IHP, March 28 2019

Soft gravitational radiation from high energy collisions: a progress report

Gabriele Veneziano

An unsolved textbook exercise

The problem of computing the GWs emitted by a binary system is (almost) as old as GR.

It has become increasingly relevant for testing GR w/ binary pulsars, for GW searches during decades, and, finally, for their detection @ LIGO/VIRGO out of the coalescence of BH-BH & NS-NS binaries.
Tools: Effective 1 Body (EOB), numerical relativity

Most of the time these processes are in the non-relativistic regime, with the exception of the merging itself when relativistic speeds ($v/c \sim 0.3$-0.6) can be reached.

Much less attention has been devoted in the past to a more academic (but simpler?) problem.

Consider the collision of two massless or highly relativistic ($\gamma = E/m \gg 1$) gravitationally interacting particles in the regime in which they deflect each other by a small angle $\theta_s = \theta_E$

$$\theta_s \equiv \theta_E = \frac{8GE}{b} \equiv \frac{2R}{b} \ ; \ c = 1$$

Problem: compute the GW spectrum associated with this collision to lowest order in θ_E.

How can it possibly be an unsolved problem?

(Andrei Gruzinov, private conversation, 2014)

What we do know

1. The zero frequency limit (ZFL)

We have a solid prediction (Smarr 1977) for

$dE^{GW}/d\omega d^2\Omega$ as $\omega \to 0$

Obtained either by a classical or by a quantum argument. Latter uses a well-known soft graviton limit (see Part II)

The result (2->2 after integrating over angles) is classical (c=1 throughout):

$$\frac{dE^{GW}}{d\omega} \to \frac{Gs}{\pi}\theta_s^2 \log(4e\theta_s^{-2}) \quad ; \quad \omega \to 0 \quad ; \quad \theta_s \ll 1$$

NB: typical magnitude of $dE/d\omega$: $Gs\theta_s^2$

2. Work in the seventies

P. D'Eath; D'Eath and Payne ~ 1978

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High-speed black-hole encounters and gravitational radiation

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(Received 15 March 1977)

Encounters between black holes are considered in the limit that the approach velocity tends to the speed of light. At high speeds, the incoming gravitational fields are concentrated in two plane-fronted shock regions, which become distorted and deflected as they pass through each other. The structure of the resulting curved shocks is analyzed in some detail, using perturbation methods. This leads to calculations of the gravitational radiation emitted near the forward and backward directions. These methods can be applied when the impact parameter is comparable to $Gc^{-2}M\gamma^2$, where M is a typical black-hole mass and γ is a typical Lorentz factor (measured in a center-of-mass frame) of an incoming black hole. Then the radiation carries power/solid angle of the characteristic strong-field magnitude c^5G^{-1} within two beams occupying a solid angle of order γ^{-2}. But the methods are still valid when the black holes undergo a collision or close encounter, where the impact parameter is comparable to $Gc^{-2}M\gamma$. In this case the radiation is apparently not beamed, and the calculations describe detailed structure in the radiation pattern close to the forward and backward directions. The analytic expressions for strong-field gravitational radiation indicate that a significant fraction of the collision energy can be radiated as gravitational waves.

S. Kovacs and K.Thorne 1977

THE GENERATION OF GRAVITATIONAL WAVES.
IV. BREMSSTRAHLUNG*†‡

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ABSTRACT

This paper attempts a definitive treatment of "classical gravitational bremsstrahlung"—i.e., of the gravitational waves produced when two stars of arbitrary relative mass fly past each other with arbitrary relative velocity v, but with large enough impact parameter that

$$(\text{angle of gravitational deflection of stars' orbits}) \ll (1 - v^2/c^2)^{1/2}.$$

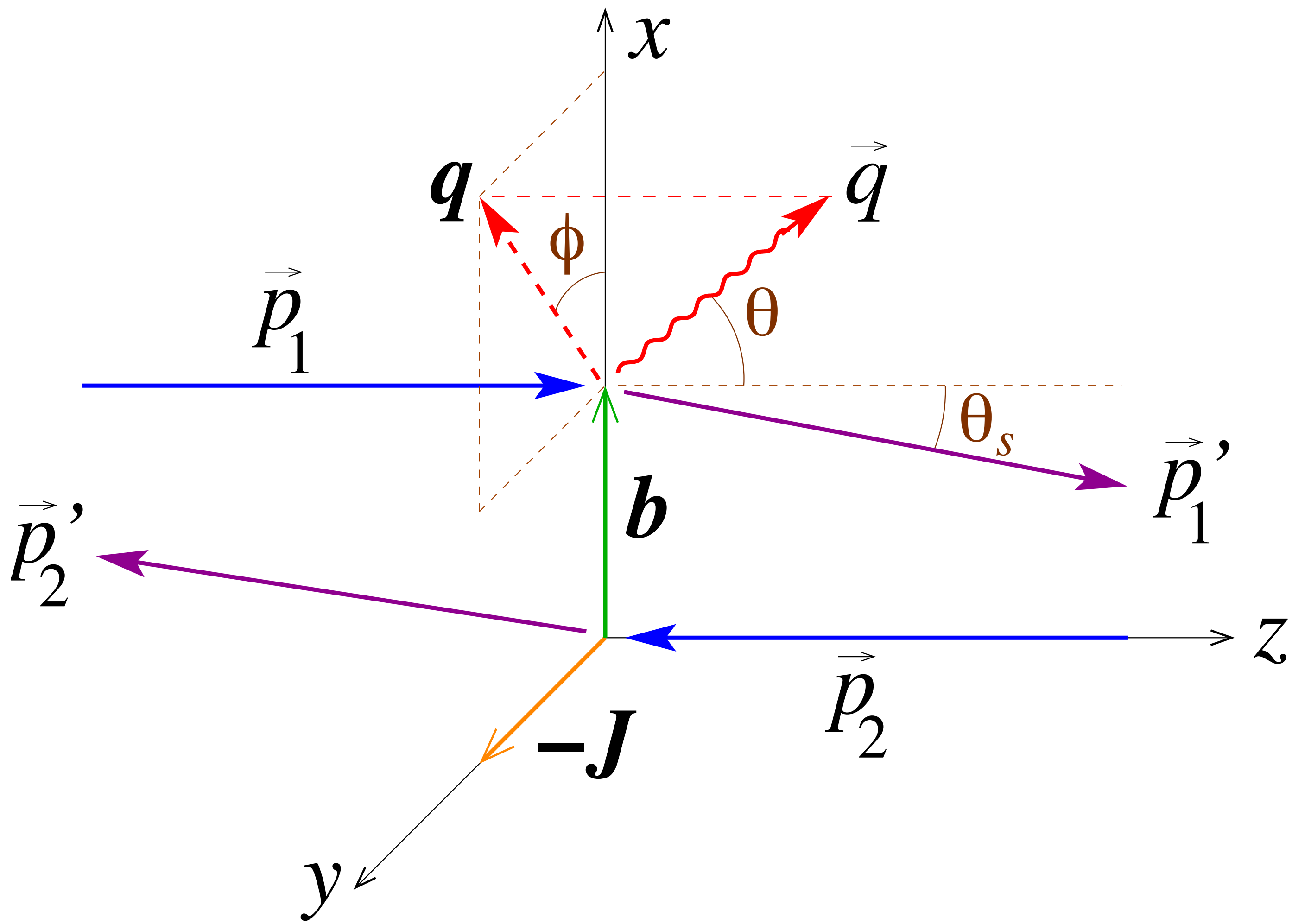
3. Numerical Relativity

(F. Pretorius, U. Sperhake, private comm. ~ 04.14)

The calculation in NR is also challenging because the deflected particles carry with them two shock waves that travel (almost) as fast as the emitted GWs & roughly in the same direction) Disentangling the two becomes very tricky for γ's >~ 3 and θ_E a bit > γ^{-1}

Hope for the future? See e.g. Pretorius & East (1807.11562) on BH & GW from axisymmetric collisions of null extended sources (beams).

The process at hand



Outline

- A classical GR approach
- A quantum approach, comparison w/ CGR
  - Rough properties of the IR spectrum & a "Hawking knee"
  - Finer properties of the deep-IR spectrum
  - An unexpected bump at $\omega b \sim 0.5$

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- A soft-theorem approach
  - Recovering the ZFL @ $O(1)$
  - The $O(\omega)$ correction
  - The $O(\omega^2)$ correction and a check
  - IR-divergences/logs .... and the bump again

Complementarity of two approaches

- The CGR approach and the quantum eikonal approach are limited to small-angle scattering but cover a wide range of GW frequencies.
- The soft-theorem approach is not limited to small deflection angles but is only valid in a much smaller frequency region.
- Consistency checks can be performed in the non-trivial overlap regime (small angle, low frequency)

Complementarity w/ other problems/calculations

- Grav.^{al} bremss. from a gravit^{al} collision occurs @ $O(G^3)$; same as a recent calculation of the 3PM conservative potential/deflection angle (Bern et al. 1901.04424, applied to EOB by Buonanno et. al. 1901.07102)
- An old paper (ACV90) computed (in m->0 limit) the 3PM correction to deflection angle using analyticity and unitarity inputs. Consistency with Bern et al. still unclear (logs m, b vs J…)
- A complete answer @ 3PM level within reach? Important for improving EOB (Damour, 1710.10599).

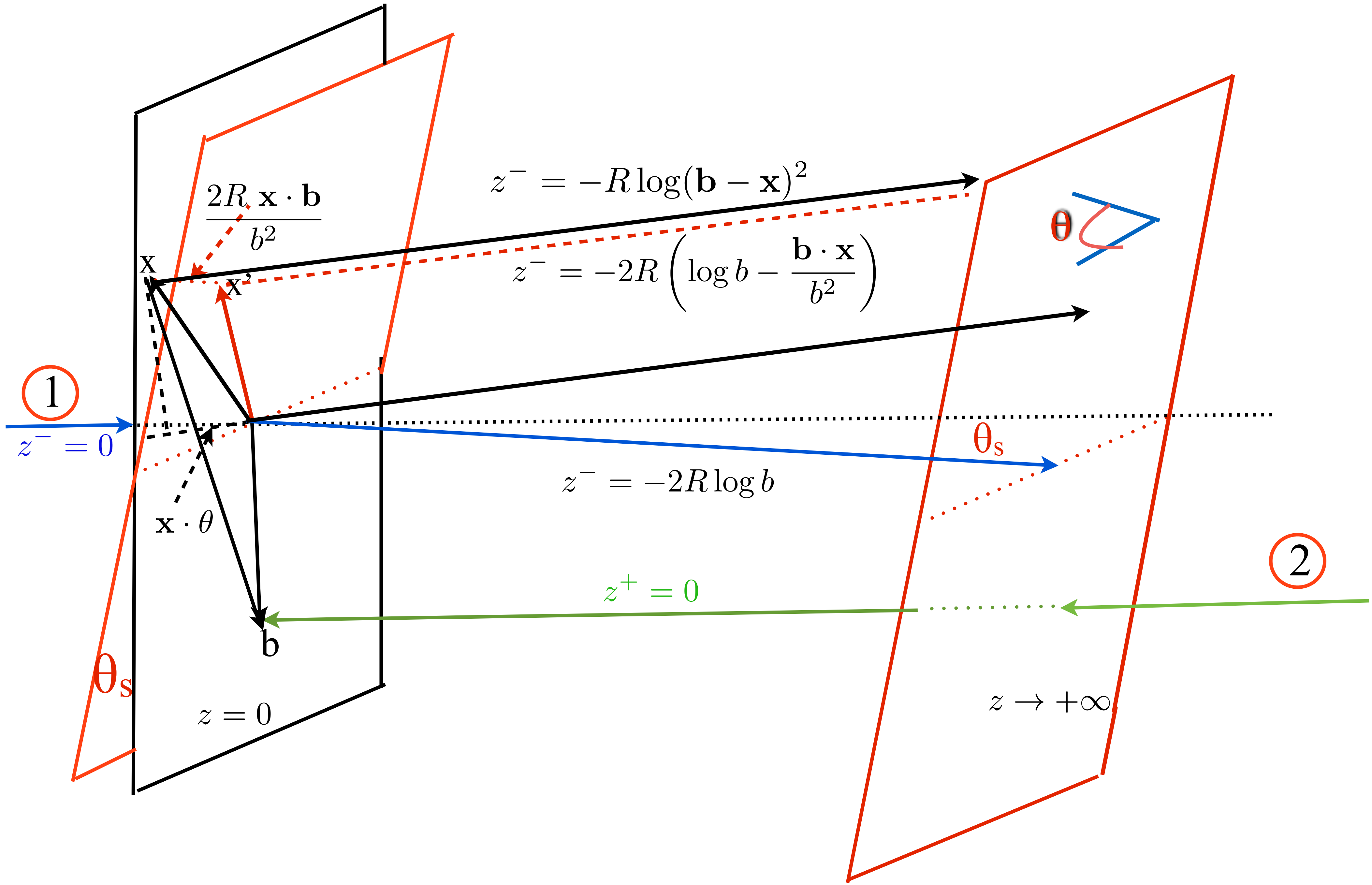
A Classical GR approach

(A. Gruzinov & GV, 1409.4555)

Based on Huygens superposition principle.

For gravity this includes in an essential way the gravitational time delay in an Aichelburg-Sexl shock-wave metric.

In pictures



$z^- = -R\log(\mathbf{b}-\mathbf{x})^2$
$\frac{2R\,\mathbf{x}\cdot\mathbf{b}}{b^2}$
x
x'
$z^- = -2R\left(\log b - \frac{\mathbf{b}\cdot\mathbf{x}}{b^2}\right)$
θ
1
$z^- = 0$
θ_s
$z^- = -2R\log b$
$\mathbf{x}\cdot\theta$
2
$z^+ = 0$
b
θ_s
$z = 0$
$z \to +\infty$

In formulae

Frequency + angular spectrum ($s = 4E^2$, $R = 4GE$)

$$\frac{dE^{GW}}{d\omega \; d^2\tilde{\theta}} = \frac{GE^2}{\pi^4}|c|^2 \; ; \; \tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \boldsymbol{\theta}_s \; ; \; \boldsymbol{\theta}_s = 2R\frac{\boldsymbol{b}}{b^2}$$

$$c(\omega, \tilde{\boldsymbol{\theta}}) = \int \frac{d^2x \; \zeta^2}{|\zeta|^4} \; e^{-i\omega \mathbf{x}\cdot\tilde{\boldsymbol{\theta}}} \left[e^{-2iR\omega\Phi(\mathbf{x})} - 1 \right]$$

$$\zeta = x + iy \qquad \Phi(\mathbf{x}) = \frac{1}{2}\ln\frac{(\mathbf{x}-\mathbf{b})^2}{b^2} + \frac{\mathbf{b}\cdot\mathbf{x}}{b^2}$$

$$c(\omega, \boldsymbol{\theta}) = \int \frac{d^2x \; \zeta^2}{|\zeta|^4} e^{-i\omega \mathbf{x}\cdot\boldsymbol{\theta}} \left[e^{-iR\omega \ln\frac{(\mathbf{x}-\mathbf{b})^2}{b^2}} - e^{+2iR\omega\frac{\mathbf{b}\cdot\mathbf{x}}{b^2}} \right]$$

Re ζ^2 and Im ζ^2 correspond to the usual (+,x) GW polarizations, ζ^2, ζ^{*2} to the two circular ones.

Subtracting the deflected shock wave (cf. P. D'Eath) is crucial!

Postponing the discussion of this spectrum let me jump to a sketch of:

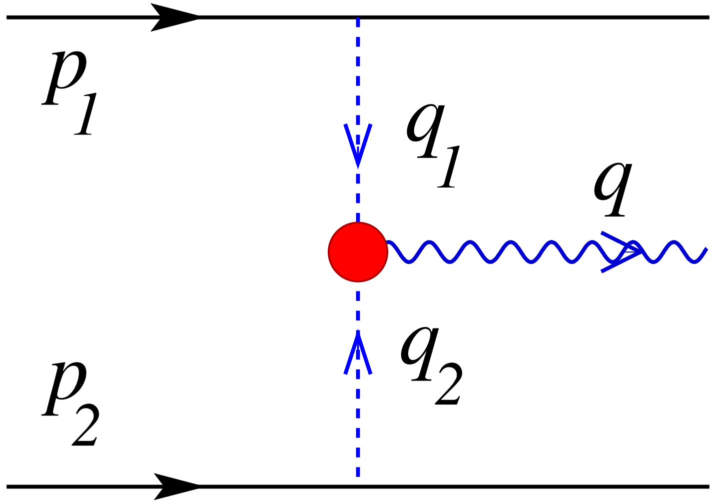
A quantum treatment in eikonal approach

(Ciafaloni, Colferai & GV, 1505.06619, **CC&Coradeschi & GV, 1512.00281**, Ciafaloni & Colferai, 1612.06923, 1709.08405, **Ciafaloni, Colferai & GV, 1812.08137**)

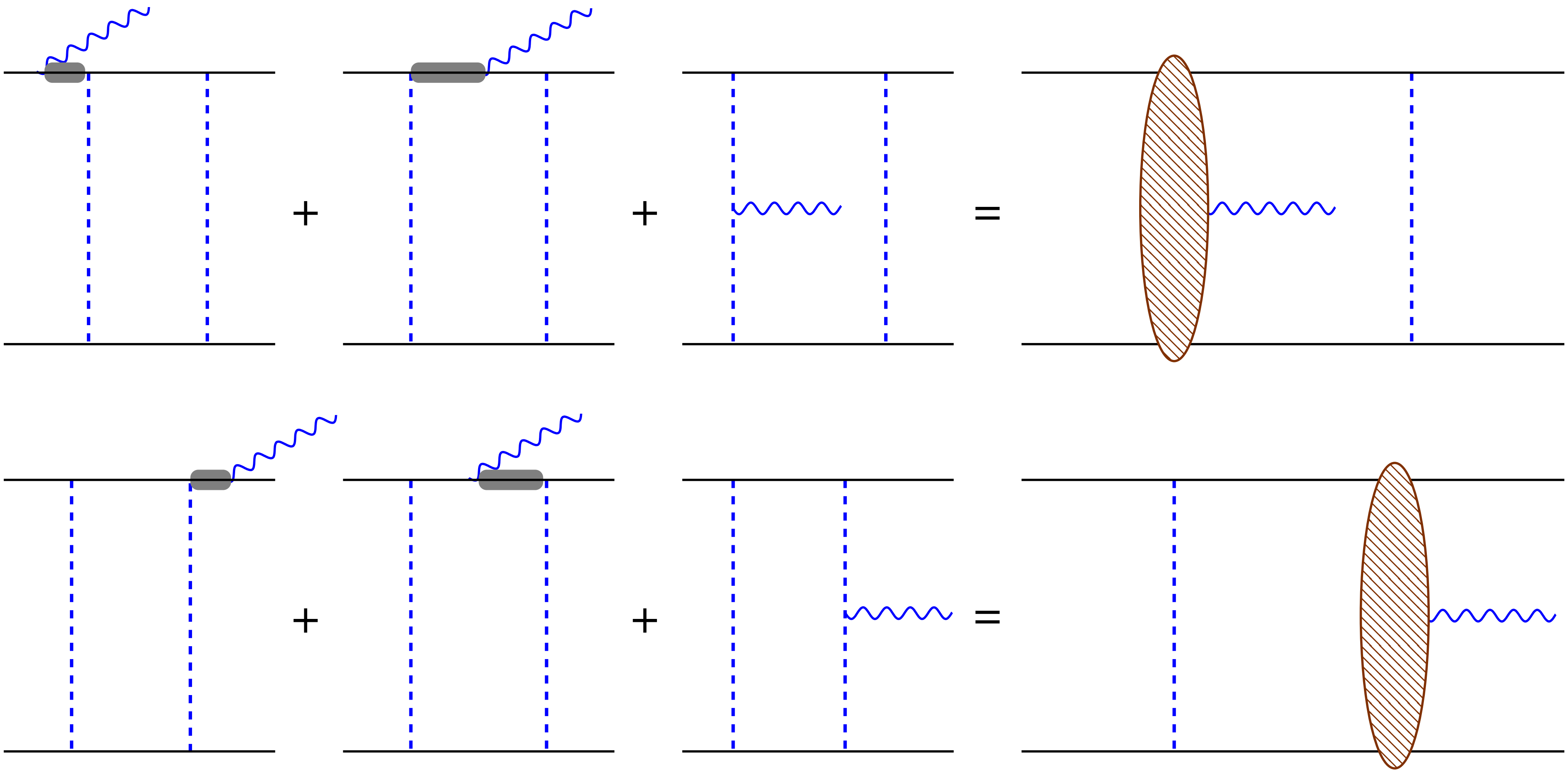
In CC(C)V (1505.06619 & 1512.00281) the same problem has been addressed at the quantum level improving on an earlier (ACV07) treatment.

1. The usual soft-graviton recipe (emission from external legs) has to be amended since the internal exchanged gravitons are almost on shell. Emission from such internal lines is important for not-so-soft gravitons (hence for the GW energy).

NB. Emission from internal legs also in low-E thrms



2. Emission from external and internal legs throughout the whole ladder has to be taken into account.



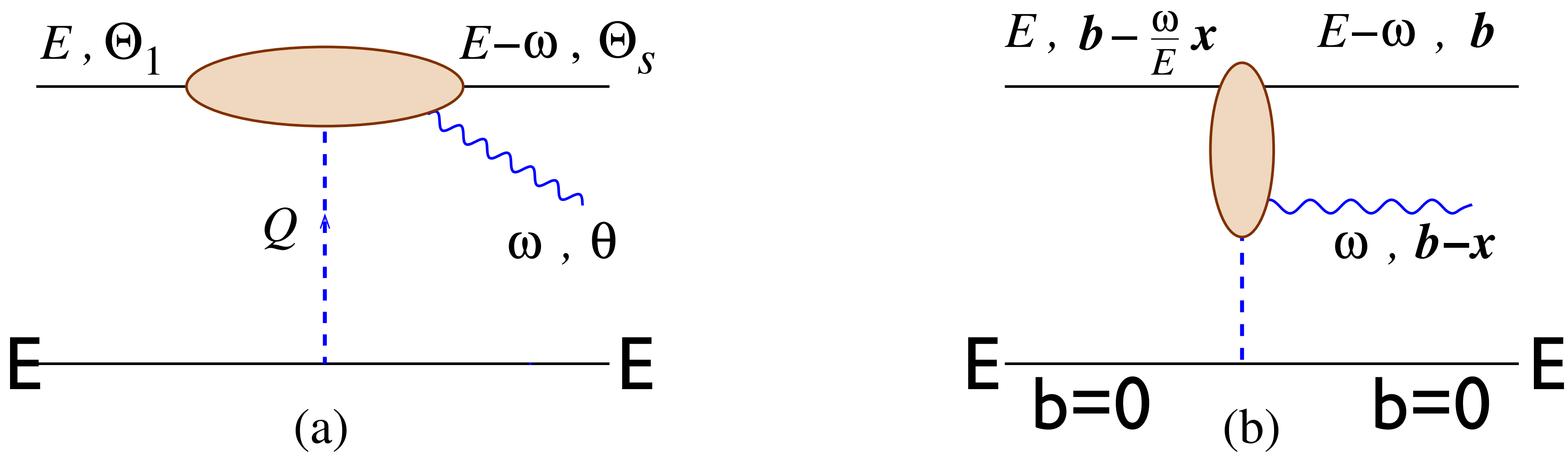
3. One should finally take into account the (finite) difference between the (infinite) Coulomb phase of the final 3-particle state and that of an elastic 2-particle state. When this is done, the classical result of G+V is exactly recovered for hω/E -> 0! Indeed, using E/J conservation (see figure):

$$E\ \Delta t(\mathbf{b}) \to (E-\hbar\omega)\Delta t(\mathbf{b}+\frac{\hbar\omega\mathbf{x}}{E-\hbar\omega}) + \hbar\omega\Delta t(\mathbf{b}-\mathbf{x})$$

$$\sim E\ \Delta t(\mathbf{b}) + \hbar\omega\left(\Delta t(\mathbf{b}-\mathbf{x}) + \mathbf{x}\cdot\frac{\partial\Delta t(\mathbf{b})}{\partial\mathbf{b}} - \Delta t(\mathbf{b})\right)$$

$$\Phi(\mathbf{x}) = \frac{1}{2}\ln\frac{(\mathbf{x}-\mathbf{b})^2}{b^2} + \frac{\mathbf{b}\cdot\mathbf{x}}{b^2}$$

E & J conservation



Rough analytic properties & emergence of Hawking knee

For $b^{-1} < \omega < R^{-1}$ the GW-spectrum is almost flat in ω

$$\frac{dE^{GW}}{d\omega} \sim \frac{4G}{\pi}\theta_s^2 E^2 \log(\omega R)^{-2}$$

Below $\omega = b^{-1}$ it "freezes" reproducing the ZFL

$$\frac{dE^{GW}}{d\omega} \to \frac{4G}{\pi}\theta_s^2 E^2 \log(\theta_s^{-2})$$

Above $\omega = R^{-1}$ drops, becomes "scale-invariant"

Hawking!

$$\frac{dE^{GW}}{d\omega} \sim \theta_s^2 \frac{E}{\omega}$$

This gives a $\log \omega^*$ in the "efficiency" for a cutoff at ω^*

A guess about ω^*

At $\omega \sim R^{-1}\,\theta_s^{-2}$ the above spectrum becomes $O(Gs\,\theta_s^4)$ i.e. of the same order as terms we neglected.
Also, if continued above $R^{-1}\,\theta_s^{-2}$, a so-called "Dyson bound" ($dE/dt < 1/G$) would be violated. Using $\omega^* \sim R^{-1}\,\theta_s^{-2}$ we find (to leading-log accuracy) the suggestive result:

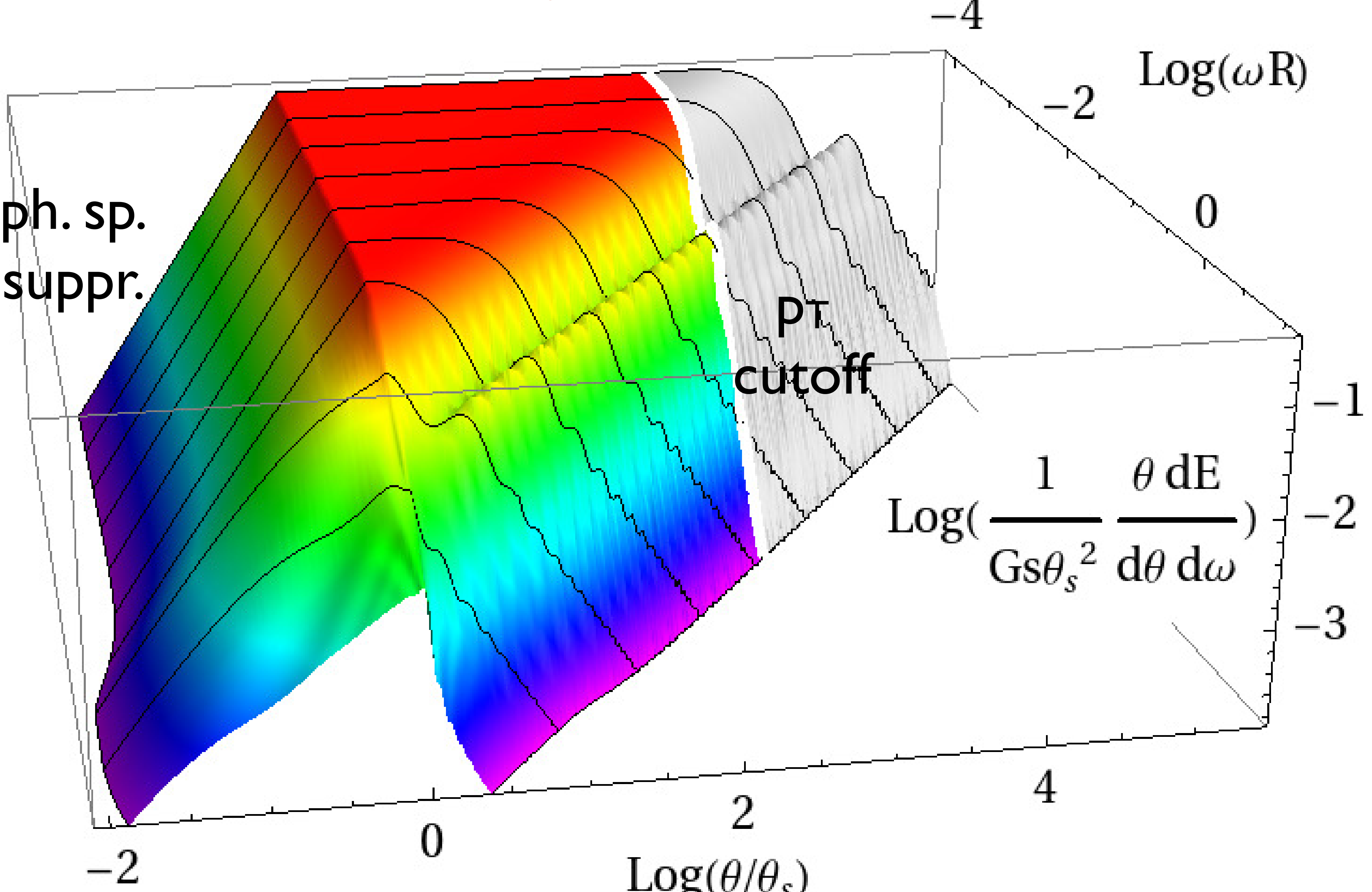
$$\frac{E^{GW}}{\sqrt{s}} = \frac{1}{2\pi}\,\theta_s^2\,\log(\theta_s^{-2})$$

For $\omega > \omega^*$ G+V argued for a $G^{-1}\omega^{-2}$ spectrum which (extrapolating to $\theta_s \sim 1$) turns out to be that of a time-integrated BH evaporation!

Rough numerical results on the GW spectra

M. Ciafaloni, D. Colferai, F. Coradeschi & GV, 1512.00281

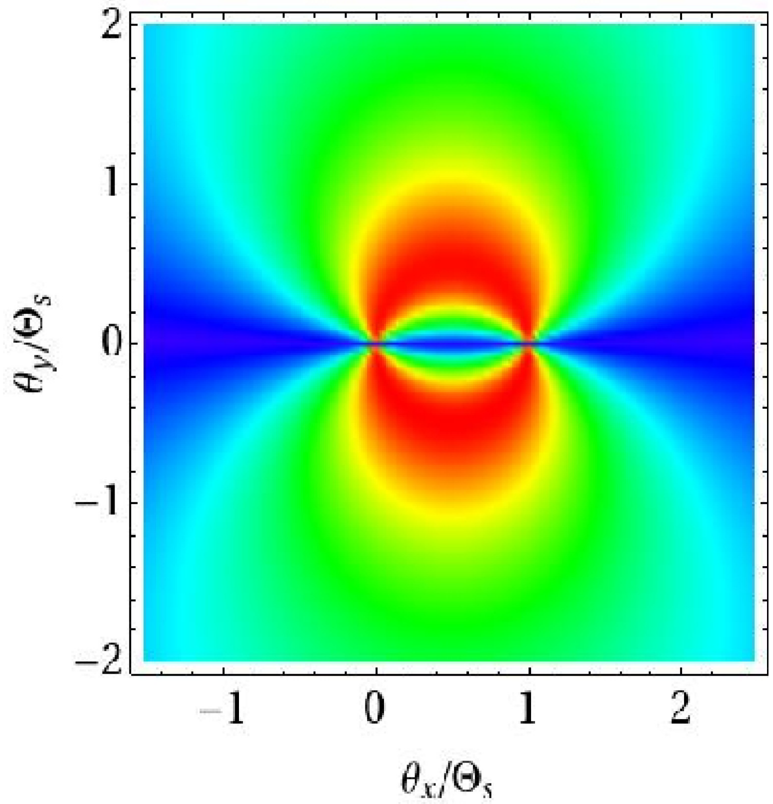
$\theta_s = 10^{-3}$



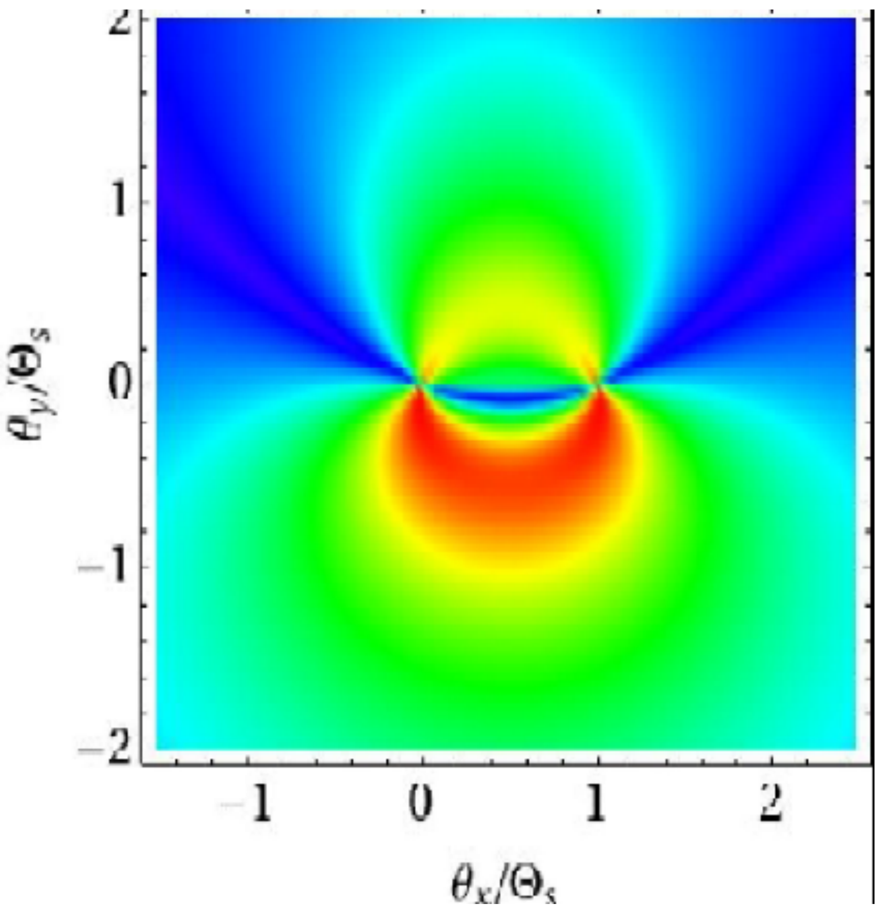
M. Ciafaloni, D. Colferai & GV, 1505.06619

Angular (polar and azimuthal) distribution

$\omega R = 10^{-3}$



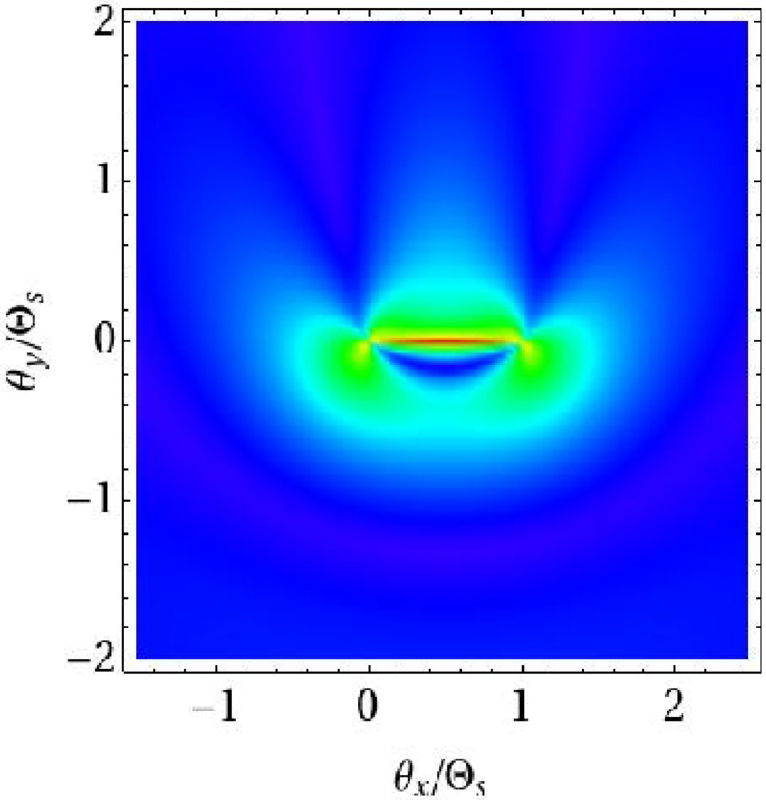
$\omega R = 0.125$



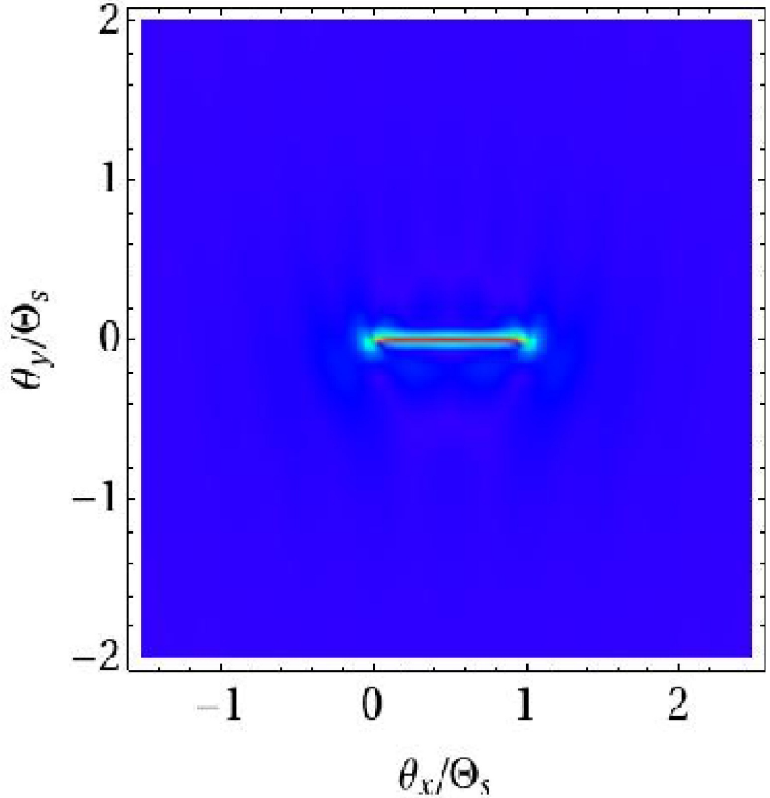
M. Ciafaloni, D. Colferai, F. Coraldeschi & GV, 1512.00281

Angular (polar and azimuthal) distribution

$\omega R = 1.0$



$\omega R = 8.0$



Selected for PRD's picture gallery…

Back to analytic spectra

$$\frac{dE^{GW}}{d\omega} \to \frac{4G}{\pi} \theta_s^2 E^2 \log(\theta_s^{-2}) \qquad (\omega b \ll 1)$$

$$\frac{dE^{GW}}{d\omega} \sim \frac{4G}{\pi}\theta_s^2 E^2 \log(\omega R)^{-2} \quad (\omega b \gg 1 \gg \omega R)$$

suggest naive (monotonic) interpolation around ωb~ 1, e.g.

$$\frac{dE^{GW}}{d\omega} \sim \frac{4G}{\pi}\theta_s^2 E^2 \log\left(\frac{b^2}{R^2(1+\omega^2 b^2)}\right) \sim \frac{4G}{\pi}\theta_s^2 E^2 \left[\log\left(\frac{b^2}{R^2}\right) - O(\omega^2 b^2)\right]$$

This turns out not to be the case...

Finer features of the deep-infrared spectrum

(Ciafaloni, Colferai & GV-1812.08137)

A careful study of the region $\omega R < 1$, but with ωb generic, shows that:

- At $\omega b < (\ll) 1$ there are corrections of order $\omega b \log(\omega b)$, $(\omega b)^2 \log^2(\omega b)$ (higher logs θ_s-suppressed).

- First noticed by Sen et al. in the context of soft thrms in D=4. Here they come from the mismatch between the two- and three-body Coulomb phase.

- These logarithmically enhanced sub and sub-sub leading corrections disappear at $\omega b > 1$ so that the previously found $\log(1/\omega R)$ behavior (for $\omega b > 1 > \omega R$), as well as the Hawking knee, remain valid.

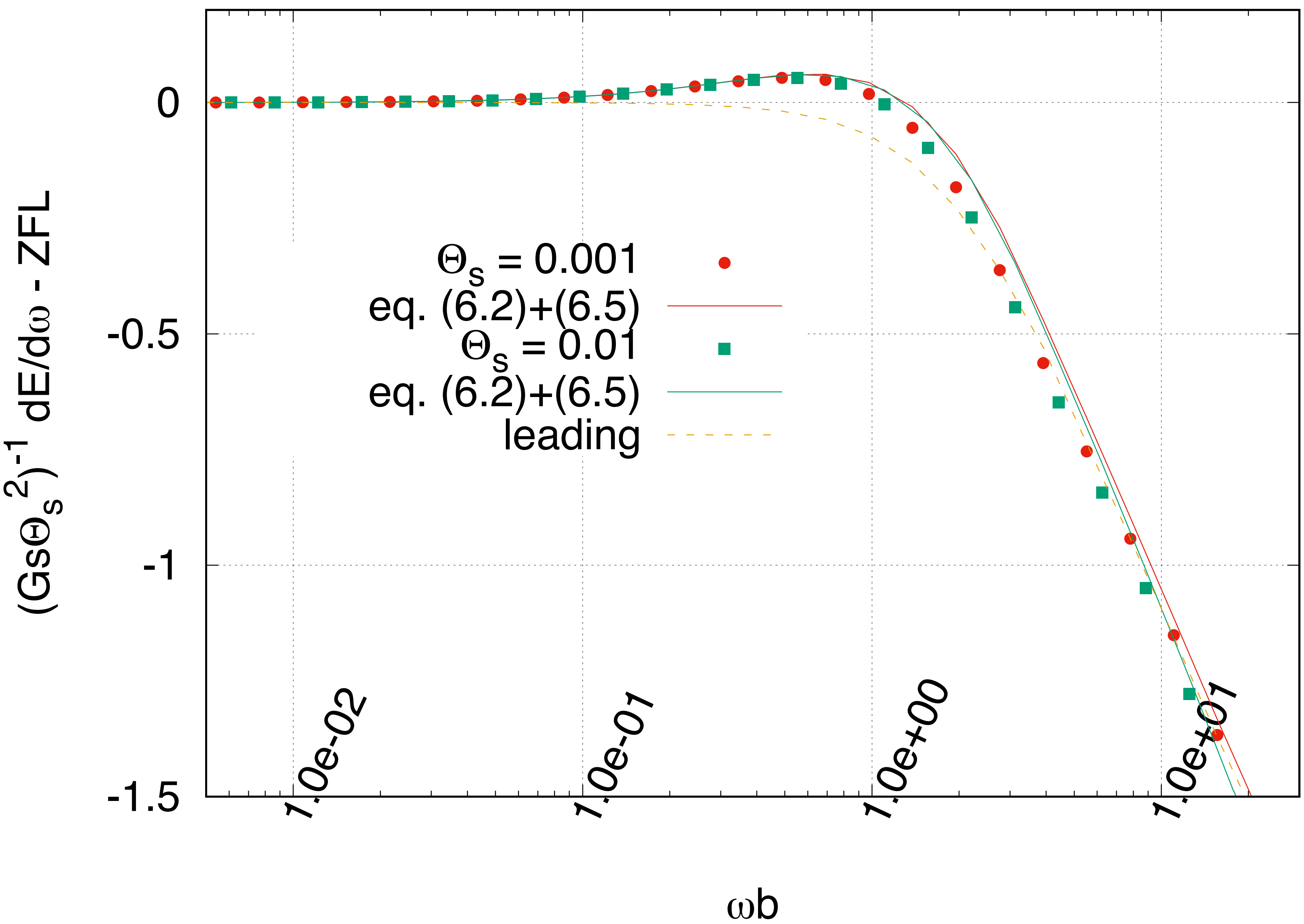
- The $\omega b \log(\omega b)$ correction only appears for circularly polarized (definite helicity) GWs but disappear either for the (more standard) + and x polarizations, or after summing over them, or finally after integration over the azimuthal angle.

- They are in complete agreement with what had been previously found by A. Sen and collaborators using soft-graviton theorems to sub-leading order (see part II). These authors claim that such logarithmically enhanced corrections lead to observable effects on the gravitational wave-forms.

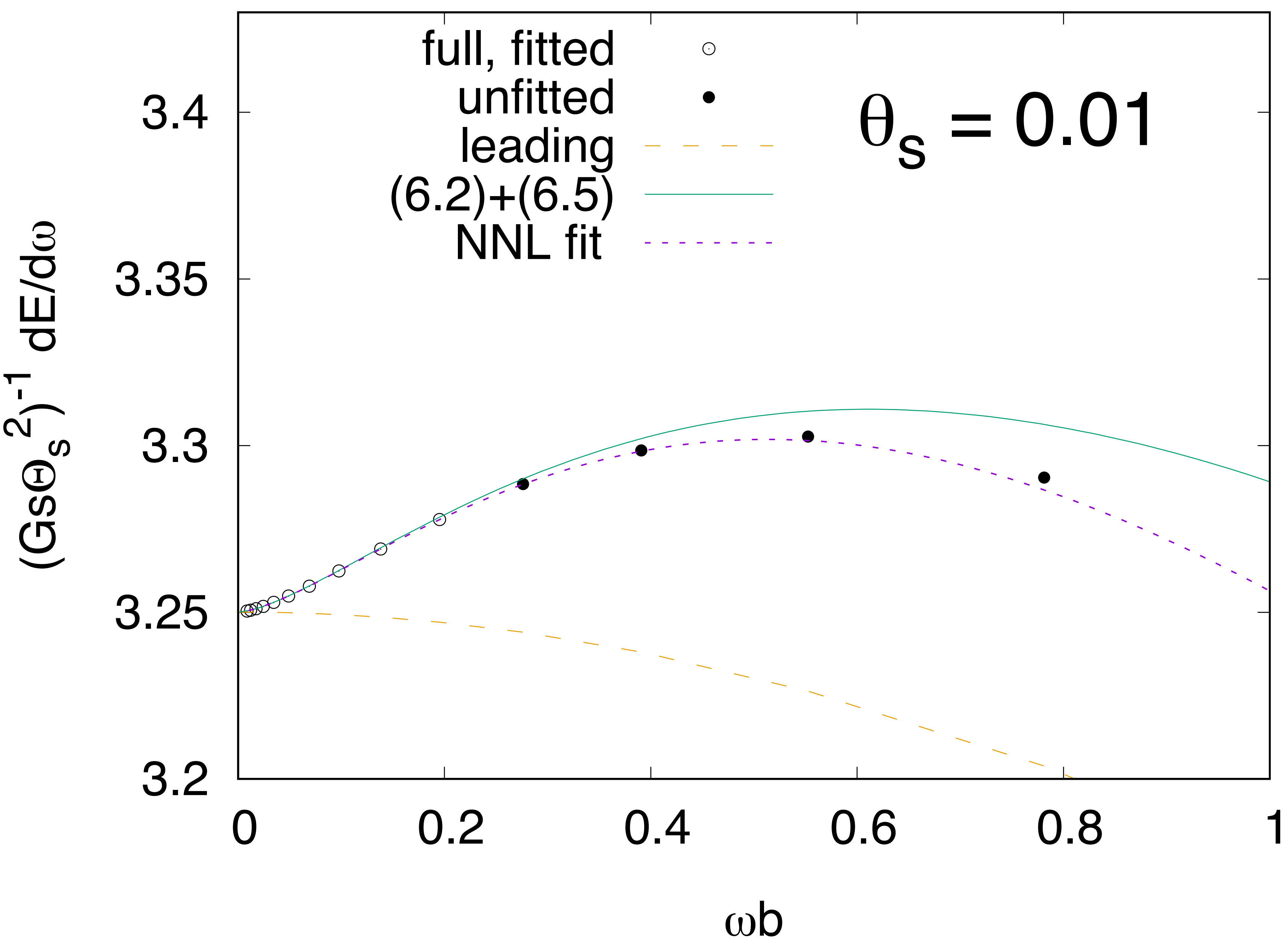
- The leading $(\omega b)^2\log^2(\omega b)$ correction to the total flux is positive and produces a bump at $\omega b \sim 0.5$.
- Could not be compared to Sen et al. who only considered $\omega b \log(\omega b)$ corrections.
- Now confirmed by Sahoo(private comm. by AS) but…
- Can be compared successfully with ABV-19 if Sen et al. recipe is adopted to $O(\omega^2)$ (see part II).

New numerical results

(Ciafaloni, Colferai & GV-1812.08137)



(GsΘs²)⁻¹ dE/dω - ZFL
0
-0.5
-1
-1.5
1.0e-02
1.0e-01
1.0e+00
1.0e+01
ωb
Θs = 0.001
eq. (6.2)+(6.5)
Θs = 0.01
eq. (6.2)+(6.5)
leading



full, fitted
unfitted
leading
(6.2)+(6.5)
NNL fit
θs = 0.01
(GsΘs2)-1 dE/dω
ωb
3.2
3.25
3.3
3.35
3.4
0
0.2
0.4
0.6
0.8
1

Part II

Beyond the ZFL via soft theorems

(Laddha & Sen, 1804.09193;
Sahoo & Sen, 1808.03288,
Addazi, Bianchi & GV, 1901.10986)

Low-energy (soft) theorems for photons and gravitons (Low, Weinberg, ... sixties) had a revival recently (Strominger, Cachazo, Bern, Di Vecchia, Bianchi...). In the case of a soft graviton of momentum q we have (for spinless hard particles)

$$\mathcal{M}_{N+1}(p_i;q) \approx \sum_{i=1}^{N} \left[\frac{p_i h p_i}{q p_i} + \frac{p_i h J_i q}{q p_i} - \frac{q J_i h J_i q}{2 q p_i} \right] \mathcal{M}_N(p_i) \equiv S_0 + S_1 + S_2$$

$$J_i^{\mu\nu} = p_i^{\mu} \partial / \partial p^i_{\nu} - p_i^{\nu} \partial / \partial p^i_{\mu}$$

NB: sub and sub-sub leading terms may need corrections at loop level & from IR sing.s @ D=4.

Recovering the ZFL (m=0 case)

Keeping just the leading term in the x-section:

$$\int \frac{d^3q}{2|q|(2\pi)^3} \sum_{s=\pm 2} \left| \sum_{i=1}^{N} \frac{p_i h_s p_i}{q p_i} \right|^2 |\mathcal{M}_N(p_i)|^2$$

sum over polarizations gives the integrand

$$\sum_{i,j} \frac{p_i^\mu p_i^\nu}{q p_i} \Pi_{\mu\nu,\rho\sigma} \frac{p_j^\rho p_j^\sigma}{q p_j} = \sum_{i,j} \frac{(p_i p_j)^2}{q p_i q p_j} .$$

$$B_0 = \frac{8\pi G}{\hbar} \int \frac{d^3q}{2|q|(2\pi)^3} \sum_{i,j} \frac{(p_i p_j)^2}{q p_i q p_j} = -\frac{2G}{\pi\hbar} \log\frac{\Lambda}{\lambda} \sum_{i,j} (p_i p_j) \log\frac{|p_i p_j|}{\mu^2}$$

$$\frac{dE_0^{GW}}{d\omega} = \hbar\omega \frac{dN_0}{d\omega} = -\frac{2G}{\pi} \sum_{i,j} (p_i p_j) \log\frac{|p_i p_j|}{\mu^2}$$

$$B_0 = \frac{8\pi G}{\hbar} \int \frac{d^3q}{2|q|(2\pi)^3} \sum_{i,j} \frac{(p_i p_j)^2}{qp_i qp_j} = -\frac{2G}{\pi\hbar} \log\frac{\Lambda}{\lambda} \sum_{i,j} (p_i p_j) \log\frac{|p_i p_j|}{\mu^2}$$

$$\frac{dE_0^{GW}}{d\omega} = \hbar\omega \frac{dN_0}{d\omega} = -\frac{2G}{\pi} \sum_{i,j} (p_i p_j) \log\frac{|p_i p_j|}{\mu^2}$$

Result does not depend on μ and is free of mass (collinear) divergences. For 2->2 scattering:

$$\frac{dE^{GW}}{d\omega}(\omega = 0) = \frac{4G}{\pi}\left(s \log s + t \log(-t) + u \log(-u)\right)$$

At small deflection angle (|t| << s):

$$\frac{dE^{GW}}{d\omega} \to \frac{Gs}{\pi} \theta_E^2 \log(4e\theta_E^{-2}) \quad ; \quad \omega \to 0$$

Remark

When we go beyond the ZFL, i.e. to $O(\omega)$, $O(\omega^2)$, we have to be careful about sub-leading terms "hidden" behind the leading one. For instance:

$$\sum_{i,j} \frac{p_i^\mu p_i^\nu}{qp_i} \Pi_{\mu\nu,\rho\sigma} \frac{p_j^\rho p_j^\sigma}{qp_j} = \sum_{i,j} \frac{(p_i p_j)^2}{qp_i qp_j} .$$

gets an additional constant -4 on its r.h.s.
Different expressions can have the same ZFL and yet differ at sub-leading level.
Turns out to be important for a check to be discussed below.

Next-to-Leading (details)

$$B_1 = 8\pi G \int \frac{d^3q}{2|q|(2\pi)^3} \sum_{i,j} \sum_{s=\pm 2} \left[\frac{(p_i h^s p_i)(p_j h^{(-s)} J_j q)}{q p_i q p_j} + (i \leftrightarrow j) \right]$$

summing over polarizations

$$B_1 = 8\pi G \int \frac{d^3q}{2|q|(2\pi)^3} \sum_{i,j} \frac{p_i p_j}{q p_i q p_j} [p_i \overrightarrow{J}_j + p_j \overleftarrow{J}_i] q$$

For given i,j the relevant integral

$$I^\mu_{ij} = \int \frac{d^3q}{2|q|(2\pi)^3} \frac{p_i p_j q^\mu}{q p_i q p_j} = \int \frac{d^4q}{(2\pi)^3} \delta_+(q^2) \frac{p_i p_j q^\mu}{q p_i q p_j} \; ; \; \delta_+(q^2) = \delta(q^2)\Theta(-q_0)$$

has collinear divergences. These are nicely avoided through a little trick (additional terms vanish after sum)

$$I^\mu_{ij} \to \tilde{I}^\mu_{ij} = \int \frac{d^4q}{(2\pi)^3} \delta_+(q^2) \frac{[(p_i p_j) q^\mu - (q p_j) p_i^\mu - (q p_i) p_j^\mu]}{(p_i q)(p_j q)}$$

We also add a $\delta(qP + 2E\omega_0)$ (w/ P the c.o.m. momentum) to fix the c.o.m. $\omega = \omega_0$ in a covariant way. Quantity in sq. brackets orthogonal to p_i, p_j. Then we get

$$\frac{dB_1}{d\omega} = -2\frac{G\sqrt{s}}{\pi}\sum_{ij}\frac{\log\left[\frac{-s(p_ip_j)}{2(Pp_i)(Pp_j)}\right]}{(p_ip_j)\tilde{s}_{ij}}\left[(p_ip_j)P^\mu - (Pp_j)p_i^\mu - (Pp_i)p_j^\mu\right]\left[p_i\overrightarrow{J}_j + p_j\overleftarrow{J}_i\right]_\mu$$

$$\tilde{s}_{ij} = -\Pi^2 = s + \frac{2(Pp_i)(Pp_j)}{p_ip_j}$$

(note absence of singularities when latter vanishes)

It can be simplified further to give:

$$\frac{dE_1}{d\omega} = -2\frac{G\sqrt{s}\hbar\omega}{\pi}\sum_{ij}\frac{\log\left[\frac{-s(p_ip_j)}{2(Pp_i)(Pp_j)}\right]}{\tilde{s}_{ij}}\left[(p_ip_j)P - (Pp_j)p_i - (Pp_i)p_j\right]^\mu\left(\frac{\overleftarrow{\partial}}{\partial p_i} + \frac{\overrightarrow{\partial}}{\partial p_j}\right)_\mu$$

To be sandwiched (divided) between (by) $S_{if}^+ S_{fi}$

Vanishing of O(ω) correction for 2->2

$$\frac{dE_1}{d\omega} = -2\frac{G\sqrt{s}\hbar\omega}{\pi}\sum_{ij}\frac{\log\left[\frac{-s(p_ip_j)}{2(Pp_i)(Pp_j)}\right]}{\tilde{s}_{ij}}\left[(p_ip_j)P-(Pp_j)p_i-(Pp_i)p_j\right]^\mu\left(\frac{\overleftarrow{\partial}}{\partial p_i}+\frac{\overrightarrow{\partial}}{\partial p_j}\right)_\mu$$

Terms with i = j do not contribute. Terms with (i, j = 1, 2 and 3,4) vanish because projector = 0. For (i,j = 1,3) the derivatives only contribute when acting on (p_1p_3): this produces a p_1 or p_3 which get killed by the contraction. In this last step a careful definition of the partial derivatives is needed…see below

The result (recall that we summed over pol.^{s}!) agrees with those obtained in the eikonal approach and also with Sen et al. for the log-enhanced term.

Side remark (if time allows)
How do we define the partial derivatives?

Q: In a process $1+2 \to 3+4 + gr(q)$ which 4-point function remains after gr-emission from an external leg?
A: If the emission is from 1 or 2 the 4-point-f. is evaluated at $s_{34} = (p_3+p_4)^2$ while for emission from 3 or 4 it is evaluated at $s_{12} = (p_1+p_2)^2 = (p_3+p_4+q)^2$
In explicit examples (one to be discussed later) this leads to a simple recipe for the derivatives

replace

$$
\begin{aligned}
s &\rightarrow -\Delta_s^2 \;;\; \Delta_s = \frac{1}{2}(p_1 + p_2 - p_3 - p_4) \\
t &\rightarrow -\Delta_t^2 \;;\; \Delta_t = \frac{1}{2}(p_1 + p_4 - p_2 - p_3) \\
u &\rightarrow -\Delta_u^2 \;;\; \Delta_u = \frac{1}{2}(p_1 + p_3 - p_2 - p_4)
\end{aligned}
$$

Then apply the derivatives as if the four momenta were independent. This gives

$$
\begin{aligned}
\partial_1^\mu &= -\Delta_s^\mu \partial_s - \Delta_t^\mu \partial_t - \Delta_u^\mu \partial_u \\
\partial_2^\mu &= -\Delta_s^\mu \partial_s + \Delta_t^\mu \partial_t + \Delta_u^\mu \partial_u \\
\partial_3^\mu &= +\Delta_s^\mu \partial_s + \Delta_t^\mu \partial_t - \Delta_u^\mu \partial_u \\
\partial_4^\mu &= +\Delta_s^\mu \partial_s - \Delta_t^\mu \partial_t + \Delta_u^\mu \partial_u
\end{aligned}
$$

These rules satisfy some desired properties such as

$$\sum_{i=1}^{4} p_i \partial_i = 2(s\partial_s + t\partial_t + u\partial_u)$$

$$\sum_{i=1}^{4} J_i^{\mu\nu} = -\Delta_s^{\mu}\Delta_s^{\nu}\partial_s - \Delta_t^{\mu}\Delta_t^{\nu}\partial_t - \Delta_u^{\mu}\Delta_u^{\nu}\partial_u - (\mu \leftrightarrow \nu) = 0$$

The sub-sub leading correction

The calculation is much more involved, but the final result takes a simple, elegant form

$$B_2|\mathcal{S}_{if}|^2 = \mathcal{S}^\dagger_{if}\frac{G\omega^2}{\pi}(C_1+C_2+C_3)\,\mathcal{S}_{fi}$$

$$C_1 = -3\sum_i \overleftarrow{D}_i \sum_j \overrightarrow{D}_j + 4\sum_i(\overleftarrow{D}_i+\overrightarrow{D}_i)^2$$

$$C_2 = \sum_{i\neq j}\frac{P^2}{\tilde{s}_{ij}}\log\frac{P^2 p_i p_j}{2Pp_iPp_j}[p_ip_j(\overleftrightarrow{\partial_{ij}})^2 - 2p_i(\overleftrightarrow{\partial_{ij}})p_j(\overleftrightarrow{\partial_{ij}})]$$

$$C_3 = \sum_{i\neq j}\frac{2}{p_ip_j\tilde{s}_{ij}}\left[1+\frac{P^2}{\tilde{s}_{ij}}\log\frac{P^2p_ip_j}{2Pp_iPp_j}\right](p_ip_j)^2\left(Q^\mu_{ij}(\overleftrightarrow{\partial_{ij}})_\mu\right)^2$$

$$D_i = p_i\partial_i \quad (\text{no sum})$$

$$\overleftrightarrow{\partial}_{ij\,\nu} \equiv \overleftarrow{\partial}_{i\nu} + \overrightarrow{\partial}_{j\nu} \qquad Q^\mu_{ij} \equiv \left(P^\mu - \frac{Pp_j}{p_ip_j}p^\mu_i - \frac{Pp_i}{p_ip_j}p^\mu_j\right)$$

Specializing to a 2->2 process

Using again the same recipe for the partial derivatives:

$$B_2|\mathcal{S}_{if}|^2 = \frac{dE_2^{GW}}{d(\hbar\omega)}|\mathcal{S}_{if}|^2 = 2\frac{G\hbar\omega^2}{\pi} \times$$
$$\mathcal{S}_{if}^\dagger \left\{ \overleftarrow{D}^2 + \overrightarrow{D}^2 + [st + us\log\left(-\frac{u}{s}\right)]\overleftrightarrow{\Delta}_{st}^2 + [su + ts\log\left(-\frac{t}{s}\right)]\overleftrightarrow{\Delta}_{su}^2 \right\} \mathcal{S}_{fi}$$

$$D \equiv s\partial_s + t\partial_t + u\partial_u \ : \ \overleftrightarrow{\Delta}_{st} \equiv \left(\overleftarrow{\partial}_s - \overleftarrow{\partial}_t - \overrightarrow{\partial}_s + \overrightarrow{\partial}_t\right) \ldots$$

The above combinations of derivatives are unambiguous. They act on either $A(s,t)$ or on $A'(s,u)$ or on $A''(t,u)$ yielding the same result for the same physical amplitude.

Example I

A tree-level 2->2 amplitude, e.g. single graviton exchange in a+b->a+b (w/ a ≠ b)

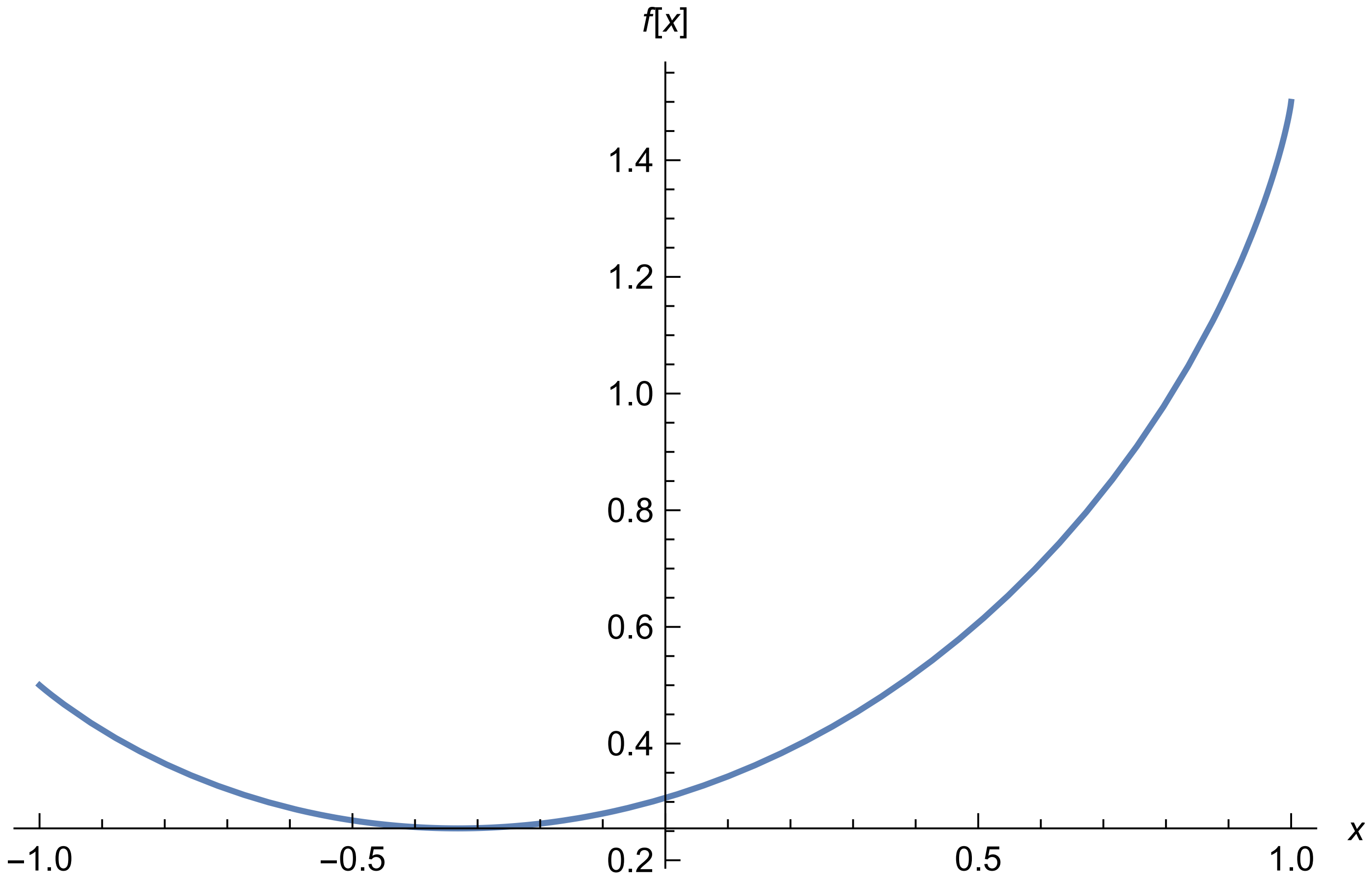
$$A(s,t) = -\frac{su}{t} = \frac{s^2}{t} + s = \frac{u^2}{t} + u$$

we find after a little algebra:

$$\frac{dE_2^{GW}}{d\omega} = \frac{4G(\hbar\omega)^2}{\pi} f(x) \quad ; \quad x = cos\theta_s$$

Corrections to ZFL look quantum and $O(h^2\ \omega^2/Q^2)$
But if we use $Q = h/b$ they become $O(\ \omega^2 b^2)$ (i.e. classical?)

$$f(x) = 1 - \frac{2}{1-x} + \frac{2}{1+x} - \frac{1+x}{1-x}\left(\frac{2}{1-x} - \frac{1-x}{2}\right)\log\left(\frac{1+x}{2}\right)$$
$$+\frac{1-x}{1+x}\left(\frac{2}{1+x} + \frac{1+x}{2}\right)\log\left(\frac{1-x}{2}\right).$$



$$f(x) = 1 - \frac{2}{1-x} + \frac{2}{1+x} - \frac{1+x}{1-x}\left(\frac{2}{1-x} - \frac{1-x}{2}\right)\log\left(\frac{1+x}{2}\right)$$
$$+\frac{1-x}{1+x}\left(\frac{2}{1+x} + \frac{1+x}{2}\right)\log\left(\frac{1-x}{2}\right).$$

This result has been checked via a long, explicit calculation in N=8 SUGRA. The same f(x) came out (!) …except for the 1 being replaced by a -1!
The origin of the discrepancy took a while to be understood: it is due to that -4 I have mentioned earlier.

Example II : Resummed eikonal a la ACV.

Because of phase O(action/h) derivatives act, to leading order, on the exponent (Cf. WKB). The powers of h cancel and we get a classical contribution. Unfortunately, the infinite Coulomb phase does NOT drop out.

The reason is quite clear: the derivative operators in J_i feel the change of the Coulomb phase due to the change of the hard momenta. Such a change is itself IR divergent. However, also the final soft graviton contributes an IR div. Coulomb phase which is exactly as needed for the cancellation (Cf. CCV18).

The standard soft-graviton recipe misses it and should be amended.

If we follow Sen et al's recipe for dealing with the Coulomb IR logs we can match the result with the one obtained in CCV-18 (for the unpolarized, angle-integrated flux).

We get, like CCV18, a positive correction of order $(\omega b)^2 \log^2(\omega b)$ (but, unlike in CCV18, with a precise coefficient in front) confirming the already mentioned bump in the spectrum around $\omega b = 0.5$.

Summarizing

- GW's from ultra-relativistic collisions is an interesting (though probably academic) theoretical problem.

- It is challenging both analytically and numerically, both classically and quantum mechanically.

- The ZFL (for $dE^{GW}/d\omega$) is classical & well understood. In order to go beyond the ZFL two approaches have been followed (besides the CGR one of G+V):

- The first follows the eikonal ACV approach, is limited (so far) to small deflection angles, but extends to frequencies somewhat beyond $1/R \gg 1/b$
- It is free from IR infinities which, however, bring about logarithmic enhancements at $\omega < 1/b$ and are responsible for a peak in the flux around $\omega b = 0.5$.

- The second goes via the soft-graviton theorems. It is not limited to small-angle scattering but is restricted to the $\omega b < 1$ regime.
- Because of IR divergences in 4D, the non-leading soft terms are ill defined and need modifications.

- At sub-(and now sub-sub?)-leading level a recipe due to Sen and collaborators looks to be confirmed by the eikonal-approach-based results.
- At sub-sub-leading level that same recipe confirms the CCV-18 prediction of a bump in the flux @ $\omega b \sim 0.5$
- Eventually, one would like to extend these results to arbitrary masses and kinematics and to combine them with recent ones on the conservative gravitational potential at 3PM level, leading hopefully to a full understanding of gravitational scattering and radiation at that level.
- With such a motivation in mind I'm pleased to announce:

Workshop on

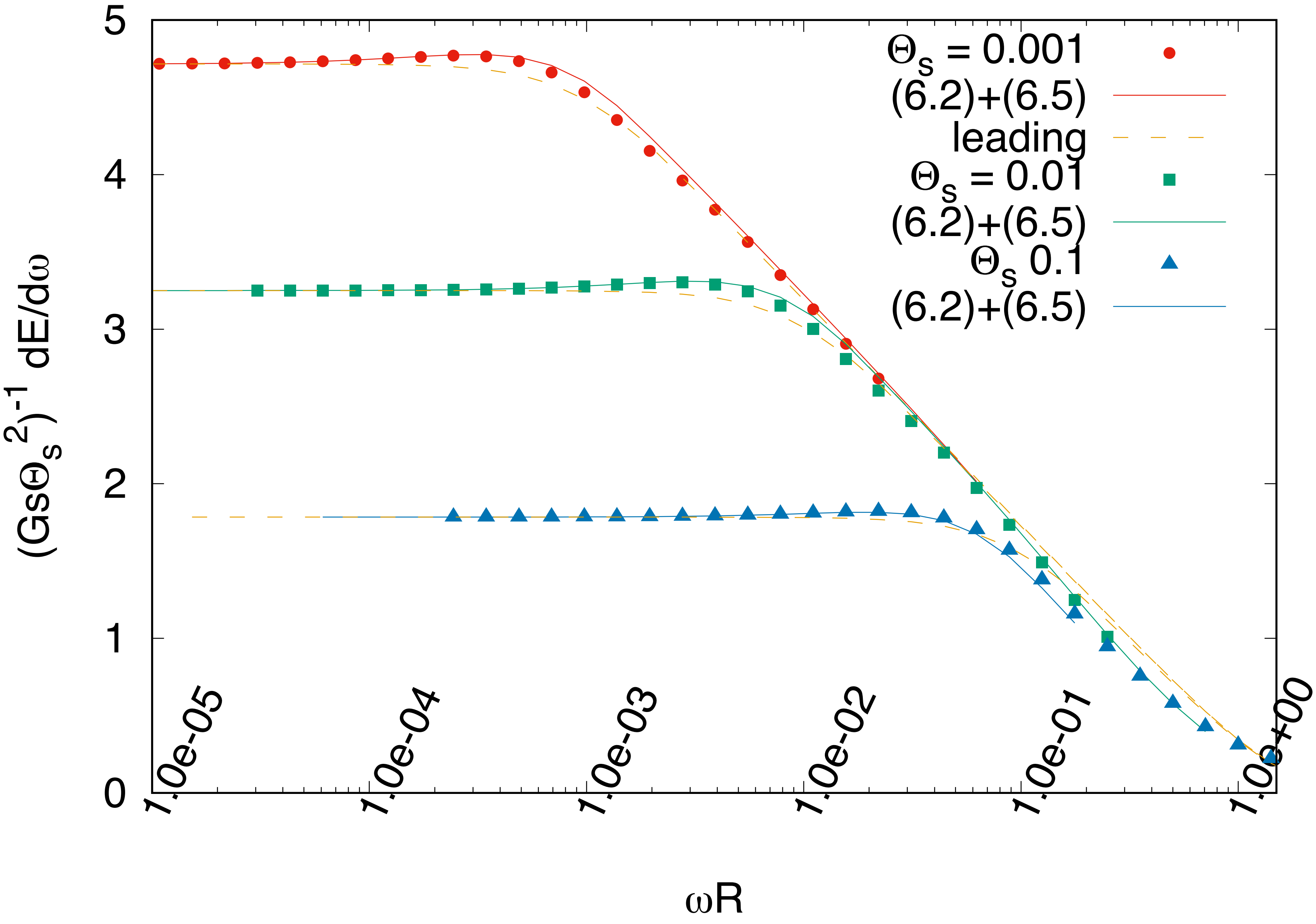
Gravitational scattering, inspiral, and radiation

(GGI, May 18-July 5, 2020)

Thank you!

Abstract

I will review recent developments on soft gravitational radiation from ultra-relativistic collisions. Calculations based on recent developments in the eikonal approach and in soft-graviton theorems will be compared. We find excellent agreement (in their common region of applicability) and are led to predict an unexpected bump in the spectrum of the gravitational energy flux at wavelengths comparable to the impact parameter of the collision.



Θ_s = 0.001
(6.2)+(6.5)
leading
Θ_s = 0.01
(6.2)+(6.5)
Θ_s 0.1
(6.2)+(6.5)
$(Gs\Theta_s^2)^{-1}$ dE/dω
ωR
0
1
2
3
4
5
1.0e-05
1.0e-04
1.0e-03
1.0e-02
1.0e-01
1.0e+00

Previous results (CCCV 1512.00281)