Proving conformal invariance in critical scalar theories in any dimension

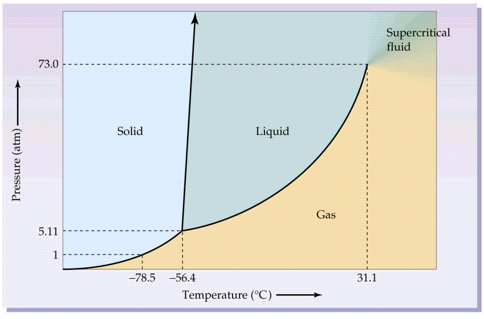
Nicolás Wschebor, [1]
work in collaboration with
B. Delamotte, G. De Polsi and M. Tissier

[1]Instituto de Física, Facultad de Ingeniería, Universidad de la República
J.H.y Reissig 565, 11000 Montevideo, Uruguay,

CPHT, École Polytechnique, November 2017

Introduction (I)

- We all know the typical phase diagram of simple materials.
- For example, the diagram for the CO2, includes one line of liquid-gas first order phase transition and a critical point.
- Other physical systems have similar phase diagrams.



→ The ferro-paramagnetic transition have similar phase diagram when varying the temperature and the external magnetic field.

Introduction (II)

- Far away from the critical point, mean-field-like approximations (or high/low temperature, etc.) are well under control.
- Near the critical point the system is strongly correlated:
  - Collective phenomena take place and many properties of the system are universal: independent of the details of the microscopic interactions.
    Example: the liquid-vapor transition and the uniaxial ferro-paramagnetic transition shares many quantitative properties in the critical regime.
  - The system shows scale invariance: at distances much larger than the atomic scale the dynamics is invariant under dilatations.
- Wilson explained scale invariance in critical phenomena by using the Renormalization Group.
- Almost at the same period, Polyakov conjectured that in a critical point the system is invariant under the full set of conformal symmetries (transformations that preserve angles).

Introduction (III)

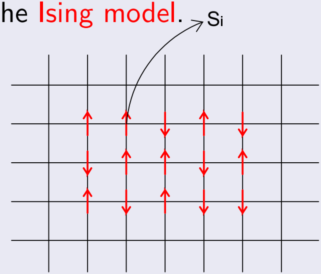
- Conformal invariance allowed the exact solution of many planar statistical mechanical models.
- In recent years, impressive progress have been made in the understanding of critical properties of many three-dimensional statistical mechanical models by using conformal invariance.
- However, until recently, conformal symmetry of three-dimensional critical systems was just a conjecture.
- In the present talk we will present a proof that in most critical phenomena,

  scale invariance $\Rightarrow$ conformal invariance.

- Finally, we discuss briefly a more rigorous proof in the case of the liquid-vapor or Ising critical regime.
- Most of the talk based on [Phys.Rev. E93 (2016) no.1, 012144].

The Ising model (I)

- To simplify matters, let us focus on the simplest interacting statistically-mechanical system: the Ising model.
- Let us take "spins" $S_i = \pm 1$ in a lattice in dimension d.
- Lenz (1920) proposed this to Ising as a toy model in order to describe the ferro-paramagnetic transition.

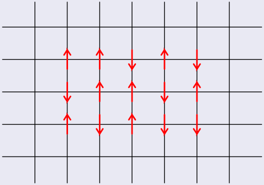


- It is described with a Hamiltonian with the form:

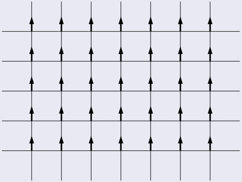
$$H = -J \sum_{\langle i,j \rangle neighbors} S_i S_j - B \sum_i S_i$$

- $J > 0$ favors the alignment of spins (ferromagnetic interactions).
- States of less energy for $B = 0$: all spins aligned.
- On the other hand, the noise tend to disorient spins.
- For $d > 1$ this provokes a phase transition.

For $B = 0$ there is a ferromagnetic phase and a paramagnetic phase:



- If $T > T_c$ paramagnetic phase.
- If $T \gg T_c$ it is possible to perform a high temperature expansion.

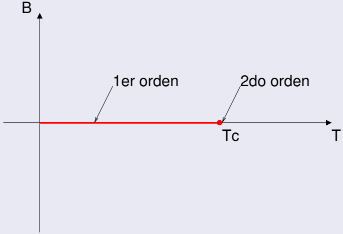


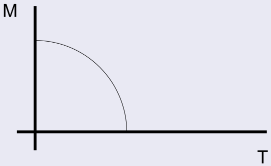
- If $T < T_c$ ferromagnetic phase.
- If $T \ll T_c$ it is possible to perform a low temperature expansion.

- In a similar way real gases can be analyzed by expanding in the density.

The Ising model (III)

- The phase diagram for $d > 1$ has the following shape:
- $T < T_c$: transition with a latent heat (1st order).
- $T = T_c$: transition without a latent heat (2d order).





- The magnetization for $B = 0^+$, $M = \langle S_i \rangle$ verifies

$$M(T) = \begin{cases} M_0(T) \neq 0 & si\ T < T_c \\ 0 & si\ T \geq T_c \end{cases}$$

- For $T \to T_c^-$: $M(T) \sim cst.(T_c - T)^{\beta}$
  with $\beta \to$ critical exponent.

The Ising model (IV)

- The exact solution for $d = 2$ and $B = 0$ (Onsager, 1944) was the first non-trivial example of exact treatment of a phase transition.
- Until very recently no exact solution for $d > 2$.
- Many approximate methods employed (as mean field).
- They work well when correlations among spins are small.
- In particular, this is fine far away from the critical point.

The Ising model (V)

Critical regime: $T \sim T_c$

- Mean field describes properly the critical case if $d > 4$ (Ginzburg, 1960).
- When $d < 4$, the critical case is the most difficult case (maybe not!).
- A breakthrough took place with Wilson work (1974):
  - He extended the theory to continuous space dimensions d.
  - He realized that an expansion in $d = 4 - \epsilon$ could be controlled.
  - In particular $d = 3$ turned to be "not so far" from $d = 4$.
  - High order expansions in ϵ turned to be a good approximation.
- The exact solution in the $d = 3$ case is the most paradigmatic open problem in equilibrium statistical mechanics.
- Recently (since 2012), Rychkov *et al.* seem to have made a big step towards an "exact" solution of the critical Ising model in $d = 3$.

The Ising and the liquid-vapor transition

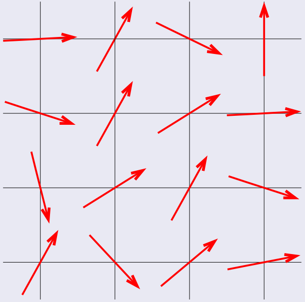
- The Ising phase diagram looks like the ferro-paramagnetic one.
- Nevertheless, Ising does not describe quantitatively the isotropic ferro-para transition.
- Surprisingly, it does describe quantitatively the liquid-vapor transition.
- Reason: the liquid-vapor is controlled by a single scalar: the density.
- In fact, we can formulate a lattice gas with the variables:

$$N_i = \frac{S_i + 1}{2}$$

- At low temperatures, Ising favors dense molecules (liquid) and at high temperatures dilute molecules (gas).
- On the other hand: ferro-para is controlled by a 3-component quantity: the magnetization.

Generalization: $O(N)$ models

- Ising: S_i has a single component.
- A natural generalization (Heisenberg) considers $\vec{S}_i$ with $|\vec{S}_i| = 1$ in a space of dimension N.
- Example: for $N = 2$ we have:



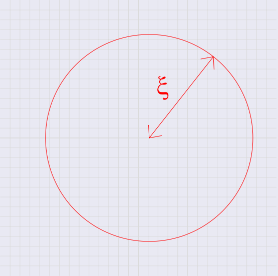
Different universality classes

Near $T = T_c$ various systems have identical behavior.

Examples:

- Ising ($N = 1$) $\rightarrow$ liquid-vapor, binary alloys, etc.
- Heisenberg ($N = 2$) $\rightarrow$ Superfluid Helium.
- Heisenberg ($N = 3$) $\rightarrow$ ferro-paramagnetic.

What happens near T_c? (I)



- Let us call a the lattice step.
- Consider the correlation function: $G(\vec{x}-\vec{y}) = \langle S(\vec{x})S(\vec{y})\rangle - \langle S(\vec{x})\rangle\langle S(\vec{y})\rangle$.
- When $T \sim T_c$ and $|\vec{x}-\vec{y}| \gg a$,

$$G(\vec{x}-\vec{y}) \propto \frac{e^{-|\vec{x}-\vec{y}|/\xi}}{|\vec{x}-\vec{y}|^{d-2+\eta}}$$

- $\xi \propto \frac{1}{|T-T_c|^{\nu}}$ is the correlation length.
- $\eta, \nu \rightarrow$ other critical exponents.
- If $T \sim T_c$, $\xi \gg a \rightarrow$ "discreteness" of the system becomes invisible.
- The systems acquires the symmetries of continuum space (translations and rotations).
- Even more, when $a \ll |\vec{x}-\vec{y}| \ll \xi$ the systems becomes scale invariant.

What happens near T_c? (II)

- Wilson: if $a \ll |\vec{x} - \vec{y}| \ll \xi$ one can average spins over distances smaller than any $a' > a$ as long as $a' \ll |\vec{x} - \vec{y}|$.
- The predictions for $a' \ll |\vec{x} - \vec{y}|$ can not depend on our choice of a' (Renormalization Group).
- Using this property, he could solve critical Ising and Heisenberg models in $d = 4 - \epsilon$ for $\epsilon \ll 1$.
- Exponents calculated at order ϵ^5: good precision for $d = 3$.
- In fact, one really uses the Ginzburg-Landau model that describes a field theory with Hamiltonian:

$$H = \int d^d x \left\{ \frac{1}{2}(\nabla\varphi)^2 + \frac{u}{4!}\left(\varphi^2 - \varphi_0^2\right)^2 \right\}$$

- In the lattice, Ginzburg-Landau tend to Ising when $u \to \infty$ (modulo a re-scaling of φ).
- More generally, both models are in the same universality class.

Polyakov's conjecture (I)

Polyakov's conjecture [A.M.Polyakov, JETP Lett. 12 (1970) 381.]

- Polyakov: at $T = T_c$ critical systems are not only scale invariant but, more generally conformal invariant.
- Conformal transformations: those conserving angles.
- Conjecture proven for $d = 2$ [A.B. Zamolodchikov, JETP Lett. 43 (1986) 34.].
- For $d = 3$, until recently, no proof.
- For $d = 2$: conformal invariance allows the exact solution of many statistical models.
- However, the $d = 2$ conformal group is very peculiar.

Conformal invariance and Renormalization Group

- Many speculated in the use of RG to prove conformal invariance (for ex. [L. Schäfer, J. Phys. A9 (1976) 377.]).
- For $d = 4 - \epsilon$ many results that prove that scale invariance "implies" conformal invariance. [J. Polchinski, Nucl. Phys. B303 (1987) 226; Jack and Osborn 1990; Luty, Polchinski and Rattazzi, 2013]

Polyakov's conjecture (II)

The role of unitarity

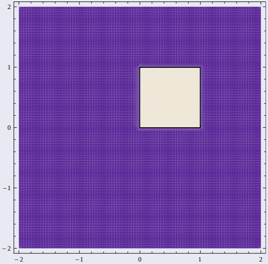
- The Ginzburg-Landau model can be Wick-rotated to Minkowski space.
- Migdal [A.A. Migdal, Phys. Lett. B 37 (1971) 386.] proposed that unitarity of the Quantum Field Theory associated to the Wick-rotated Ginzburg-Landau model could imply conformal invariance in scale invariant models ("conformal bootstrap").
- In fact, the Zamolodchnikov's $d = 2$ proof works that way.

Example of conformal transformation

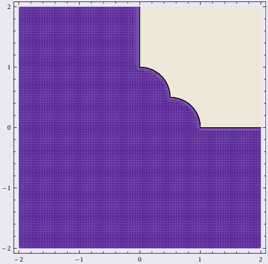
→ An example of non-trivial conformal transformation is the inversion:

$$\vec{x} \to \alpha \frac{\vec{x}}{x^2}$$

→ Original system.



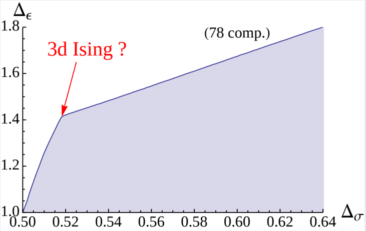
→ System after inversion.



→ Physical motivation for conformal invariance: In a critical system there are not rulers but only (small) protractors.

Consequences in $d = 3$ (I)

- By exploiting unitarity, various properties of the Operator Product Expansion and and supposing conformal invariance in $d = 3$, Rychkov *et al.* are able to prove rigorous inequalities on critical exponents [S. El-Showk *et al.*, Phys. Rev. D 86 (2012) 025022 and J. Stat. Phys (2014).]

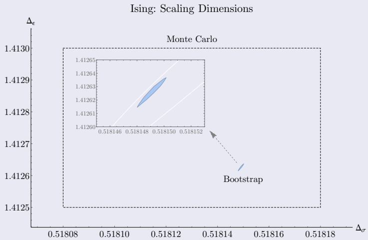


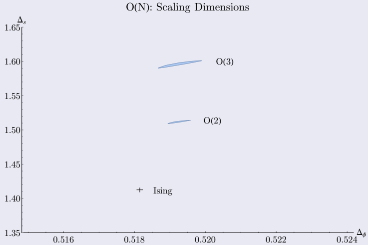
- $\Delta_\sigma = (d - 2 + \eta)/2$ y $\Delta_\epsilon = d - 1/\nu$
- Surprise: Ising seems to be in a corner of inequalities.
- Admitting this to be be true, Rychkov *et al.* find the best precision in critical exponents never achieved in 2014:

ref	method	ν	η
Guida (1998)	ϵ-exp	0.63050(250)	0.03650(500)
Campostrini (2002)	HT	0.63012(16)	0.03639(15)
Hasenbusch (2010)	MC	0.63002(10)	0.03627(10)
Rychkov (2014)	conformal	0.62999(5)	0.03631(3)

Consequences in $d = 3$ (II)

- One can go beyond the "cusp" hypothesis by including inequalities for other correlators.
- Kos and collaborators obtain, in this way rigorous "islands" for critical exponents [F. Kos *et al.*, JHEP 1608 (2016) .]
- The same procedure has been employed for $O(N)$ model [F. Kos *et al.*, JHEP 1608 (2016).].
- For Ising, in this way Kos *et al.* go even further:





ref	method	ν	η
Rychkov (2014)	conformal cusp	0.62999(5)	0.03631(3)
Kos (2016)	conformal island	0.629971(4)	0.0362978(20)

Proving Polyakov's conjecture (I)

- Recently, we proved together with B. Delamotte and M. Tissier Polyakov's conjecture in any dimension for Ising. [Phys.Rev. E93 (2016) no.1, 012144]
- The proof has two parts:
  - We used the Wilson's RG to prove a sufficient condition for the validity that

    Scale invariance $\Rightarrow$ conformal invariance.
  - After that, we prove that this sufficient condition is fulfilled in the Ising case.
- I present now (without proof) the sufficient condition and explain its meaning.
- I will analyze why, on general grounds, one can expect such condition to be fulfilled.
- Finally I sketch the rigorous proof in the Ising case.

Proving Polyakov's conjecture (II)

- Scale invariance implies that two-point correlation functions behaves for $|x-y| \gg a$ as:

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim \frac{cst.}{|x-y|^{2d_{\mathcal{O}}}}$$

  where $\mathcal{O}(x)$ is a local operator expressed as a function of $\varphi(x)$ and its derivatives.

- Examples: $\mathcal{O}_1(x) = \varphi(x)$ or $\mathcal{O}_2(x) = \partial_\mu(\varphi^2)(x)$ or $\mathcal{O}_3(x) = \partial_\mu(\varphi^2)(x)\sum_\nu(\partial_\nu\varphi(x))^2$, etc.
- The sufficient condition for scale invariance implying conformal invariance is

  > There must not be any local vectorial operator $V_\mu(x)$ with $d_{V_\mu} = d-1$ and even in $\varphi \to -\varphi$ (excluding operators being total derivatives).

- A similar sufficient condition had been formulated by Polchinski (within different hypothesis) [J. Polchinski, Nucl. Phys. B303 (1987) 226]

Some simple examples (I)

- Let us define $\tilde{d}_{V_\mu} = d_{V_\mu} - d$.
- Is is the dimension of the integrated operator $\int d^d x V_\mu(x)$.
- We need then to see if there are operators with $\tilde{d}_{V_\mu} = -1$.

Examples:

1) The φ^4 model in $d = 4$ (scaling dimension=canonical dimension):

The vector operators with smallest scaling dimension are:

$$\begin{aligned} \partial_\mu \phi^2 &\qquad \tilde{d}_{V_\mu} = 3 - 4 = -1 \qquad (\textit{but total derivative}!) \\ \partial_\mu \phi^4 &\qquad \tilde{d}_{V_\mu} = 5 - 4 = 1 \qquad (\textit{but total derivative}!) \\ \partial_\mu \partial^2 \phi^2 &\qquad \tilde{d}_{V_\mu} = 5 - 4 = 1 \qquad (\textit{but total derivative}!) \\ \phi\, \partial_\mu \phi (\partial_\nu \phi)^2 &\qquad \tilde{d}_{V_\mu} = 7 - 4 = 3 \end{aligned} \tag{1}$$

Trivial generalization to $O(N)$.

Ising and $O(N)$ models are conformal invariant at all orders of ϵ-expansion.

2) Euclidean gauge-fixed QED

$$S = \int_x \left[\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{\alpha}{2} (\partial_\mu A_\mu)^2 \right]. \tag{2}$$

There exists a vector operator with dimension -1 (up to a normalization): $C \int_x A^\mu (\partial_\nu A_\nu)$.
A direct calculation of $\Sigma_k^\mu[A_\nu]$ shows that $C = \alpha d + 4 - d$. We retrieve that conformal invariance holds only for $\alpha d + 4 - d = 0$.

A "physicist proof" of conformal invariance (I)

For general critical systems, one observes that conformal invariance is almost inevitable for many reasons:

1) Isotropy

If $\tilde{d}_{V_\mu} < 0$, one would have a relevant operator with a preferred direction $\Rightarrow$ violation of isotropy!

But for most microscopic models, not physically reachable

Example: If the microscopic lattice has cubic symmetry.

2) Estimating $\tilde{d}_{V_\mu}$ in $d = 3$ with ϵ- expansion

$\rightarrow$ We calculated $\tilde{d}_{V_\mu}$ for Ising and Heisenberg for the two vector operators with lower dimensions obtaining (the first is absent for Ising, $N = 1$):

$$\tilde{d}^{(a)}_{V_\mu} = 3 - \frac{6\epsilon}{N+8} + \mathcal{O}(\epsilon^2)$$

$$\tilde{d}^{(b)}_{V_\mu} = 3 + \mathcal{O}(\epsilon^2)$$

Much larger than $\tilde{d}_{V_\mu} = -1$ for $d = 3$...

But Can-we really trust ϵ-expansion for vectorial operators?

A conspiracy could take place...

3) Other estimates of $\tilde{d}_{V_\mu}$

We also estimated $\tilde{d}_{V_\mu}$ with other approximations:

[De Polsi, Tissier, NW, preliminary]

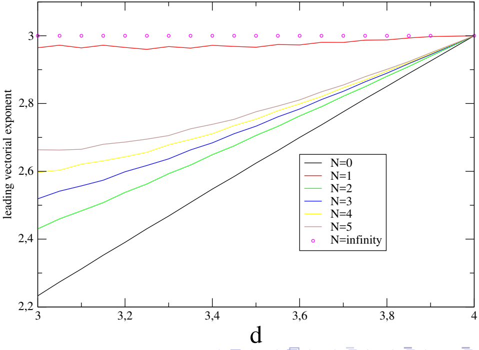
$\rightarrow$ Large-N limit gives (for all d):

$$\tilde{d}^{(a)}_{V_\mu} = \tilde{d}^{(b)}_{V_\mu} = 3 + \mathcal{O}(1/N)$$

$\rightarrow$ Order $\mathcal{O}(\partial^3)$ of the Derivative Expansion of NPRG gives:

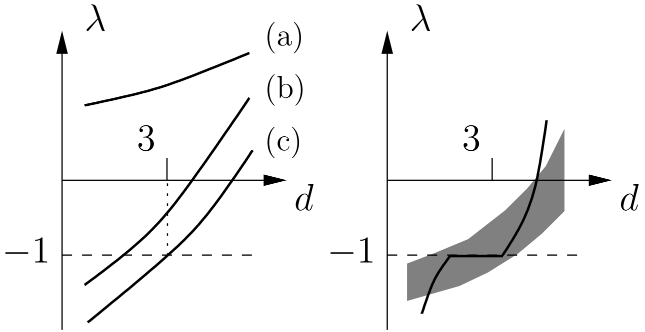
PRELIMINARY!

[De Polsi, Tissier, NW, preliminary]



A "physicist proof" of conformal invariance (III)

3) Imagine that $\tilde{d}_{V_\mu} = -1$ precisely at $d = 3$...



By continuity in d, conformal symmetry would take place also in $d = 3$!
But Non-rigorous: non-analyticity in $\tilde{d}_{V_\mu}$ could take place (or not?).

- Let us consider the Ginzburg-Landau model (in the same universality class).
- Let us consider generic local vector operator even in φ (including total derivatives).
  Example: $V_\mu(x) = \partial_\mu(\varphi^2(x))$ or $V_\mu(x) = \varphi\, \partial_\mu \varphi \sum_\nu (\partial_\nu \varphi)^2$.
- We need to give an upper bound for a two-point correlation function of such operator.
  Note: ϵ-expansion $\to$ we expect that $\tilde{d}_{V_\mu}$ is larger than -1.
- Such operators can be discretized in a lattice. Example:

$$V_\mu(i) = \varphi_i(\varphi_{i+\mu} - \varphi_{i-\mu}) \sum_\nu (\varphi_{i+\nu} - \varphi_{i-\nu})^2$$

A "rigorous" proof for Ising universality class (II)

- There are rigorous well-known bounds for correlation functions in those models. [Griffiths, J. Math. Phys. 8 (1967) 478; Kelly, Sherman, J. Math. Phys. 9 (1968) 466; Lebowitz, Commun. Math. Phys. 35 (1974) 87.]
- Many of those bounds compare correlation functions with the gaussian case ($u = 0$).
  Example: for $T \geq T_c$ and $B = 0$, there are bounds comparing to Wick's type formulas:

$$\langle \varphi_i \varphi_j \varphi_k \varphi_l \rangle \leq \langle \varphi_i \varphi_j \rangle \langle \varphi_k \varphi_l \rangle + \langle \varphi_i \varphi_k \rangle \langle \varphi_j \varphi_l \rangle + \langle \varphi_i \varphi_l \rangle \langle \varphi_j \varphi_k \rangle$$

- Using similar bounds for arbitrary correlation function one can show that: $(T \geq T_c)$

$$0 \leq \langle \varphi^n(x) \varphi^m(y) \rangle_c \leq \begin{cases} cst. \langle \varphi(x)\varphi(y) \rangle^2, & \textit{if n and m even} \\ cst. \langle \varphi(x)\varphi(y) \rangle, & \textit{if n and m odd} \end{cases}$$

- Using scale invariance, these correlation functions are power-laws.

A "rigorous" proof for Ising universality class (III)

- Accordingly, for n and m even:

$$\left|\langle \partial_\mu(\varphi^n(x))\partial_\nu(\varphi^m(y))\rangle\right| \le cst.\partial^x_\mu \partial^y_\mu \left(\frac{cst.}{|x-y|^{2(d-2+\eta)}}\right) \le \frac{cst.}{|x-y|^{2(d-1+\eta)}} \tag{3}$$

- Any discretization of the operator $\partial_\mu(\varphi^n(x))$ will give the same critical exponent.
  Example: Operators

$$\varphi^2(x+a) - \varphi^2(x-a),$$
$$2\varphi(x)\Big(\varphi(x+a) - \varphi(x-a)\Big)or$$
$$\Big(\varphi(x+b) + \varphi(x-b)\Big)\Big(\varphi(x+a) - \varphi(x-a)\Big)$$

have the same continuum limit.

A "rigorous" proof for Ising universality class (IV)

- Now, any vector operator can be discretized as a difference of such operators.
- Using triangle inequality for any vector operator $V_\mu(x)$,

$$\left|\langle V_\mu(x) V_\nu(y)\rangle\right| \leq \frac{cst.}{|x-y|^{2(d-1+\eta)}}$$

- This implies that $d_{V_\mu} \geq d-1+\eta$.
- Moreover $\eta > 0$ in any interacting theory (true for $d<4$). Accordingly, $\tilde{d}_{V_\mu} > -1$.

> The critical Ising model is conformal invariant (for distances $\gg a$).

- Very recently we extended this result for $O(N)$ models when $N=2,3$ or 4. [De Polsi, Tissier, NW, preliminary]

> Heisenberg model is conformal invariant for $N=2,3$ or 4 (for distances $\gg a$).

Conclusions

- An "exact" solution of the critical Ising model in $d = 3$ seems approaching.
- This the most important paradigmatic open problem in equilibrium statistical mechanics.
- It describes the critical regime of the liquid-vapor transition.
- In the recent progresses play a very important role the symmetries of the model in the neighborhood of $T = T_c$.
- In particular, the conformal symmetry plays a major role.
- We gave an almost mathematical proof with B. Delamotte and M. Tissier that this symmetry is a consequence of scale invariance in this (or similar) models.
- Perspectives:
  - We would like to generalize this result to other models (Example: Potts model).
  - We are trying to exploit conformal symmetry in the frame of NPRG (or Wilson) RG equations.