The Coulomb branch integral and mock modular forms

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Based on

1. G.K. and Manschot hep-th/1707.06235

2. G.K., Manschot, Moore, Nidaiev hep-th/1810.xxxxx

3. G.K. hep-th/1810.xxxxx

- **Topologically twisted theories** are over 30 years old by now but we still have many things to learn about them. This is important because of
  - their mathematical implications (topological invariants)
  - the fact they allows us to extract exact results
- There is a lot of recent activity in the world of
  - Kapustin-Witten theory (the $\mathcal{N} = 4$ GL-twist)
  - Vafa-Witten theory (I might have something to say in the end)
  - Donaldson-Witten theory (the $\mathcal{N} = 2$ twist)
- This talk is about some new techniques and results on the Donaldson-Witten (DW) theory [Witten 1988]

Ultimately I am interested in calculating VEVs of various operators in DW theory.

But first I need to make a detour and give motivation and background.

Notation and motivation

Cohomological field theories are defined on arbitrary Riemanian manifolds X in $d = 2, 4, 6...$

They are QFTs whose physical observables belong to the cohomology of X

$$\boxed{\text{observables } \in H^{\bullet}(X)}$$

Our theory of interest is defined on any 4-manifold X but we will make some restrictions for simplicity

X compact Kähler surface with quadratic form

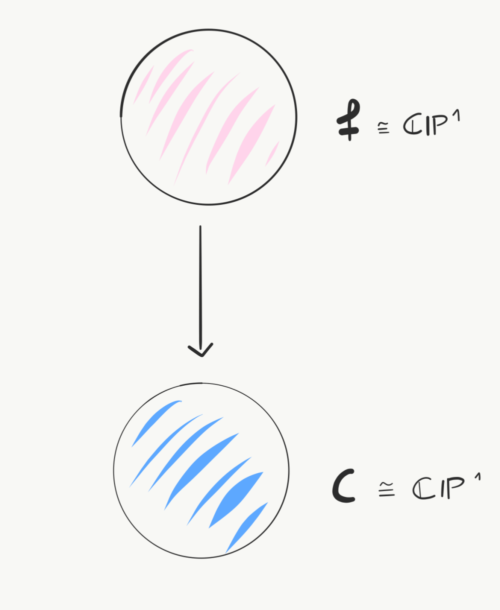
$$Q : H^2(X) \to \mathbb{Z}$$

and bilinear form

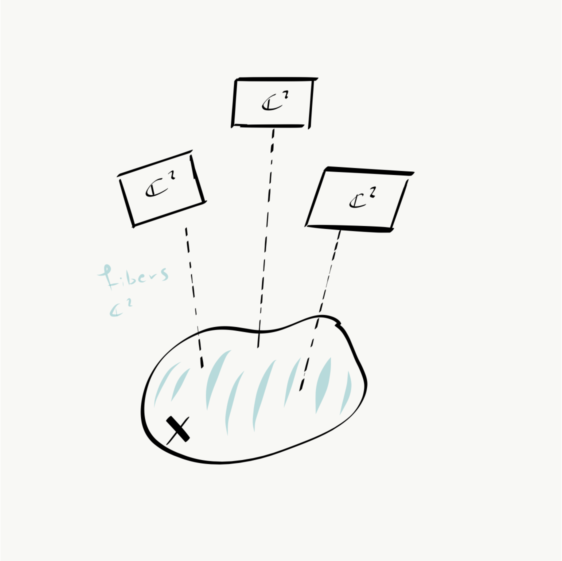
$$B : H^2(X) \times H^2(X) \to \mathbb{Z}$$

$$B(a, b) \equiv \int_X a \wedge b$$

Example: Hirzebruch surface $\mathbb{F}_l$



$E \to X$ a holomorphic vector bundle of rank 2 equipped with connection 1-form A



Also we denote

- J Kähler form
- The moduli space of instantons over X

  $\mathcal{M}(c_1, c_2) := \left\{\text{gauge fields } A \mid F_A^+ = 0 \text{ modulo gauge equiv.}\right\}$
- $\overline{\mathcal{M}}(c_1, c_2)$ is an appropriate compactification of $\mathcal{M}(c_1, c_2)$

We can now construct **Donaldson's polynomial invariants** using homology of X

They are integrals of cohomology classes of $\overline{\mathcal{M}}(c_1, c_2)$ that can be interpreted as correlation functions of topologically twisted theories.

- Donaldson map $\mu : H_i(X) \to H^{4-i}(\overline{\mathcal{M}}(c_1, c_2))$

- Donaldson invariant of degree $d = 4l + 2m$

$$P_{c_2}^{X,d}(p, \boldsymbol{x}) = \int_{\overline{\mathcal{M}}(c_1,c_2)} \mu(p)^l \wedge \mu(\boldsymbol{x})^m$$

$p \in H_0(X)$ and $\boldsymbol{x} \in H_2(X)$

We can package the various $P_{c_2}^{X,d}(p,\boldsymbol{x})$ into a generating function

$$\Phi_{c_1}(p,\boldsymbol{x}) = \sum_{c_2} q^{c_2} \sum_{d} P_{c_2}^{X,d}\left(\frac{p^l}{l!},\frac{\boldsymbol{x}^m}{m!}\right) a^l b^m$$

$$= \sum_{c_2} q^{c_2} \int_{\overline{\mathcal{M}}(c_1,c_2)} e^{\mu(ap+b\boldsymbol{x})}$$

If $b_2^+(X) > 1$ Donaldson invariants are true invariants of the **smooth-structure** of the 4-manifold X.

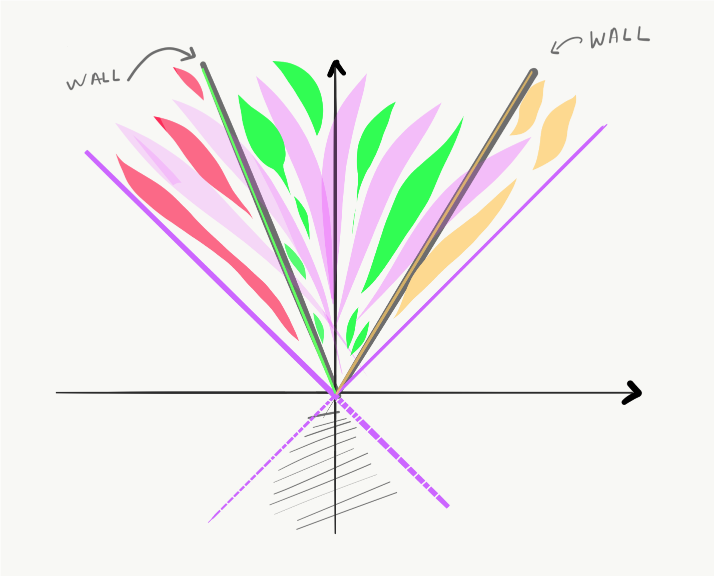
They are independent of the metric we equip X with.

Now let me tell you something interesting that Göttsche and Göttsche and Zagier discovered.

If $b_2^+(X) = 1$ then Donaldson invariants are only **piece-wise smooth invariants**, they depend on the choice of J

$$\Phi_{c_1}(p, \boldsymbol{x}) \longrightarrow \Phi^J_{c_1}(p, \boldsymbol{x})$$

Consider $\mathcal{C}_+ := H^2(X, \mathbb{R})_+$, that is the positive cone of X



- This is exactly the case we are interested in. This is how the physics of the **Coulomb branch** $\mathcal{B}$ plays a role.

- For $b_2^+(X) > 1$ the contribution of the Coulomb branch vanishes - left only with SW invariants

- This is due to Witten's [1988] physical formulation of Donaldson theory and Moore-Witten's [1997] formulation of the u-plane integral.

Donaldson theory in physics arises as following:

- Consider pure $\mathcal{N} = 2$ **topologically twisted SYM** on X with gauge group $SU(2)$ or $SO(3)$

- Twist results to all fields being differential forms on X (including the supercharges)

- The theory contains a BRST-like supercharge $\mathcal{Q}$ whose cohomology provides the physical observables of the theory

$$\mathcal{Q} = \epsilon^{\dot{A}\dot{B}} \bar{Q}_{\dot{A}\dot{B}}$$

- Physical observables (descent formalism) of the theory belong to the

$$\mathcal{Q} - \text{cohomology}$$

- Correlation functions of such observables localize to integrals over

$$\overline{\mathcal{M}}(c_1, c_2)$$

- Witten: for $G = SU(2)$ such correlation functions compute Donaldson invariants (also for $G = SO(3)$)

Greg's favorite comutative diagram

$$
\begin{array}{ccc}
H_*(X) & \xrightarrow{\text{descent formalism}} & H^*(\text{field space}, \mathcal{Q}) \\
& {\scriptstyle \mu\text{-map}} \searrow \qquad \swarrow {\scriptstyle \text{localization}} & \\
& H^*(\overline{\mathcal{M}}(c_2)) &
\end{array}
$$

Aside

What happens for other $\mathcal{N} = 2$ theories with matter?

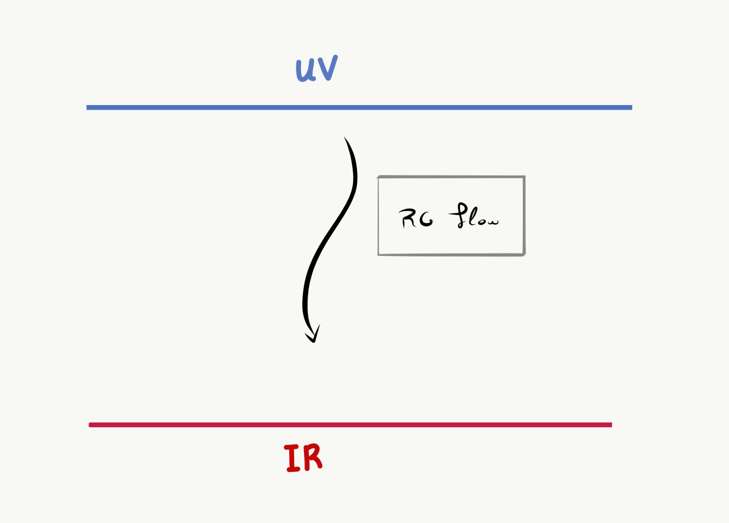
$\overline{\mathcal{M}}(c_1, c_2)$ gets replaced by $\mathscr{M}$

where $\mathscr{M}$ is the moduli space of the so-called (generalized) **Seiberg-Witten equations**

$$F^+ = M\bar{M}$$
$$\not{D}M = 0$$

[Labadista, Marino 1998] [Losev, Shatashvili, Nekrasov 1998] also [Moore, Nidaiev 2017]

Back to the pure theory



- The fields in the IR are:

  bosons: gauge field A, scalar fields $a, \bar{a}$, auxiliary field D
  fermions: 0-form η, 1-form ψ, self-dual 2-form χ

- In the low energy effective theory, the gauge group is broken down to $U(1)$ and the order parameter is

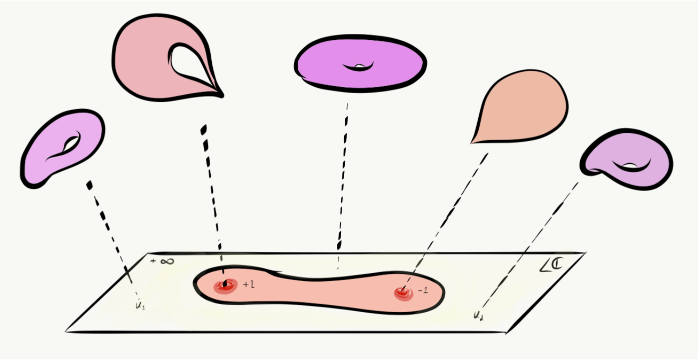
$$u = \langle \mathrm{Tr}(\phi^2) \rangle = 2a^2$$

We study the theory in the context of **Seiberg-Witten theory**.

The quantum moduli space of the theory, that is parametrized by this $u \in H_G^*(pt.)$ is

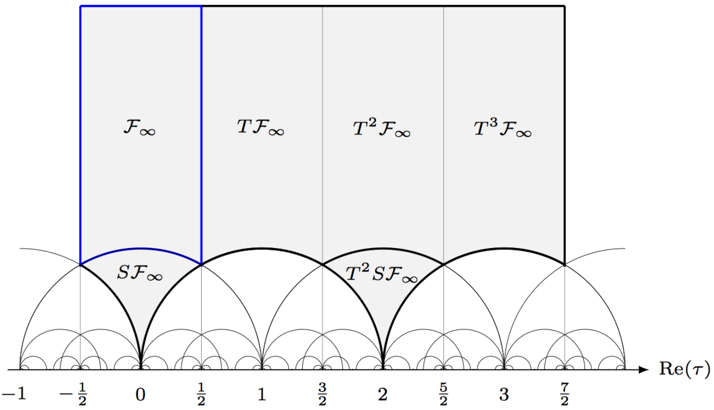
$$\mathcal{B} := \mathbb{CP}^1 \backslash \{-1, 1, \infty\}$$

the **Coulomb branch** over which is fibered a local system of elliptic curves Σ_u.



+∞
+1
−1
ℂ

Recall that we can map $\mathcal{B}$ to $\mathbb{H}/\Gamma^0(4)$ (Riemann uniformization theorem)



The partition function of the low energy DW theory reads

$$Z_{\rm DW} = \int [da\, d\bar{a}\, dA\, dD\, d\eta\, d\psi\, d\chi]\; e^{-S_{\rm DW}}$$

We can consider the partition function with the insertion of some observables

$$Z_{\mathrm{DW}}(p, \boldsymbol{x}) = \int [da\, d\bar{a}\, dA\, dD\, d\eta\, d\psi\, d\chi]\; e^{-S_{\mathrm{DW}} + 2pu + I_{-} + \boldsymbol{x}^2 G(u)}$$

$p \in H_0(X)$ and $\boldsymbol{x} \in H_2(X)$

$$Z_{\rm DW}(p,\boldsymbol{x}) = \int [da\, d\bar{a}\, dA\, dD\, d\eta\, d\psi\, d\chi]\ e^{-S_{\rm DW}+2pu+I_-+\boldsymbol{x}^2 G(u)}$$

- From the descent proceedure we have the 2-observable

$$I_-(\boldsymbol{x}) = -\frac{i}{4\pi}\int_{\boldsymbol{x}} \frac{du}{da}(F_- + D)$$

- From the RG flow we get the contact term

$$G(u) = \frac{1}{24}\left(8u - E_2\Big(\frac{du}{da}\Big)^2\right)$$

Moore and Witten after a LOT of work showed that this reproduces the generating function of **Donaldson invariants**

$$Z_{\mathrm{DW}}(p, \boldsymbol{x}) = \Phi^J_{c_1}(p, \boldsymbol{x})$$

But in CohFT we can include any $\mathcal{Q}$-exact insertions to the path integral since their vev must vanish [Witten 1988]

Let us include the following $\mathcal{Q}$-exact surface observable

[G.K., Manschot 2017]

$$I_+(\boldsymbol{x}) := \int_{\boldsymbol{x}} \left\{ \mathcal{Q}, \frac{d\bar{u}}{d\bar{a}} \chi \right\}$$

Remark: this operator appears in [Moore, Nekrasov, Shatashvili] in a slightly different context

- Does the vev of this $\mathcal{Q}$-exact observable vanish?
- What are the benefits of including it in $\Phi^J_{\boldsymbol{\mu}}(p, \boldsymbol{x})$?

Does the vev of this $\mathcal{Q}$-exact observable vanish?

- Witten says it should [1988]
- Witten and Moore et. al. say maybe not [1992]

I quote from Kapustin's lectures on the $\mathcal{N} = 4$ GL-twisted theory [2007]:

[...], one considers only observables which are annihilated by $\mathcal{Q}$ (and are gauge-invariant) modulo those which are $\mathcal{Q}$-exact. This is consistent because any correlator involving $\mathcal{Q}$-closed observables, one of which is $\mathcal{Q}$-exact, vanishes.

Does the vev of this $\mathcal{Q}$-exact observable vanish?

Yes. At least in DW theory. But it is completely non-trivial.

[G.K., Manschot, Moore, Nidaiev, in preparation]

I will come back to this point.

What are the benefits of including it in $\Phi^J_{\boldsymbol{\mu}}(p, \boldsymbol{x})$?

- localization of Coulomb branch integral to the boundary of $\mathcal{B}$
- no need for lattice reduction
- evaluation for any $J \in H^2(X, \mathbb{R})$
- connection to mock modular forms and moonshine
- $\mathcal{Q}$-exact operator is a good new operator for other quasi(?)-topological theories like the $N_f = 4$ theory and the topologically twisted AD3 theory

Let us look at the Coulomb branch integral

- We are interested in this correlator

$$\Phi^J_{\boldsymbol{\mu}}(p,\boldsymbol{x}) = \int [da\, d\bar{a}\, dA\, dD\, d\eta\, d\psi\, d\chi]\ e^{-S_{\mathrm{DW}}+2pu+I_-+I_++\boldsymbol{x}^2 G(u)}$$

- And after some work it can be shown that for $\pi_1(X)=0$ it equals

$$\boxed{\Phi^J_{\boldsymbol{\mu}}(p,\boldsymbol{x}) = \int_{\mathbb{H}/\Gamma^0(4)} d\tau \wedge d\bar{\tau}\ \nu(\tau) e^{2pu+\boldsymbol{x}^2 G(u)} \Psi^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})}$$

- Where we have
  - $\tau \in \mathbb{H}$ and $\boldsymbol{\rho} \in H_2(X,\mathbb{C}) \otimes \mathbb{M}_{(1,0)}(\Gamma^0(4))$
  - $\nu(\tau)$ a holomorphic function that depends only on the topology of X and the elliptic curve Σ_u
  - $\Psi_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})$ is a sum over $\mathrm{Pic}(X)$

Let us take a closer look at the Siegel-Narain theta function $\Psi^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})$. It explicitly reads

$$\Psi^J_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}) := e^{-2\pi i y \boldsymbol{b}_+} \sum_{\boldsymbol{k}\in\Lambda+\boldsymbol{\mu}} \partial_{\bar\tau}\left(\sqrt{2y}\; B(\boldsymbol{k},\underline{J})\right)$$
$$(-1)^{B(\boldsymbol{k},K_X)} e^{-\pi i\bar\tau \boldsymbol{k}_+^2 - \pi i \tau \boldsymbol{k}_-^2 - \pi i B(\boldsymbol{k}_+,\bar{\boldsymbol{\rho}}) - 2\pi i B(\boldsymbol{k}_-,\boldsymbol{\rho})}$$

with

- $\boldsymbol{k} = [F]/4\pi \in H^2(X,\mathbb{Z})$ are the $\mathrm{U}(1)$ magnetic fluxes
- $\boldsymbol{\rho} = \frac{du}{da}\frac{\boldsymbol{x}}{2\pi}$ and $\boldsymbol{b} = \mathrm{Im}(\boldsymbol{\rho})/y$
- $K_X = c_1(\Omega_X^{\mathsf{top}})$

Although the function $\Psi^J_{\boldsymbol{\mu}}(\tau, \boldsymbol{\rho})$ above is the one relevant for Donaldson invariants we can look into more general such functions in the context of Donaldson-Witten theory. Let us consider

$$\Psi^J_{\boldsymbol{\mu}}[\mathcal{K}](\tau, \boldsymbol{\rho}) := e^{-2\pi y \boldsymbol{b}_+^2} \sum_{\boldsymbol{k} \in \Lambda + \boldsymbol{\mu}} \mathcal{K}(\boldsymbol{k}) (-1)^{B(\boldsymbol{k}, K_X)} \times e^{-\pi i \bar{\tau} \boldsymbol{k}_+^2 - \pi i \tau \boldsymbol{k}_-^2 \pi i B(\boldsymbol{k}_+, \bar{\boldsymbol{\rho}}) - 2\pi i B(\boldsymbol{k}_-, \boldsymbol{\rho})}$$

where $\mathcal{K}(\boldsymbol{k})$ is a generic kernel for now (some function of the magnetic fluxes $\boldsymbol{k}$).

Integrals of the form

$$\Phi^J_{\boldsymbol{\mu}}[\mathcal{K}] = \int_{\mathbb{H}/\Gamma^0(4)} d\tau \wedge d\bar{\tau}\ \nu(\tau)\Psi^J_{\boldsymbol{\mu}}[\mathcal{K}]$$

can be solved by relating them to integrals over $\tau \in \mathbb{H}/\Gamma^0(4)$ and write the integrands as total derivatives

$$\boxed{\frac{d}{d\bar{\tau}}\widehat{\mathcal{H}}_{\boldsymbol{\mu}}[\mathcal{K}] = \nu(\tau)\ \Psi^J_{\boldsymbol{\mu}}[\hat{\mathcal{K}}](\tau,\bar{\tau})}$$

where $\widehat{\mathcal{H}}_{\boldsymbol{\mu}}[\mathcal{K}]$ transforms as a modular form of weight $(2,0)$.

Let us assume that $b_2 > 1$ and let us require the existence of an empty chamber w.r.t to the null vector J'.

Then we can write $\Psi_{\boldsymbol{\mu}}^{J}[\mathcal{K}]$ as a total derivative using **Zwegers' completed indefinite theta function** with kernel $\widehat{\mathcal{K}}$

$$\widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}[\widehat{\mathcal{K}}] = \sum_{\boldsymbol{k}\in\Lambda} \widehat{\mathcal{K}}(\boldsymbol{k})(-1)^{B(\boldsymbol{k},K_X)} q^{-\frac{\boldsymbol{k}^2}{2}}$$

Then $\widehat{\mathcal{H}}_{\boldsymbol{\mu}}^{JJ'}[\widehat{\mathcal{K}}]$ takes generically the form

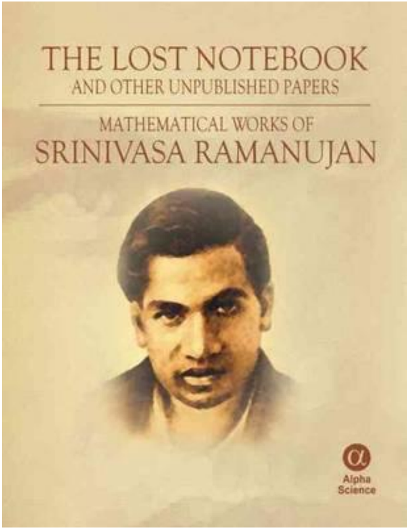
$$\widehat{\mathcal{H}}_{\boldsymbol{\mu}}^{JJ'}[\mathcal{K}] = \nu \ \widehat{\Theta}_{\boldsymbol{\mu}}^{JJ'}[\widehat{\mathcal{K}}]$$

with $\mathcal{K}$ and $\widehat{\mathcal{K}}$ related as

$$\frac{d}{d\bar{\tau}}\widehat{\mathcal{K}}(\boldsymbol{k}) = \mathcal{K}(\boldsymbol{k})e^{-2\pi y \boldsymbol{k}^2}$$

Interlude

Let me say very few words on the indefinite theta functions and **mock modular forms** that were first discovered by Ramanujan.



Zwegers' indefinite theta function is a (famous) example of a **mock modular form**. It is a map

$$\Theta : \mathbb{H} \to \mathbb{C}$$

which can be expressed as a convergent holomorphic q-series over a lattice Λ of indefinite signature. **Such a function fails to be modular**.

The way to go around this is to add a non-holomorphic function R with argument a function (the shadow of Θ)

$$g : \mathbb{H} \times \bar{\mathbb{H}} \to \mathbb{C}$$

such that

$$\widehat{\Theta} = \Theta + R(g(\tau, \bar{\tau}))$$

transforms as a non-holomorphic modular form.

Back to the Coulomb branch integral...

Exactly due to the equation

$$\frac{d}{d\bar{\tau}}\widehat{\mathcal{H}}_{\boldsymbol{\mu}}[\mathcal{K}] = \nu(\tau)\ \Psi^{J}_{\boldsymbol{\mu}}[\widehat{\mathcal{K}}](\tau,\bar{\tau})$$

which makes $\widehat{\mathcal{H}}$ have the generic form

$$\widehat{\mathcal{H}}_{\boldsymbol{\mu}}[\mathcal{K}] = \nu(\tau)\ \widehat{\Theta}^{JJ'}_{\boldsymbol{\mu}}[\widehat{\mathcal{K}}](\tau,\bar{\tau})$$

...our integral localizes to the boundaries of the Coulomb branch $\mathcal{B}$ that map to the boundaries of $\mathbb{H}/\Gamma^0(4)$. As a result we have

$$\Phi^J_{\boldsymbol{\mu}}[\mathcal{K}] = \sum_{\partial(\mathbb{H}/\Gamma^0(4))} \oint d\tau \ \widehat{\mathcal{H}}[\mathcal{K}]$$

...and the contribution from the three cusps therefore reads

$$\boxed{\Phi^J_{\boldsymbol{\mu}}[\mathcal{K}](p,\boldsymbol{x}) = 4\left[\widehat{\mathcal{H}}[\mathcal{K}](\tau,\bar{\tau})\right]_{q^0} + \left[S\mathcal{F}_\infty\right]_{q^0} + \left[T^2 S\mathcal{F}_\infty\right]_{q^0}}$$

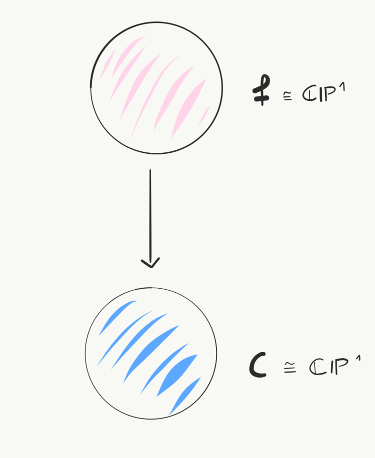
where $\mathcal{F}_\infty$ is the fundamental domain of $SL(2,\mathbb{Z})$.

The q^0 coefficients of this modular form then easily gives us the Donaldson invariants if we include p and $\boldsymbol{x}$.

Example. Let $X = \mathbb{F}_l$ with quadratic form $Q_{\mathbb{F}_l} = \begin{pmatrix} -l & 1 \\ 1 & 0 \end{pmatrix}$.
These are the Hirzebruch surfaces which can be viewed as a fibration

$$\pi : \mathbb{F}_l \to \boldsymbol{C}$$

with fiber $\boldsymbol{f} \cong \mathbb{CP}^1$ over the base $\boldsymbol{C} \cong \mathbb{CP}^1$.



By setting $J' = \boldsymbol{f}$ we can evaluate $\Phi^J_{\boldsymbol{\mu}}(p, \boldsymbol{x})$ for $\mathbb{F}_l$ using the kernel

$$\widehat{\mathcal{K}}(\boldsymbol{k}) = -\frac{du}{da}\frac{\pi i B(\boldsymbol{k},\boldsymbol{x})}{2}\left\{E(\sqrt{2y}B(\boldsymbol{k},\underline{J})) - \mathrm{sgn}(B(\boldsymbol{k},J'))\right\}$$
$$-\frac{du}{da}\frac{iB(\boldsymbol{x},\underline{J})}{2\sqrt{2y}}e^{-2\pi y\boldsymbol{k}_+^2}$$

and the result of the u-plane integral [actually for all Kähler surfaces of Kodaira dimension $-\infty$ and $K_X^2 > 0$] is

$$\Phi^J_{\boldsymbol{\mu}}(p, \boldsymbol{x}) = 4\left[\nu(\tau)e^{2pu+\boldsymbol{x}^2G(u)}\widehat{\Theta}^{JJ'}_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})\right]_{q^0}$$

Only the cusp at $i\infty$ contributes in the case of $\mathbb{F}_l$ and we arrive at the expression

$$\Phi^J_{\boldsymbol{\mu}}(p,\boldsymbol{x}) = 32i\left[(u^2-1)\frac{da}{du}e^{2pu+\boldsymbol{x}^2G(u)}\widehat{\Theta}^{J\boldsymbol{f}}_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})\right]_{q^0}$$

The expression for $\Theta^{J\boldsymbol{f}}_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})$ can be written as a **generalized Appell sum** that skipping a lot of details looks like

$$\Theta^{J\boldsymbol{f}}_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}) = \sum_{\substack{\boldsymbol{m}\in\Lambda+\boldsymbol{\mu}\\ B(\boldsymbol{m}+\boldsymbol{b},J)/B(\boldsymbol{f},J)\in[0,1)}} \frac{(-1)^{B(\boldsymbol{m},K_\ell)}q^{-\boldsymbol{m}^2/2}e^{-2\pi i B(\boldsymbol{\rho},\boldsymbol{m})}}{1-q^{-B(\boldsymbol{f},\boldsymbol{m})}e^{-2\pi i B(\boldsymbol{\rho},\boldsymbol{f})}}$$

Wall-crossing formula is obtained readily with this technique :)

$$\Delta\Phi^{J_1 J_2}_{\boldsymbol{\mu}}(p,\boldsymbol{x}) = 4\left[\nu(\tau)\widehat{\Theta}^{J_1 J_2}_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho})e^{2pu+\boldsymbol{x}^2 G(u)}\right]_{q^0} + \text{ modular transf.}$$

for arbitrary polarizations J_1 and J_2.

We have seen how the inclusion of the $\mathcal{Q}$-exact observable $I_+(\boldsymbol{x})$ yields these useful formulae which allow an explicit computation of the u-plane integral for any polarization.

$$I_+(\boldsymbol{x}) = \int_{\boldsymbol{x}} \left\{ \mathcal{Q}, \frac{d\bar{u}}{d\bar{a}} \chi \right\}$$

But some questions arise.

Is this operator well-defined ?

Is the integrand single-valued?

Does the vev of this operator vanish?

$$\langle I_+ \rangle = 0 \ ?$$

Is this operator well-defined ? **Yes!**

Is the integrand single-valued? **Yes!**

Does the vev of this operator vanish? **Yes!**

$$\langle I_+ \rangle = 0 \, !$$

CohFT rules say that for a $\mathcal{Q}$-exact observable $\mathcal{O}$ we should have

$$\langle \mathcal{O} \rangle = \int [D\mathcal{X}] \; e^{-S_{\mathsf{CohFT}}[\mathcal{X}]} \mathcal{O} = 0$$

Naively our integral $\langle I_+ \rangle$ is divergent!

In upcoming work together with Jan Manschot, Greg Moore and Iurri Nidaiev we show that this is not the case.

We derive a subtle regularization of such naively diverging integrals generalizing on work of mathematicians on regularization of Petersson inner products.

Applying this regularization to the vev of our $\mathcal{Q}$-exact operator

$$I_+(\boldsymbol{x}) = \int_{\boldsymbol{x}} \left\{ \mathcal{Q}, \frac{d\bar{u}}{d\bar{a}} \chi \right\}$$

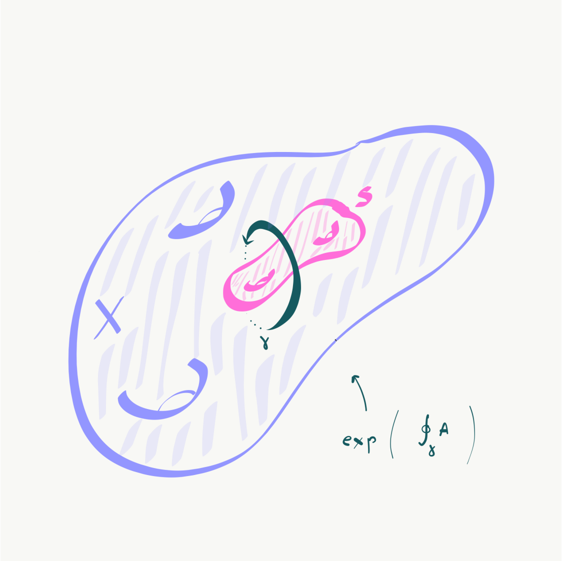
we find that indeed

$$\boxed{\int [D\mathcal{X}]\ e^{-S_{\rm DW}} I_+(\boldsymbol{x}) = 0}$$

as required by the CohFT rules set by Witten

Ramified Donaldson-Witten theory

Finally let me mention a rather simple extension to theories with **surface defects** .



- X is again a Kähler surface
- $\boldsymbol{S} \hookrightarrow X$ is an embedded complex curve of genus g

A **surface operator** amounts to a prescribed singular behavior of the gauge field restricted on $N\boldsymbol{S}$

$$A = \alpha d\theta + \text{regular}$$

where the normal coordinate to a plane on $\boldsymbol{S}$ is

$$d\theta = i\frac{dz}{z}$$

and with $\alpha \in \mathfrak{t}$ specifying the type of surface operator. Actually $\alpha \in \mathbb{T}$ and the type of surface operator is defined by a choice of lift from $\mathbb{T} \cong \mathfrak{t}/\Lambda_{\mathrm{cochar.}}$ to $\mathfrak{t}$.

The curvature of the gauge field modies to

$$F = 2\pi\alpha\delta_{\boldsymbol{S}}$$

We can interpret the theory of the surface operator as the theory of an extended vector bundle with connection whose curvature is

$$\mathscr{F} = F - 2\pi\alpha\delta_{\boldsymbol{S}}$$

For flat space BPS condition gives

$$\mathscr{F} = 0$$

while for compact 4-manifold X

$$\mathscr{F}^{+} = 0$$

which corresponds to the so-called **ramified instantons**.

Performing a similar analysis as before (following the works of [Moore, Witten 1997] and [Tan 2009]) by including the $\mathcal{Q}$-exact operator $I_+(\boldsymbol{x})$ we find [G.K. 2018]

$$\tilde{\Phi}^J_{\boldsymbol{\mu}}(p,\boldsymbol{x},\boldsymbol{S}) = 4\left[\nu(\tau)e^{2pu+\boldsymbol{x}^2G(u)+\tilde{\boldsymbol{S}}^2H(u)}\widehat{\Upsilon}^{JJ'}_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho};\alpha)\right]_{q^0} + [S\mathcal{F}_\infty]_{q^0} + [T^2S\mathcal{F}_\infty]_{q^0}$$

$$\widehat{\Upsilon}^{JJ'}_{\boldsymbol{\mu}}(\tau,\boldsymbol{\rho}) = \sum_{\tilde{\boldsymbol{k}}\in\Lambda+\boldsymbol{\mu}} \Big\{ E(\sqrt{2y}B(\tilde{\boldsymbol{k}}+\boldsymbol{b},\underline{J})) - \mathrm{sgn}(\sqrt{2y}B(\tilde{\boldsymbol{k}}+\boldsymbol{b},J')) \Big\}$$

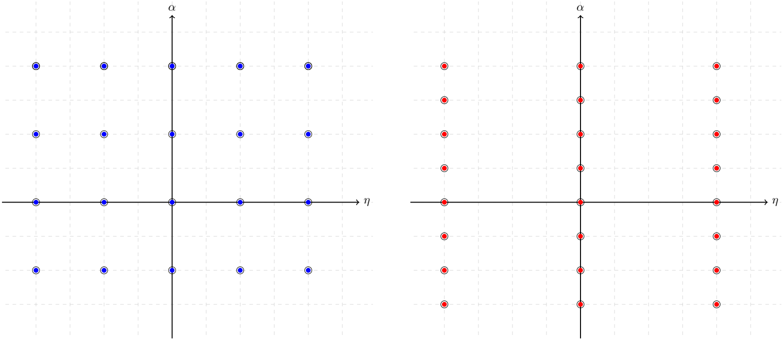
$$\times\, (-1)^{B(\tilde{\boldsymbol{k}},K_X)} q^{-\boldsymbol{k}^2/2} e^{-2\pi i B(\tilde{\boldsymbol{k}},\boldsymbol{\rho})}$$

$$\tilde{\boldsymbol{k}} := \boldsymbol{k} - \frac{\alpha}{2}\delta_{\boldsymbol{S}}$$

$$\lim_{\alpha,\eta\to 0} \widehat{\Upsilon}^{JJ'}_{\boldsymbol{\mu}} = \widehat{\Theta}^{JJ'}_{\boldsymbol{\mu}}$$

.

Modularity of the integrand of imposes the allowed charges of the surface operator.



The ramified Donaldson invariants can be expressed in terms of the classical Donaldson invariants, thus they do not provide any new information for smooth 4-manifolds. [Mrowka, Kronheimer]

Still, it is nice to be able to explicitly compute these correlators in the ramified theory as well.

Summary

- Still many things to understand about CohFTs

- $\mathcal{Q}$-exact insertion in DW theory yields new surprising results and connections to mock modular forms

- DW seems to be somewhat special; vevs of $\mathcal{Q}$-exact insertions seem to vanish

- These techniques apply for the ramified DW theory as well (surface defects)

Some cool ideas

- Higher rank theories? First steps at [G.K., Manschot] but also previous work by [Moore, Mariño]
- Other surfaces? E.g. $\mathbb{CP}^2 \# 7 \overline{\mathbb{CP}}^2$, Enriques surface?
- Other theories?

- Take the example of $\mathbb{F}_l$ and try to understand precisely what happens when the fiber shrinks to zero. The resulting function should be the generator of the quantum cohomology of the moduli space of flat connections on $\mathbb{CP}^1$

- Consider $\pi_1(X) \neq 0$. Reduce the theory to $2d$. Can you compute this quantum cohomology without lattice reduction but with the techniques desribed above? [Bershadsky et. al. 1995]

- What about a ramified version of Vafa-Witten theory? That is, study the partition function of the VW theory with embedded divisors and its modular properties. Will there be any surprises?