Superconformal symmetry and classification of black hole horizons

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Work presented includes material published in papers with

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Black Holes and Conformal Symmetry

- Many of the recent developments in physics, like AdS/CFT, rely on the observation that
- (super)symmetry enhances near extreme black hole and brane horizons and includes a conformal subgroup.

Is there an explanation for this?

Determine the Killing superalgebras of near horizon geometries

Prove a no-existence theorem for $N > 16$ near horizon geometries

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Reissner-Nordström

Consider the Reissner-Nordström black hole [Carter]

$$ds^2 = -\frac{\Lambda}{\rho^2}dt^2 + \rho^2\Lambda^{-1}d\rho^2 + \rho^2 ds^2(S^2)$$

where

$$\Lambda = \rho^2 - 2M\rho + Q^2 = (\rho - \rho_+)(\rho - \rho_-)\ , \quad \rho_\pm = M \pm \sqrt{M^2 - Q^2}$$

Introduce Eddington-Finklestein coordinates as

$$d\rho^* = \rho^2\Lambda^{-1}d\rho\ , \quad u = t + \rho^*$$

to rewrite the metric as

$$ds^2 = -\frac{\Lambda}{\rho^2}du^2 + 2dud\rho + \rho^2 ds^2(S^2)$$

Next define the coordinate $r = \rho - \rho_+$ and observe that the metric is analytic in r. Expanding around $r = 0$

$$\begin{aligned} ds^2 &= 2du\Big[dr - \frac{1}{2}\Big(r\,\frac{\rho_+ - \rho_-}{\rho_+^2} + r^2\,\frac{2\rho_- - \rho_+}{\rho_+^3} + \mathcal{O}(r^3)\Big)du\Big] \\ &+ (\rho_+^2 + 2r\,\rho_+ + r^2)ds^2(S^2) \end{aligned}$$

- The linear term in r is the surface gravity of the horizon
- If the black hole is extreme $\rho_- = \rho_+$, then a near horizon limit can be defined by writing

$$u \to \ell^{-1}u\,, \quad r \to \ell r$$

and taking $\ell \to 0$

- The near horizon geometry in this case is

$$ds^2 = 2du\left[dr - \frac{1}{2}r^2\,\frac{1}{\rho_+^2}du\right] + \rho_+^2 ds^2(S^2)$$

which is isometric to $AdS_2 \times S^2$.

- The symmetry enhances from $\mathbb{R} \times SO(3)$ to $SL(2,\mathbb{R}) \times SO(3)$
- Supersymmetry also enhances from 4 to 8 supersymmetries
- Every black hole solution that has ever been found in $d \geq 4$ dimensions which preserves some supersymmetry and has a regular horizon exhibits an $SL(2,\mathbb{R})$ symmetry!

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Lichnerowicz formula and theorem

Representation theory implies that the number of parallel (Killing) spinors of 8-d manifolds with holonomy strictly $Spin(7)$, $SU(4)$, $Sp(2)$ and $\times^2 Sp(1)$ is

$$N_p = 1,\ \ 2,\ \ 3,\ \ 4,$$

respectively.

Th:

$$\begin{aligned} N_p = \mathrm{index}(\not{D}) &\equiv \dim\mathrm{Ker}\not{D} - \dim\mathrm{Ker}\not{D}^\dagger = \frac{1}{5760}(-4p_2 + 7p_1^2) \\ &= \frac{1}{24}(-1 + b_1 - b_2 + b_3 + b_4^+ - 2b_4^-) \end{aligned}$$

where $\not{D}$ is the Dirac operator, $\not{D} : \Gamma(S^+) \to \Gamma(S^-)$, and $S^\pm$ are the chiral and anti-chiral spin bundles on the manifold.

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Proof:
To relate N_p to the index use the identity $\not{D}^2 = \nabla^2 - \frac{1}{4}R$ to establish the Lichnerowicz formula

$$\int \parallel \not{D}\epsilon \parallel^2 = \int \parallel \nabla\epsilon \parallel^2 + \frac{1}{4}\int R \parallel \epsilon \parallel^2$$

Since for such manifolds, $R = 0$, all zero modes of the Dirac operator $\not{D}\epsilon = 0$ and are parallel $\nabla\epsilon = 0$. Furthermore $\ker \not{D}^\dagger = \{0\}$ as otherwise the holonomy will reduce further to subgroups of G_2. Using these we have

$$N_p = \dim \mathrm{Ker}(\not{D}) - 0 = \dim \mathrm{Ker}(\not{D}) - \dim \mathrm{Ker}(\not{D}^\dagger) = \mathrm{index}(\not{D})$$

which establishes the statement. □

- Note that for supersymmetric quantum mechanics models on manifolds with $R \geq 0$, supersymmetry is spontaneously broken!

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Near horizon metrics

Near a smooth extreme Killing horizon an (Eddington-Finklestein) coordinate system can be adapted such that the metric is [Isenberg, Moncrief; Friedrich, et al]

$$ds^2 = 2du[dr + r\, h_I(r,y)dy^I - \frac{1}{2}r^2 f(r,y)du] + \gamma_{IJ}(y,r)dy^I dy^J$$

In the near horizon limit, $u \to \ell^{-1}u$, $r \to \ell r$, $\ell \to 0$, the leading term of the metric is

$$ds^2 = 2du[dr + r\, h_I dy^I - \frac{1}{2}r^2\, \Delta du] + \gamma_{IJ} dy^I dy^J$$

where

$$h_I = h_I(0,y)\ , \quad \Delta = f|_{r=0}\ , \quad \gamma_{IJ} = \gamma_{IJ}(0,y)$$

- The near horizon metric has two isometries generated by translations in u and the scale transformation

$$u \to \ell^{-1} u \;, \quad r \to \ell r$$

- The two Killing vectors

$$\partial_u \;, \quad -u\partial_u + r\partial_r$$

  do not commute. The algebra of isometries is NOT $\mathfrak{sl}(2,\mathbb{R})$
- This coordinate system can be adapted in the presence of other fields like Maxwell and k-form gauge potentials
- The co-dimension 2 space given by $u = r = 0$ is the spatial horizon section, $\mathcal{S}$, and it is required to be closed, i.e. compact without boundary.

Horizon conjecture

Let M be the near horizon geometry of an extreme, smooth, *Killing* horizon admitting a closed spatial horizon section $\mathcal{S}$ and preserving at least one supersymmetry, then [Gran, Gutowski, GP]

- Conjecture 1: The number of Killing spinors N of M are

$$N = 2N_- + \mathrm{Index}(D\!\!\!/_E)$$

  where $N_- \in \mathbb{Z}_{>0}$, $D\!\!\!/_E$ is a Dirac operator twisted by E defined on the horizon sections $\mathcal{S}$. E depends on the gauge symmetries of supergravity.

- Conjecture 2: If M has non-trivial fluxes and $N_- \neq 0$, then M admits a $\mathfrak{sl}(2,\mathbb{R})$ symmetry subalgebra

Established for

- d=5 minimal gauged, d=11, IIB, heterotic, (massive) IIA supergravities and non-minimal gauged $\mathcal{N}$=2 d=4 supergravity

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  where $N_{-} \in \mathbb{Z}_{>0}$, $D\!\!\!/_E$ is a Dirac operator twisted by E defined on the horizon sections $\mathcal{S}$. E depends on the gauge symmetries of supergravity.

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Remarks

- If the index vanishes, which is the case for non-chiral theories, then N is even. In particular for all odd dimensional horizons, N is even.
- The horizons with non-trivial fluxes of all non-chiral theories have a $\mathfrak{sl}(2, \mathbb{R})$ symmetry subalgebra
- If $N_- = 0$, then $N = \mathrm{index}(D\!\!\!/_E)$ and so the number of Killing spinors is determined by the topology of horizons.
- The only symmetry assumptions are (i) a time-like Killing vector field which becomes null at the horizon, and (ii) that the horizon preserves one supersymmetry.

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Consequences and Applications

These results can be applied in a variety of problems

- The existence of higher dimensional black holes with exotic topologies and geometries
  Asymptotically AdS_5 rings in minimal 5d gauged supergravity have been ruled out! [Grover, Gutowski, GP, Sabra; Grover, Gutowski, Sabra]
- Microscopic counting of entropy for black holes
  The presence of $\mathfrak{sl}(2, \mathbb{R})$ justifies the use of conformal mechanics in entropy counting.
- AdS/CFT: Provides a new method to classify all AdS backgrounds in supergravity.
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The bosonic fields of the theory [Cremmer, Julia, Scherk] are the metric g and a 4-form field strength, F, $dF=0$.
The Einstein field equation of the theory is

$$R_{MN}=\frac{1}{12}F_{ML_1L_2L_3}F_N{}^{L_1L_2L_3}-\frac{1}{144}g_{MN}F_{L_1L_2L_3L_4}F^{L_1L_2L_3L_4}\ .$$

and the field equation of the 4-form field strength is

$$d\star_{11}F-\frac{1}{2}F\wedge F=0\ ,$$

The KSE is the vanishing condition of the supersymmetry variation of the gravitino

$$\mathcal{D}_M\epsilon\equiv\nabla_M\epsilon-\big(\frac{1}{288}\Gamma_M{}^{L_1L_2L_3L_4}F_{L_1L_2L_3L_4}-\frac{1}{36}F_{ML_1L_2L_3}\Gamma^{L_1L_2L_3}\big)\epsilon=0$$

where ϵ is a 32 component Majorana $\mathfrak{spin}(10,1)$ spinor.

M-horizons

The near horizon fields of d=11 supergravity are

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j = 2du(dr + rh - \frac{1}{2}r^2\Delta du) + d\tilde{s}^2(\mathcal{S}) \,, \\ F &= \mathbf{e}^+ \wedge \mathbf{e}^- \wedge Y + r\mathbf{e}^+ \wedge d_h Y + X \,, \quad d_h Y = dY - h \wedge Y \,, \end{aligned}$$

where

$$\mathbf{e}^+ = du \,, \quad \mathbf{e}^- = dr + rh - \frac{1}{2}r^2\Delta du \,, \quad \mathbf{e}^i = e^i{}_J dy^J$$

The steps in the proof of the conjectures are as follows.

- Integration of KSEs along the lightcone directions r, u
- Independent KSEs on $\mathcal{S}$
- Horizon Dirac equations
- Two Lichnerowicz type of theorems
- Index and number of Killing spinors

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Integrability of KSEs along the lightcone

The KSE $\mathcal{D}_M\epsilon = 0$ of $d = 11$ supergravity can be integrated along to lightcone directions to give

$$\epsilon = \epsilon_+ + \epsilon_- \ , \quad \Gamma_\pm \epsilon_\pm = 0 \ ,$$

with

$$\epsilon_+ = \eta_+, \qquad \epsilon_- = \eta_- + r\Gamma_-\Theta_+\eta_+ \ ,$$

and

$$\eta_+ = \phi_+ + u\Gamma_+\Theta_-\phi_-, \qquad \eta_- = \phi_- \ ,$$

where

$$\Theta_\pm = \left(\frac{1}{4}h_i\Gamma^i + \frac{1}{288}X_{\ell_1\ell_2\ell_3\ell_4}\Gamma^{\ell_1\ell_2\ell_3\ell_4} \pm \frac{1}{12}Y_{\ell_1\ell_2}\Gamma^{\ell_1\ell_2}\right) \ ,$$

and $\phi_\pm = \phi_\pm(y)$ do not depend on r or u.

Independent KSEs

The integration along the lightcone directions has two consequences.
First after using the field equations and Bianchi identities, the remaining independent KSEs are

$$\nabla_i^{(\pm)}\phi_\pm \equiv \tilde{\nabla}_i\phi_\pm + \Psi_i^{(\pm)}\phi_\pm = 0\,,$$

where

$$\Psi_i^{(\pm)} = \mp\frac{1}{4}h_i - \frac{1}{288}\Gamma_i{}^{\ell_1\ell_2\ell_3\ell_4}X_{\ell_1\ell_2\ell_3\ell_4} + \frac{1}{36}X_{i\ell_1\ell_2\ell_3}\Gamma^{\ell_1\ell_2\ell_3} \pm\frac{1}{24}\Gamma_i{}^{\ell_1\ell_2}Y_{\ell_1\ell_2} \mp \frac{1}{6}Y_{ij}\Gamma^j\,,$$

and $\tilde{\nabla}$ the Levi-Civita connection of $\mathcal{S}$.
Second, if ϕ_- is a solution, $\nabla_i^{(-)}\phi_- = 0$, then

$$\nabla_i^{(+)}\phi'_+ = 0\,, \quad \phi'_+ = \Gamma_+\Theta_-\phi_-$$

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Horizon Dirac operators

The associated horizon Dirac operators are

$$\mathcal{D}^{(\pm)}\phi_\pm \equiv \Gamma^i \nabla_i^{(\pm)}\phi_\pm = \Gamma^i \tilde{\nabla}_i \phi_\pm + \Psi^{(\pm)}\phi_\pm \ ,$$

where

$$\Psi^{(\pm)} = \Gamma^i \Psi_i^{(\pm)} = \mp\frac{1}{4} h_\ell \Gamma^\ell + \frac{1}{96} X_{\ell_1\ell_2\ell_3\ell_4}\Gamma^{\ell_1\ell_2\ell_3\ell_4} \pm \frac{1}{8} Y_{\ell_1\ell_2}\Gamma^{\ell_1\ell_2} \ .$$

Clearly,

$$\nabla_i^{(\pm)}\phi_\pm = 0 \Longrightarrow \mathcal{D}^{(\pm)}\phi_\pm = 0$$

The converse is also true, ie

$$\nabla_i^{(\pm)}\phi_\pm = 0 \Longleftrightarrow \mathcal{D}^{(\pm)}\phi_\pm = 0$$

A maximum principle

The proof of converse for the $\mathcal{D}^{(+)}$ operator relies on the formula that if $\mathcal{D}^{(+)}\phi_+ = 0$, then

$$\tilde{\nabla}^i\tilde{\nabla}_i \parallel \phi_+ \parallel^2 - h^i\tilde{\nabla}_i \parallel \phi_+ \parallel^2 = 2\langle \nabla^{(+)i}\phi_+, \nabla^{(+)}_i\phi_+\rangle \ .$$

Using the maximum principle for the function $\parallel \phi_+ \parallel^2$ based on the compactness of $\mathcal{S}$, one concludes that

$$\nabla^{(+)}_i\phi_+ = 0 \ , \quad \parallel \phi_+ \parallel^2 = \text{const} \ .$$

which gives the proof of a Lichnerowicz type of theorem for $\mathcal{D}^{(+)}$

- A Lichnerowicz type of Theorem can also be demonstrated for the $\mathcal{D}^{(-)}$ operator

A Lichnerowicz Theorem for $\mathcal{D}^{(-)}$

This is based on a partial integration formula,

$$\int_{\mathcal{S}} \parallel \mathcal{D}^{(-)}\phi_- \parallel^2 = \int_{\mathcal{S}} \parallel \nabla^{(-)}\phi_- \parallel^2 + \int_{\mathcal{S}} \langle \mathcal{B}\phi_-, \mathcal{D}^{(-)}\phi_- \rangle + \text{FEs, BI, surf. terms}$$

where $\mathcal{B}$ depends on the fluxes and one of the FEs is

$$\tilde{R}_{ij} + \tilde{\nabla}_{(i}h_{j)} - \frac{1}{2}h_i h_j = -\frac{1}{2}Y_{i\ell}Y_j{}^{\ell} + \frac{1}{12}X_{i\ell_1\ell_2\ell_3}X_j{}^{\ell_1\ell_2\ell_3} + \delta_{ij}\left(\frac{1}{12}Y_{\ell_1\ell_2}Y^{\ell_1\ell_2} - \frac{1}{144}X_{\ell_1\ell_2\ell_3\ell_4}X^{\ell_1\ell_2\ell_3\ell_4}\right),$$

The surface terms vanish because $\mathcal{S}$ is compact without boundary. So if the field equations and Bianchi identities are satisfied, then all zero modes of $\mathcal{D}^{(-)}$ are $\nabla^{(-)}$-parallel.

Index and supersymmetry

The spin bundle splits $S = S_+ \oplus S_-$ on $\mathcal{S}$ with respect to $\Gamma_\pm$, and $\mathcal{D}^{(+)} : \Gamma(S_+) \to \Gamma(S_+)$ and its adjoint $(\mathcal{D}^{(+)})^\dagger : \Gamma(S_+) \to \Gamma(S_+)$.
$\mathcal{D}^{(+)}$ has the same principal symbol as the Dirac operator and $\mathrm{Index}(\mathcal{D}^{(+)}) = 0$ as $\dim\mathcal{S} = 9$. Thus

$$\dim\ker\mathcal{D}^{(+)} = \dim\ker(\mathcal{D}^{(+)})^\dagger \ .$$

Then $(\mathcal{D}^{(+)})^\dagger\Gamma_+ = \Gamma_+\mathcal{D}^{(-)}$ and so

$$\dim\ker(\mathcal{D}^{(+)})^\dagger = \dim\ker\mathcal{D}^{(-)}$$

Thus

$$\dim\ker\mathcal{D}^{(+)} = \dim\ker\mathcal{D}^{(-)} \ .$$

The number of supersymmetries of a near horizon geometry is the number of parallel spinors of $\nabla^{(\pm)}$ and so from the Lichnerowicz theorems and the index

$$N = \dim\ker\mathcal{D}^{(+)} + \dim\ker\mathcal{D}^{(-)} = 2\dim\ker\mathcal{D}^{(-)} = 2N_- .$$

This proves that the number of supersymmetries preserved by M-horizon geometries is even.

Construction of ϕ_+ spinors from ϕ_- spinors

Recall that if $\nabla^{(-)}\phi_- = 0$, then

$$\nabla^{(+)}\phi_+ = 0\,, \quad \phi_+ = \Gamma_+\Theta_-\phi_-\,.$$

To find a second supersymmetry, $\phi_+ \neq 0$. Indeed after a partial integration argument and some use of the maximum principle

$$\mathrm{Ker}\,\Theta_- \neq \{0\} \Longleftrightarrow F = 0, h = \Delta = 0$$

So if $\mathrm{Ker}\,\Theta_- \neq \{0\}$, the near horizon geometries have vanishing fluxes and are products $\mathbb{R}^{1,1} \times S^1 \times X^8$, where X^8 has holonomy contained in $Spin(7)$.

- For horizons with non-trivial fluxes if $\phi_- \neq 0$, then $\phi_+ = \Gamma_+\Theta_-\phi_- \neq 0$

$\mathfrak{sl}(2,\mathbb{R})$ symmetry

Every near horizon geometry with non-trivial fluxes admits at least two Killing spinors given by

$$\epsilon_1 = \phi_- + u\phi_+ + ru\Gamma_-\Theta_+\phi_+ \ , \quad \epsilon_2 = \phi_+ + r\Gamma_-\Theta_+\phi_+ \ , \quad \phi_+ = \Gamma_+\Theta_-\phi_-$$

These give rise to 3 Killing vector bi-linears given by

$$\begin{aligned} K_{12} &= -2u \parallel \phi_+ \parallel^2 \partial_u + 2r \parallel \phi_+ \parallel^2 \partial_r + W^i\tilde{\partial}_i \ , \\ K_{22} &= -2 \parallel \phi_+ \parallel^2 \partial_u \ , \\ K_{11} &= -2u^2 \parallel \phi_+ \parallel^2 \partial_u + (2 \parallel \phi_- \parallel^2 + 4ru \parallel \phi_+ \parallel^2)\partial_r + 2uW^i\tilde{\partial}_i \ , \end{aligned}$$

where W is a Killing vector on $\mathcal{S}$ which leaves all the data invariant.
They satisfy the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra

$$[K_{AB}, K_{A'B'}] = \parallel \phi_+ \parallel^2 (\epsilon_{AA'}K_{BB'} + \epsilon_{BB'}K_{AA'} + \epsilon_{BA'}K_{AB'} + \epsilon_{AB'}K_{BA'}) \ .$$

- If $W_i = \langle \Gamma_+\phi_-, \Gamma_i\phi_+ \rangle = 0$, the near horizon geometry is $AdS_2 \times_w \mathcal{S}$

Killing superalgebras

The Killing superalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{g}_0 = \mathbb{R}\langle V_{K_{\mathbf{mn}}}\rangle$ and $\mathfrak{g}_1 = \mathbb{R}\langle Q_{\epsilon_{\mathbf{m}}}\rangle$ of supersymmetric backgrounds are defined as follows [Gauntlett, Myers, Townsend; Figueroa-O'Farrill]:

$$\{Q_{\epsilon_{\mathbf{m}}}, Q_{\epsilon_{\mathbf{n}}}\} = V_{K_{\mathbf{mn}}} \ , \quad [V_{K_{\mathbf{mn}}}, Q_{\epsilon_{\mathbf{p}}}] = Q_{\mathcal{L}_{K_{\mathbf{mn}}}\epsilon_{\mathbf{p}}} \ , \quad [V_{K_{\mathbf{mn}}}, V_{K_{\mathbf{pq}}}] = V_{[K_{\mathbf{mn}}, K_{\mathbf{pq}}]} \ ,$$

where $K_{\mathbf{mn}} = \langle \Gamma_0 \epsilon_{\mathbf{m}}, \Gamma_M \epsilon_{\mathbf{n}} \rangle dx^M$ is the 1-form bilinear, $[K_{\mathbf{mn}}, K_{\mathbf{pq}}]$ is the Lie commutator of two vector fields and

$$\mathcal{L}_X \epsilon = \nabla_X \epsilon + \frac{1}{8} dX_{MN} \Gamma^{MN} \epsilon \ ,$$

is the spinorial Lie derivative of ϵ with respect to the vector field X.

- $V_{K_{\mathbf{mn}}} = V_{\mathbf{mn}}$ are the even generators and $Q_{\epsilon_{\mathbf{m}}} = Q_{\mathbf{m}}$ are the odd ones.

Killing superalgebras

The main question that arises is how to compute the Killing superalgebras of near horizon geometries without knowing explicitly the geometry of the backgrounds

- Achieved by imposing the super-Jacobi identities

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Horizon superalgebras

The KSAs of near horizon geometries are the following.

Horizon KSAs in type II and $d = 11$

N	$\mathfrak{g}/\mathfrak{c}$
$2n$	$\mathfrak{osp}(n\|2)$
$4n, n > 1$	$\mathfrak{sl}(n\|2)$
$8n, n > 1$	$\mathfrak{osp}^*(4\|2n)$
16	$\mathfrak{f}(4)$
14	$\mathfrak{g}(3)$
8	$\mathfrak{D}(2, 1, \alpha)$
8	$\mathfrak{sl}(2\|2)/1_{4\times 4}$

Table: The spatial horizon section is closed . There may be a central term $\mathfrak{c}$

N=2 horizons

There are two Killing spinors ϵ_A and 3 vector bilinear $K_{AA'}$. After an explicit computation

$$\begin{aligned}
\mathcal{L}_{K_{12}}\epsilon_1 &= -\|\phi_+\|^2\epsilon_1 + \tilde{\mathcal{L}}_W\epsilon_1 \,, \quad \mathcal{L}_{K_{12}}\epsilon_2 = \|\phi_+\|^2\epsilon_2 + \tilde{\mathcal{L}}_W\epsilon_2 \,, \\
\mathcal{L}_{K_{22}}\epsilon_1 &= -2\|\phi_+\|^2\epsilon_2 \,, \\
\mathcal{L}_{K_{11}}\epsilon_1 &= u(-2\|\phi_+\|^2\phi_- + 2\|\phi_-\|^2\Gamma_-\Theta_+\phi_+ - \not{W}\Gamma_-\phi_+) + 2u\tilde{\mathcal{L}}_W\epsilon_1 \,, \\
\mathcal{L}_{K_{11}}\epsilon_2 &= 2\|\phi_+\|^2u\epsilon_2 + 2\|\phi_-\|^2\Gamma_-\Theta_+\phi_+ - \not{W}\Gamma_-\phi_+ + 2u\tilde{\mathcal{L}}_W\epsilon_2 \,.
\end{aligned}$$

The super-Jacobi identity requires that $[\{Q_1, Q_1\}, Q_1] = \mathcal{L}_{K_{11}}\epsilon_1 = 0$. This implies that

$$\tilde{\mathcal{L}}_W\phi_+ = 0 \,, \;\; 2\tilde{\mathcal{L}}_W\phi_- - 2\|\phi_+\|^2\phi_- + 2\|\phi_-\|^2\Gamma_-\Theta_+\phi_+ - \not{W}\Gamma_-\phi_+ = 0$$

which eventually leads to $\tilde{\mathcal{L}}_W\epsilon_1 = \tilde{\mathcal{L}}_W\epsilon_2 = 0$ and finally to

$$\begin{aligned}
\mathcal{L}_{K_{12}}\epsilon_1 &= -\|\phi_+\|^2\epsilon_1 \,, \quad \mathcal{L}_{K_{12}}\epsilon_2 = \|\phi_+\|^2\epsilon_2 \,, \\
\mathcal{L}_{K_{22}}\epsilon_1 &= -2\|\phi_+\|^2\epsilon_2 \,, \quad \mathcal{L}_{K_{11}}\epsilon_2 = 2\|\phi_+\|^2\epsilon_1 \,.
\end{aligned}$$

The Killing superalgebra of $N=2$ near horizon geometries is

$$\{Q_A, Q_B\} = V_{AB}\ , \quad [V_{AB}, Q_C] = -\epsilon_{CA}Q_B - \epsilon_{CB}Q_A$$
$$[V_{AB}, V_{A'B'}] = \epsilon_{AA'}V_{BB'} + \epsilon_{BB'}V_{AA'} + \epsilon_{BA'}V_{AB'} + \epsilon_{AB'}V_{BA'}$$

where $\|\phi_+\|^2 = 1$.

Therefore

- $\mathfrak{g} = \mathbb{R}\langle V_{AB}, Q_C\rangle = \mathfrak{osp}(1|2)$

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Therefore

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$N = 2k$ horizon superalgebras

Take k-copies of the Killing spinors

$$\begin{aligned} \epsilon_1^s &= \phi_-^s + u\phi_+^s + ru\Gamma_-\Theta_+\phi_+^s \ , \\ \epsilon_2^s &= \phi_+^s + r\Gamma_-\Theta_+\phi_+^s \ , \end{aligned}$$

After some extensive work with super-Jacobi identities, the 1-form bilinears are

$$K_{AB}^{rs} = \langle \phi_+^r, \phi_+^s \rangle K_{AB} + \epsilon_{AB} Z^{rs} \ , \quad Z^{rs} = -Z^{sr} = (Z^{rs})^i \tilde{\partial}_i$$

with

$$[K_{AB}, Z^{rs}] = 0$$

and

$$\begin{aligned} \mathcal{L}_{K_{12}^{rs}} \epsilon_1^t &= -\langle \phi_+^r, \phi_+^s \rangle \epsilon_1^t + \tilde{\mathcal{L}}_{Z^{rs}} \epsilon_1^t \ , \quad \mathcal{L}_{K_{22}^{rs}} \epsilon_1^t = -2\langle \phi_+^r, \phi_+^s \rangle \epsilon_2^t \ , \\ \mathcal{L}_{K_{12}^{rs}} \epsilon_2^t &= \langle \phi_+^r, \phi_+^s \rangle \epsilon_2^t + \tilde{\mathcal{L}}_{Z^{rs}} \epsilon_2^t \ , \quad \mathcal{L}_{K_{11}^{rs}} \epsilon_2^t = 2\langle \phi_+^r, \phi_+^s \rangle \epsilon_1^t \end{aligned}$$

So it remains to determine $\tilde{\mathcal{L}}_{Z^{rs}} \epsilon_A^t$

As $\tilde{\mathcal{L}}_{Z^{rs}}$ preserves the type of Killing spinors closure requires

$$\tilde{\mathcal{L}}_{Z^{rs}} \epsilon^p_A = \alpha^{rsp}{}_q \epsilon^q_A \ , \quad p \neq r, s \ .$$

for some α.

The super-Jacobi identities for Q^r_A, Q^s_B and Q^p_A for $B \neq A$ and the spinorial Lie derivative identity

$$\tilde{\mathcal{L}}_{Z^{rs}} \langle \epsilon^p_2, \epsilon^q_2 \rangle = \langle \tilde{\mathcal{L}}_{Z^{rs}} \epsilon^p_2, \epsilon^q_2 \rangle + \langle \epsilon^p_2, \tilde{\mathcal{L}}_{Z^{rs}} \epsilon^q_2 \rangle = 0 \ ,$$

give

$$\alpha^{rspq} = \alpha^{[rspq]} \ .$$

So α is a 4-form.

Therefore the Killing superalgebra of near horizon geometries can be written as

$$\{Q^r_A, Q^s_B\} = \delta^{rs} V_{AB} + \epsilon_{AB} \tilde{V}^{rs} \ , \quad [V_{AB}, Q^r_C] = -\epsilon_{CA} Q^r_B - \epsilon_{CB} Q^r_A \ ,$$
$$[\tilde{V}^{rs}, Q^t_A] = -(\delta^{tr} Q^s_A - \delta^{ts} Q^r_A - \alpha^{rst}{}_\ell Q^\ell_A) \ ,$$

where V_{AB} generators of $\mathfrak{sl}(2,\mathbb{R})$, $A, B = 1, 2$; V^{rs} generators associated to Z^{rs}, $r, s = 1, \ldots, N/2$; and $\langle \phi^r_+, \phi^s_+ \rangle = \delta^{rs}$.
Define the representation D as

$$[V^{rs}, Q^t_A] = D(V^{rs})_p{}^t Q^p_A$$

- If there is a a such that $a_{rs} D(V^{rs}) = 0$, the generators $a_{rs} D(V^{rs})$ are central.
- Given $p_r Q^r_A$ and $q_r Q^r_A$, then $p_r q_s V^{rs}$ infinitesimally rotates $p_r Q^r_A$ to $q_r Q^r_A$. In particular, D rotates any vector in $\mathbb{R}^{\frac{N}{2}}$ to any other as it acts with $\mathfrak{so}(2)$ rotations on their plane, ie D acts transitively on the $S^{\frac{N}{2}-1}$ sphere in $\mathbb{R}^{\frac{N}{2}}$!

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Lie algebras of groups acting transitively on spheres [Montgomery, Samelson]

Algebra	Sphere	$N/2$
$\mathfrak{so}(k)$	S^{k-1}	k
$\mathfrak{u}(k)$	S^{2k-1}	$2k$
$\mathfrak{su}(k)$	S^{2k-1}	$2k$
$\mathfrak{sp}^*(k) \oplus \mathfrak{sp}^*(1)$	S^{4k-1}	$4k$
$\mathfrak{sp}^*(k) \oplus \mathfrak{u}(1)$	S^{4k-1}	$4k$
$\mathfrak{sp}^*(k)$	S^{4k-1}	$4k$
$\mathfrak{g}_2$	S^6	7
$\mathfrak{spin}(7)$	S^7	8
$\mathfrak{spin}(9)$	S^{15}	16

Table: $\mathfrak{spin}(9)$ cannot be realized as a symmetry of the internal space of near horizon geometries as there are no such maximally supersymmetric backgrounds.

- Some cases are ruled out, like $\mathfrak{su}(k)$, as they do not satisfy the super-Jacobi identities. The remaining cases give the superalgebras stated.

Isometry algebras of horizons

Table: Horizon Killing superalgebras in type II and 11d

N	$\mathfrak{g}/\mathfrak{c}$	$\mathfrak{t}_0/\mathfrak{c}$	$\dim \mathfrak{c}$
$2n$	$\mathfrak{osp}(n\|2)$	$\mathfrak{so}(n)$	0
$4n,\ n>2$	$\mathfrak{sl}(n\|2)$	$\mathfrak{u}(n)$	0
$8n, n>1$	$\mathfrak{osp}(4\|2n)$	$\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)$	0
16	$\mathfrak{f}(4)$	$\mathfrak{spin}(7)$	0
14	$\mathfrak{g}(3)$	$\mathfrak{g}_2$	0
8	$\mathfrak{D}(2,1,\alpha)$	$\mathfrak{so}(3)\oplus\mathfrak{so}(3)$	0
8	$\mathfrak{sl}(2\|2)/1_{4\times 4}$	$\mathfrak{su}(2)$	≤ 3

A theorem for $N > 16$ horizons

Th: There are no near horizon geometries in type II and $d = 11$ supergravities that preserve $N > 16$ supersymmetries.

The proof proceeds in several steps

- For near horizon geometries that preserve $N > 16$ supersymmetries, the spatial horizon sections $\mathcal{S}$ are G/H homogeneous spaces
- The Killing superalgebras specify the groups G that act transitively on $\mathcal{S}$
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Homogeneity

The Killing vectors on $\mathcal{S}$ constructed as bilinears can be rewritten as

$$W^{rs} = \langle \phi^r_-, \Gamma^i \Gamma_- \phi^s_+ \rangle \tilde{\partial}_i = \delta^{rs} W + Z^{rs} \ , \quad [W, Z^{rs}] = 0$$

and $\phi^s_+ = \Gamma_+ \Theta_- \phi^s_-$.

- To show that $\mathcal{S}$ is homogeneous it suffices to show that W^{rs} span the tangent space $T\mathcal{S}$ of $\mathcal{S}$ at every point

Define the subspace

$$\mathcal{W}_p = \mathbb{R}\langle W^{rs}|_p \rangle \subseteq T_p\mathcal{S}$$

If the W^{rs} do not span $T_p\mathcal{S}$, then there is a $v \in \mathcal{W}_p^\perp$, $v \neq 0$, such that

$$v^i W^{rs}_i|_p = v^i \langle \phi^r_-, \Gamma_i \Gamma_- \phi^s_+ \rangle|_p = \langle \phi^r_-, \not{v} \Gamma_- \phi^s_+ \rangle|_p = 0$$

Let $\mathfrak{S}$ be the spin bundle over $\mathcal{S}$ whose sections are ϕ_- spinors and define the span of Killing spinors

$$\mathcal{K}_p = \mathbb{R}\langle \phi^r_-|_p \rangle$$

Clearly $\mathcal{K}_p \subseteq \mathfrak{S}_p$.

For solutions preserve more than half of the supersymmetry, $\dim \mathcal{K}_p > \dim \mathcal{K}_p^{\perp}$.
On the other hand $\not{v}\Gamma_-\phi_+^s = \not{v}\Gamma_-\Gamma_+\Theta_-\phi_-^s = 2\not{v}\Theta_-\phi_-^s$, and

$$\not{v}\Theta_-|_p : \mathcal{K}_p \to \mathcal{K}_p^{\perp}$$

But for all near horizon geometries with non-trivial fluxes $\operatorname{Ker}\Theta_- = \{0\}$ and so $\operatorname{Ker}\not{v}\Theta_- = \{0\}$ for $v \neq 0$ as $\not{v}^2 = v^2\mathbf{1} \neq 0$.

Thus $\not{v}\Theta_-$ is an injection which is not possible as $\dim \mathcal{K}_p > \dim \mathcal{K}_p^{\perp}$. This leads to a contradiction unless $v = 0$ and so $\mathcal{W}_p$ spans all $T_p\mathcal{S}$.

- $\mathcal{S}$ must be a homogeneous space G/H with $\mathfrak{Lie}(G) = \mathfrak{t}_0 \oplus \mathfrak{so}(2)$.
- For $AdS_2 \times_w \mathcal{S}$ backgrounds $\check{W} = 0$ and so $\mathfrak{Lie}(G) = \mathfrak{t}_0$.

Conjecture: all supersymmetric backgrounds that preserve more than half of supersymmetry are homogeneous Meessen; proof for type II and $d = 11$ given by [Figueroa-O'Farrill, Hustler]

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Focus on the $AdS_2 \times_w \mathcal{S}$ backgrounds with $N > 16$.

The Killing superalgebra computation reveals that the following possibilities can arise for $\mathfrak{t}_0$

$$\begin{array}{l} \mathfrak{so}(N/2) \quad (N = 18, 20, 22, 24, 26, 28, 30) \\ \mathfrak{u}(N/4) \quad (N = 20, 24, 28) \\ \mathfrak{sp}(3) \oplus \mathfrak{sp}(1) \quad (N = 24) \end{array}$$

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On the other hand the classification of the homogeneous spaces gives

Table: 8-dimensional compact, simply connected, homogeneous spaces

	$M^8 = G/H$
(1)	$SU(3)$, group manifold
(2)	$\frac{Sp(3)}{Sp(2)\times Sp(1)} = \mathbb{HP}^2$, symmetric space
(3)	$\frac{U(5)}{U(4)\times U(1)} = \mathbb{CP}^4$, symmetric space, not spin
(4)	$\frac{Spin(9)}{Spin(8)} = S^8$, symmetric space
(5)	$\frac{Sp(2)}{T^2}$, $T^2 \subset Sp(2)$ maximal torus
(6)	$\frac{G_2}{SO(4)}$, symmetric space
(7)	$\frac{SU(4)}{S(U(2)\times U(2))} = G_2(\mathbb{C}^4) = \frac{SO(6)}{SO(4)\times SO(2)} = G_2(\mathbb{R}^6)$, Grassmannian, symmetric space
(8)	$\frac{SU(2)\times SU(2)\times SU(2)}{\Delta_{k,l,m}(U(1))}$
(9)	$S^2 \times S^6$
(10)	$S^2 \times \mathbb{CP}^3$
(11)	$S^2 \times \frac{SU(3)}{T^2}$
(12)	$S^2 \times G_2(\mathbb{R}^5)$, not spin
(13)	$S^3 \times S^5$
(14)	$S^3 \times \frac{SU(3)}{SO(3)}$, not spin
(15)	$S^4 \times S^4$
(16)	$S^4 \times \mathbb{CP}^2$, not spin
(17)	$\mathbb{CP}^2 \times \mathbb{CP}^2$, not spin
(18)	$S^2 \times S^2 \times S^4$
(19)	$S^2 \times S^3 \times S^3$
(20)	$S^2 \times S^2 \times S^2 \times S^2$
(21)	$S^2 \times S^2 \times \mathbb{CP}^2$, not spin

Conclusion

- The horizon conjecture implies that the emergence of $\mathfrak{sl}(2,\mathbb{R})$ symmetry near black hole horizons is a property of supergravity and independent of the model or the particular solution considered.
- The Killing superalgebras of near horizon geometries, and so those of warped AdS_2 backgrounds, have been classified.
- There are no near horizon geometries in type II and $d = 11$ supergravities that preserve $N > 16$ supersymmetries
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