq n$$

Relativistic QFT

- Instant form :
  hard to apply the selection rule because z-projection of angular momentum generally changes under Lorentz transformations
- Front form :
  selection rule applies since z-projection of angular momentum is frame-independent

Selection rule for the orbital angular momentum in LF

In all renormalizable QFT, interaction vertex $H_I \propto P^\perp$ at most :

- $$|\Delta\, l^3| \leq 1 \quad \text{at every vertex}$$
- In the n-th order perturbative expansion, the change of between initial and final state orbital angular momentum is constrained by

$$|\Delta\, l^3| \leq n$$

Examples of OAM selection rule in LF

- QED :

  $V(-\frac{1}{2} \to +\frac{1}{2} + 1;\ \Delta l^3 = -2) = 0$

- QCD :

  $$V(- \to ++;\ \Delta l^3 = -3) = 0$$
  $$V(- \to +++;\ \Delta l^3 = -4) = 0$$
  $$V(- \to -++;\ \Delta l^3 = -2)$$
  $$V(- \to ---;\ \Delta l^3 = 2) = 0$$

- $2 \to n$ gluon scattering at tree level, $\mathscr{M}(+,+,...,+) = 0$:

  $\mathscr{M}(+,+,...,+) = \mathscr{M}(-- \to +...+;\ \Delta l^3 = -n-2) = 0$ since there are only n triple gluon vertices for tree-level g^n amplitude

Summary

- Studied relativistic spin states defined by different choices of Lorentz boosts – light-front choice is special because light-front boost generators are *kinematical*
- Showed that the $A^+ = 0$ light-front gauge condition is preserved under Lorentz transformations – only 2 *physical* transverse polarizations for gauge bosons (analogy of Coulomb gauge in the non-relativistic regime)
- Proved that the z-projection of angular momentum for any LF spin state (elementary/composite) is invariant under kinematical transformations of the Poincaré group
  $\Rightarrow$ Frame-independent angular momentum conservation in all scattering amplitudes
- Found an upper bound on the change of orbital angular momentum in scattering processes at any fixed order in perturbation theory

Unitary irreducible representations of the Poincaré group

- In $d = 3 + 1$ dimensions, the Poincaré group has two Casimir operators, $P^2 = m^2$ and W^2
- $W^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} P_\nu M_{\alpha\beta}$ (Pauli-Lubanski pseudovector), generates the little group, the maximal subgroup of the Lorentz group which leaves p^μ invariant
- Wigner's theorem:
  elementary particles classified with m^2 and W^2 transform in
  unitary irreducible representations of the the Poincaré group
- The maximum compact subgroup of the Little group defines the spin group of elementary particles :
  $SO(3)$, for $m^2 \neq 0$ particles
  $SO(2)$, for $m^2 = 0$ particles
- Spin covariant vector is defined as

$$s^\mu(p) = \frac{1}{s} \frac{\langle p; s^i | \, W^\mu \, | p; s^i \rangle}{\langle p; s^i | p; s^i \rangle}$$

Relativistic spin of a massive particle

Start with the rest frame of a massive particle in which

$$\mathring{p}^\mu \equiv \overset{\begin{matrix}0 & 1 & 2 & 3\end{matrix}}{\begin{bmatrix} m & 0 & 0 & 0 \end{bmatrix}} \text{ and } W^\mu = m \overset{\begin{matrix}0 & 1 & 2 & 3\end{matrix}}{\begin{bmatrix} 0 & S^1 & S^2 & S^3 \end{bmatrix}}:$$

- S^i are the spin generators in 3 spatial dimensions, and $\left(S^i\right)^2 = s(s+1)$
- $W^2 = -m^2 s(s+1)$ defines the spin representation s of a massive particle in a relativistic theory
- Spin is uniquely labeled in the rest frame by s^3 (used interchangeably with helicity λ):

$$S^3 \left|\mathring{p}; \lambda = s^3\right\rangle = \lambda \left|\mathring{p}; \lambda\right\rangle \quad \text{for } \lambda = -s, -s+1, ..., 0, ..., s-1, s$$

Simple proof of $\langle J^3 \rangle_L(p) = j^3$

Compare the action of (M^{+-}, P^+, M^{12}) on $M^{+\perp}$ and $P^\perp$

$$e^{iM^{12}\phi}\, M^{+\perp}\, e^{-iM^{12}\phi} = cos\phi\, M^{+\perp} - sin\phi\, \varepsilon^{\perp\perp'} M^{+\perp'}$$

$$e^{i\frac{M^{+-}\omega}{2}}\, M^{+\perp}\, e^{-i\frac{M^{+-}\omega}{2}} = e^{\omega} M^{+\perp}$$

$$e^{iP^+x^-}\, M^{+\perp}\, e^{-iP^+x^-} = M^{+\perp}$$

with

$$e^{iM^{12}\phi}\, P^{\perp}\, e^{-iM^{12}\phi} = cos\phi\, P^{\perp} - sin\phi\, \varepsilon^{\perp\perp'} P^{\perp'}$$

$$e^{i\frac{M^{+-}\omega}{2}}\, P^{\perp}\, e^{-i\frac{M^{+-}\omega}{2}} = P^{\perp}$$

$$e^{iP^+x^-}\, P^{\perp}\, e^{-iP^+x^-} = P^{\perp}$$

The action of $M^{+\perp} \approx P^\perp \Rightarrow \langle J^3 \rangle_L(p) = j^3$