Exact Summation of Chiral Logs in $2D$: Quasi-renormalizable QFTs, Dixon's Elliptic Functions, and all that

K.M. Semenov-Tian-Shansky

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Petersburg
Nuclear
Physics
Institute

НИЦ КИ

Outline

1. Introduction.
2. Recurrence relations for leading logarithms (LLs) from unitarity, analyticity and crossing symmetry in $D = 4$.
3. Pseudofactorials and a dream of quasi-renormalizable QFTs.
4. Bi-quartic theory in $D = 2$.
5. Solutions of the recurrence relations for LLs for bi-quartic theory in $D = 2$ and examples of quasi-renormalizable QFTs.
6. Summary and Outlook.

Based on
M. Polyakov, A. Smirnov, K.S. and A. Vladimirov, arXiv:1811.08449 [hep-th], accepted for publication in Theor. Math. Phys.
J. Linzen, M. Polyakov, K.S. and N. Sokolova, JHEP **1904** (2019) 007, [arXiv:1811.12289 [hep-ph]].

LLogs in χPT

$$\mathcal{L}_{\chi\mathrm{PT}} = \mathcal{L}_2 + \mathcal{L}_4 + \dots .$$

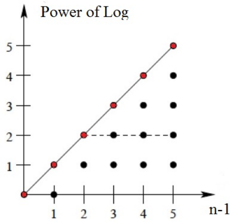
- $2 \to 2$ scattering amplitude:

$$A(s,t) = \underbrace{c_1 \frac{E^2}{\Lambda^2}}_{\text{only } \mathcal{L}_2 \text{ parameters}} + \frac{E^4}{\Lambda^4}\Bigg(\underbrace{c_2 \log\left(\frac{\mu^2}{E^2}\right)}_{\text{1-loop only } \mathcal{L}_2 \text{ parameters}} + \underbrace{c_3}_{\substack{\text{1-loop } \mathcal{L}_2 \text{ parameters} \\ \text{tree-level } \mathcal{L}_4 \text{ parameters}}} \Bigg) + O\left(\frac{E^6}{\Lambda^6}\right)$$

Leading Logs

$n \equiv$ [number of loops $+1$]

$$A(s,t) = \sum_{n=1}^{\infty} \omega_n \left(\frac{E^2}{\Lambda^2}\right)^n \log^{n-1}\left(\frac{\mu^2}{E^2}\right) + \mathrm{NLL}$$



Do we really have to care about LLogs in EFTs?

Renormalizable QFT

$$A(s) = \alpha + \alpha^2(a_1 \log|s| + b_1) + \alpha^3(a_2 \log^2|s| + b_2 \log|s| + c) + \ldots; \quad \alpha \sim \frac{\alpha_0}{\log|s|}$$

- LLog approximation gives leading asymptotic behavior.

Case of χPT

$$A(s) = \frac{s}{F^2} + \frac{s^2}{F^4}(a_1 \log|s| + b_1) + \frac{s^3}{F^6}(a_2 \log^2|s| + b_2 \log|s| + c) + \ldots;$$
$$s \ll F^2; \quad s \log|s| \sim s.$$

- In χPT LLog can not compete with power-like corrections. No reason to be particularly interested in their resummation... But!

Motivation: chiral expansion of PDFs and GPDs

- Additional dimensional parameter can make it necessary to sum up chiral logs.
- Non-local quark-antiquark operator on the light-cone ($n^2 = 0$):

$$O(\lambda) = \bar{q}\left(\frac{1}{2}\lambda n\right)\gamma_+ q\left(-\frac{1}{2}\lambda n\right)$$

- The dependence of GPDs on the soft momenta and/or pion mass can be controlled by χPT N. Kivel and M. Polyakov'02.
- Pion PDF:

$$q(x) = q^{\rm reg}(x) + \sum_{\substack{n\geq 1\\ \rm odd}} D_n \left[a_\chi \log\left(\frac{1}{a_\chi}\right)\right]^n \delta^{(n-1)}(x); \qquad a_\chi = (m_\pi/4\pi F_\pi)^2$$

Reorganization required for $a_\chi \sim x$.
- GPDs and CFFs ($\xi = \frac{x_{\rm Bj}}{2-x_{\rm Bj}}$):

$$\mathcal{A}(\xi,t) = \int_{-1}^{1} dx \frac{H(x,\xi,t)}{\xi - x} = \mathcal{A}^{\rm reg}(\xi,t) + \sum_{k=1}^{\infty} \mathcal{A}_k \frac{1}{\xi^k}\left[b_\chi \log(1/b_\chi)\right]^k;$$

$$b_\chi = |t|/4\,(4\pi F_\pi)^2$$

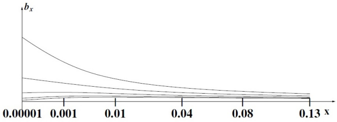
Transverse size of pion

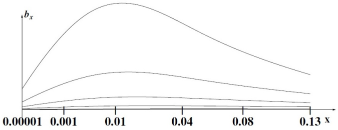
- Impact parameter distribution of quarks:

$$q(x, b^2) = \int \frac{d^2\Delta}{(2\pi)^2} q(x, \Delta^2) e^{i(b\cdot\Delta)}.$$

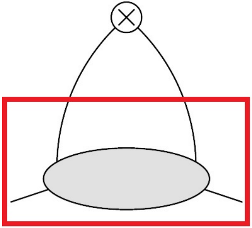
- Chiral inflation of the pion radius I. Perevalova, M. Polyakov, A. Vall and A. Vladimirov'11:
  - $x \ll \frac{m_\pi}{F}$ the radius grows as $\frac{1}{x^\alpha}$;
  - $x \sim \frac{m_\pi}{F}$ the radius grows as $\frac{1}{x}$.

- Gribov diffusion Ansatz: $$q(x, \Delta^2) = q(x) e^{-\alpha' \Delta^2 \log \frac{1}{x}}$$ v.s. the resummation of chiral logs:





- Graphs contributing to the leading singular terms:



- Key issue: LL structure 2 $\to$ 2 scattering amplitude in massless EχT.

Some references:

- D. Kazakov'88 and M. Buchler, G. Colangelo'04: generalization of the RG-group methods for EFTs.
- M. Bissegger, A. Fuhrer'07: 5-loop LLs in massless O(4)/O(3) σ-model;
- N. Kivel, M. Polyakov, A. Vladimirov'08, J. Koschinski , M. Polyakov, A. Vladimirov'10: arbitrary-loop LLs in massless ϕ^4-type EFTs.

LLs in massless EFT in $D = 4$: preliminaries

J. Koschinski , M. Polyakov, A. Vladimirov'10:

- Action (ϕ^4-type theory; all fields are massless):

$$S = \int d^4x \left[\frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - V(\phi, \partial\phi) \right].$$

- The lowest chiral order of interaction is 2κ: *e.g.* $\phi^2 \partial^{2\kappa} \phi^2$
- The Lagrangian is invariant under some particular global group G (isospin).

- *E.g.* $O(N+1)/O(N)$ σ-model belongs to this class: $G \to O(N)$, $\kappa = 1$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \phi^a \partial^\mu \phi^a) = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{8F^2}(\phi^a \phi^a)\partial^2(\phi^b \phi^b) + \mathcal{O}(\phi^6),$$

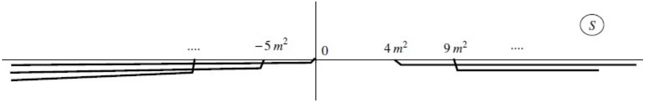
Main object of study: $2 \to 2$ scattering amplitude

- Consider the PW expansion of $2 \to 2$ scattering amplitude:

$$\langle \phi^c \phi^d | S - I | \phi^a \phi^b \rangle = 2\pi i (4\pi)^4 \delta(\ldots) \sum_I P_I^{abcd} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell \left(1 + \frac{2t}{s}\right) t_\ell^I(s),$$

- P_I^{abcd} are projectors on invariant isospin subspaces; $P_\ell(\ldots)$ are the Legendre polynomials.
- LLog structure of PW amplitudes:

$$t_\ell^I(s) = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\hat{S}^n}{2I+1} \sum_{i=0}^{n-1} \alpha_{n,i}^{I,\ell} \log^i \left(\frac{\mu^2}{s}\right) \log^{n-i-1} \left(\frac{\mu^2}{-s}\right) + O(NLL).$$



- F is the coupling in the Lagrangian; $\hat{S} = \frac{s^\kappa}{(4\pi F)^2}$ is the dimensionless expansion parameter.
- Both left and right cuts in the complex s plane contribute.

Recurrence relation for $\omega^I_{n\ell}$ from unitarity, analyticity and crossing I

- The LL coefficients are given by

$$\omega^I_{n\ell} = \sum_{i=0}^{n-1} \alpha^{I,\ell}_{n,i}.$$

- Only 2-particle unitarity is relevant! 2-particle unitarity (s-channel cut):

$$\mathrm{Disc}\, t^I_\ell(s)\Big|_{s>0} = |t^I_\ell(s)|^2.$$

- Implication of the s-channel cut unitarity:

$$\sum_{i=0}^{n-1}(n-i-1)\alpha^{I,\ell}_{n,i} = \frac{1}{2(2\ell+1)} \sum_{i=1}^{n-1} \omega^I_{i\ell}\omega^I_{n-i,\ell}.$$

Recurrence relation for $\omega^{I}_{n\ell}$ from unitarity, analyticity and crossing II

- Analyticity + crossing :

$$T^I(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \left(\frac{\delta^{II'}}{s'-s} + \frac{C_{su}^{II'}}{s'-4m^2+t+s} \right) \mathrm{Disc}\, T^{I'}(s',t).$$

- Isospin crossing matrices ($s \leftrightarrow u$-crossing):

$$T^I(s,t,u) = C_{su}^{II'}\, T^{I'}(u,t,s); \qquad C_{su}^{II'} = P_I^{abcd} P_{I'}^{bdac} \frac{1}{d_I}.$$

- 2-particle unitarity + analyticity: Roy equations:

$$\mathrm{Disc}\; t^I_\ell(s)\Big|_{s<0} = \sum_{\ell'=0}^{\infty} C_{su}^{II'} \frac{2(2\ell'+1)}{s} \int_0^{-s} ds'\, P_\ell\left(\frac{s+2s'}{-s}\right) P_{\ell'}\left(\frac{2s+s'}{-s'}\right) |t^I_\ell(s')|^2.$$

Recurrence relation for $\omega^I_{n,\ell}$ from unitarity, analyticity and crossing III

- Close form of the recurrence relation:

$$\omega^I_{n\ell} = \frac{1}{n-1} \sum_J \sum_{k=1}^{n-1} \sum_{\ell'=0}^{\kappa \cdot n} \frac{1}{2} \left(\delta^{IJ} \delta^{\ell \ell'} + C^{IJ}_{st} \Omega^{\ell' \ell}_{\kappa \cdot n} + C^{IJ}_{su} (-1)^{\ell + \ell'} \Omega^{\ell' \ell}_{\kappa \cdot n} \right) \frac{\omega^J_{k,\ell'} \omega^J_{n-k,\ell'}}{2\ell' + 1}.$$

- $(\kappa \cdot n + 1) \times (\kappa \cdot n + 1)$ matrices $\Omega^{\ell \ell'}_{\kappa \cdot n}$ perform the crossing for the PWs.

$$\left(\frac{z-1}{2} \right)^{\kappa \cdot n} P_\ell \left(\frac{z+3}{z-1} \right) = \sum_{\ell'=0}^{\kappa \cdot n} \Omega^{\ell \ell'}_{\kappa \cdot n} P_{\ell'}(z).$$

- Reminder for the indices:

n : number of loops $+1$;

I, J : label isospin invariant subspaces;

ℓ, ℓ' : PW OAM quantum numbers

- Initial conditions $\omega^I_{n\ell}$ with $\ell = 0, 1, \ldots \kappa \cdot n$ come from the tree-level calculation of PWs.

Results for $O(N+1)/O(N)$ σ-model

The Lagrangian of the $O(N+1)/O(N)$ σ-model:

$$\mathcal{L}_2 = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{8F^2}(\phi^a\phi^a)\partial^2(\phi^b\phi^b) + \mathcal{O}(\phi^6),$$

The projectors on 3 isospin spaces:

$$P_0^{abcd} = \frac{\delta^{ab}\delta^{cd}}{N}; \quad P_1^{abcd} = \frac{\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}}{2}; \quad P_2^{abcd} = \frac{\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}}{2} - \frac{1}{N}\delta^{ab}\delta^{cd}.$$

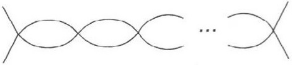
Boundary conditions from tree-level calculation: $\omega_{10}^0 = N-1;\ \omega_{11}^1 = 1;\ \omega_{10}^2 = -1$

(and all others are zero).

$$\frac{\omega_{n0}^0}{N-1} = \left\{1,\ \frac{N}{2} - \frac{1}{9},\ \frac{N^2}{4} - \frac{61N}{144} + \frac{59}{144},\ \frac{N^3}{8} - \frac{631N^2}{2700} + \frac{46279N}{194400} - \frac{13309}{194400}\right\}$$

How to check consistency?

- $N=1$ - free theory.
- Explicit calculation up to 3-loop order M. Bissegger, A. Fuhrer'07.
- Large N limit is well understood.



Case of renormalizable theory: $O(N)$-symmetric ϕ^4

- Consider

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i - \frac{\lambda_0}{4!}(\phi^2)^2.$$

- LLog approximation can be obtained by solving 1-loop RG equation:

$$\mu^2 \frac{\partial}{\partial \mu^2}\lambda(\mu^2) = \beta(\lambda) = \frac{N+2}{8}\lambda^2(\mu^2) + \mathcal{O}(\lambda^2) \Rightarrow A(s,t) = \lambda(\mu^2) = \frac{\lambda_0}{1 - b_1 \lambda_0 \log(\mu^2/s)}$$

- Same solution comes from the recursive equations.
- The sum of the crossing matrices $\frac{1}{2}(1 + C_{st} + C_{su})$ is just the 1-loop β-function coefficient.

$$\omega_n = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \underbrace{\frac{N+2}{8}}_{b_1} \omega_k \omega_{n-k} \quad \Rightarrow A(s,t) = \sum_{n=1}^{\infty} \lambda_0^n \log^{n-1}\left(\frac{\mu^2}{s}\right)\omega_n$$

Simplified form of the recurrence relation

Problem formulation:

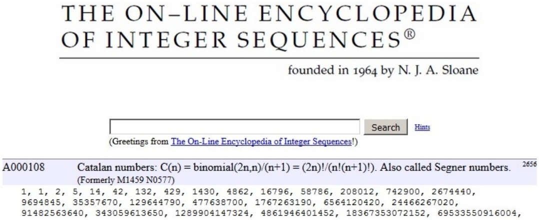
$$f(z) = \sum_{n=1}^{\infty} f_n z^{n-1}$$

$$f_n = \frac{1}{n-1} \sum_{k=1}^{n-1} A(n,k) f_k f_{n-k}, \quad f_1 = 1.$$

- $A(n,k)$ function: Greek "$\mathcal{A}\nu\alpha\delta\rho o\mu\acute{\eta}$" for "recursion" (courtesy of N. Stefanis).
- Singularities of $f(z)$ closest to the origin play the crucial role for the asymptotic behavior of f_n for $n \to \infty$.

Some remarkable cases I: Catalan numbers

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}, \quad C_0 = 1, \quad A(n+1,k) = n; \quad f(z) = \sum_{n=0}^{\infty} C_n z^n = \frac{2}{1+\sqrt{1-4z}}.$$



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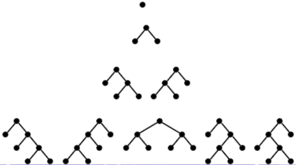
Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A000108 Catalan numbers: C(n) = binomial(2n,n)/(n+1) = (2n)!/(n!(n+1)!). Also called Segner numbers. 2656
(Formerly M1459 N0577)

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, 18367353072152, 69533550916004,

- Plenty of combinatoric applications.
- *E.g.* number of rooted binary trees with n internal nodes and $n+1$ external nodes



Some remarkable cases II: Bessel functions

A. Vladimirov, 2010:

$$A(n,k) = \frac{n-1}{n+\nu} \quad \nu - \text{parameter.}$$

- The solution:

$$f(z) = \sqrt{\frac{\nu+1}{z}}\,\frac{J_{\nu+1}(2\sqrt{z(\nu+1)})}{J_{\nu}(2\sqrt{z(\nu+1)})}.$$

- For $\nu > -1$ poles along $z > 0$ axis and a cut along $z < 0$;
- For $\nu < -1$ poles along $z < 0$ axis and a cut along $z > 0$;
- N.b. $\nu = \frac{1}{2}$: $f(z) = \frac{1-\sqrt{6z}\cot\sqrt{6z}}{2z}$ *c.f.* D. Kazakov: summing up UV-divergencies in the supersymmetric gauge theories ($D = 6, 8, 10$ SYM).

Some motivation from A.A. Migdal

- A.A.'Migdal'1977, 78: 2-point function of large-N_c QCD as the sum of infinite number of pole terms with spectrum given by roots of the Bessel functions.
- Some motivation to revive of the approach: AdS/CFT. Same spectrum reported (see J.Erlich et al.'2006)

Padé approximation: definition and basic properties

Let $f(z)$ be analytic function defined by its Taylor series:

$$f(z) = \sum_{n=1}^{\infty} f_n z^{n-1}.$$

Padé approximant:

$$[M/N]_f(z) = \frac{P_N(z)}{Q_M(z)} = f(z) + O(z^{M+N+1})$$

- Uniqueness;

Important class of functions:

Stieltitjes functions in a cut plane:

$$\mathrm{Im}[f(z)]\mathrm{Im}[z] \geq 0$$

- Keep this property at any order of diagonal Padé;
- Nice behavior of zeroes of the denominators;
- May prove uniform convergence of diagonal Padé in the cut plane;

More remarkable examples: factorials and pseudofactorials

Case of renormalizable theory

- $A(n,k) = 1$: factorial:
  $\{a_n\} \equiv \{f_n (n-1)!\} = \{0!, 1!, 2!, \dots\}$

$$f'(z) = -f^2(z); \quad f(0) = 1.$$

Quasi-renormalizable theory ??

- $A(n,k) = (-1)^{n+1}$
  R. Bacher and P. Flajolet'09: pseudofactorial sequence:
  $\{a_n\} \equiv \{f_n (n-1)!\} = \{1, -1, -2, 2, 16, -40, -320, 1040, \dots\}$.

$$f'(z) = -f^2(-z); \quad f(0) = 1.$$

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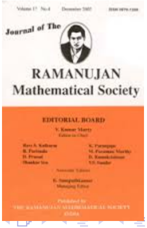
A098777 Pseudo-factorials: a(0)=1, a(n+1) = (-1)^(n+1) * Sum_{k=0..n} binomial(n,k) * a(k)*a(n-k), n>=0. 4

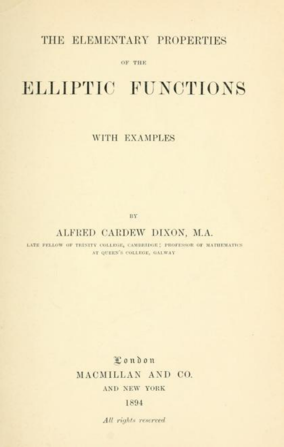
1, -1, -2, 2, 16, -40, -320, 1040, 12160, -52480, -742400, 3872000, 66457600, -411136000, -8202444800, 58479872000, 1335009280000, -10791497728000, -277035646976000, 2502527565824000, 71391934873600000, -712816377856000000, -22367684235100160000, 244597236078018560000 (list; graph; refs; listen; history; text; internal format)



Dixon's elliptic functions

Dixon's elliptic functions (Dixon'189X) (elliptic = meromorphic, doubly periodic)

$$\begin{cases} \mathrm{sm}'(z) = \mathrm{cm}^2(z) \\ \mathrm{cm}'(z) = -\mathrm{sm}^2(z) \end{cases} \qquad \mathrm{sm}(0) = 0; \quad \mathrm{cm}0 = 1 \qquad \boxed{\mathrm{sm}^3(z) + \mathrm{cm}^3(z) = 1.}$$


THE ELEMENTARY PROPERTIES

OF THE

ELLIPTIC FUNCTIONS

WITH EXAMPLES

BY

ALFRED CARDEW DIXON, M.A.

LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE; PROFESSOR OF MATHEMATICS AT QUEEN'S COLLEGE, GALWAY

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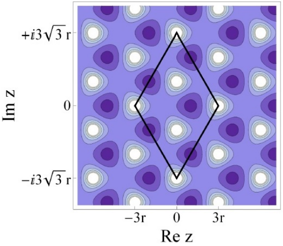
Dixon's elliptic functions and pseudofactorial

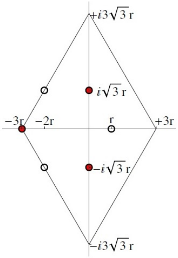
The equation is equivalent to $(g(z) \equiv f(-z))$

$$\begin{cases} f'(z) = -g^2(z) \\ g'(z) = f^2(z), \end{cases} \qquad f(0) = 1; \quad g(0) = 1.$$

- This system has a simple first integral: $f^3(z) + g^3(z) = 2$.

Integration gives: $$f(z) = 2^{\frac{1}{3}} \mathrm{sm}\left(\frac{\pi_3}{6} - 2^{\frac{1}{3}} z\right),$$ with $$\frac{\pi_3}{6} = \frac{1}{6} B\left(\frac{1}{3}, \frac{1}{3}\right).$$





- Real period: $\omega_3 \equiv 6r = 2^{-1/3}\pi_3$. Invariance under rotations by $\pm\frac{2\pi}{3}$.
- Can be expressed through the Weierstrass $\wp$-function.

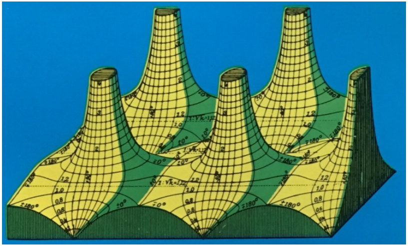
Weierstrass elliptic $\wp$-function

- Inverse of the first kind elliptic integral:

$$z = \int_{\infty}^{\wp(z;g_2,g_3)} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}.$$

- Weierstrass elliptic $\wp$-function satisfies the equation:

$$\left[\wp'(z;g_2,g_3)\right]^2 = 4\wp^3(z;g_2,g_3) - g_2\wp(z;g_2,g_3) - g_3; \quad \wp(z\to 0;g_2,g_3) = \frac{1}{z^2}.$$



O(N)-symmetric bi-quartic theory in $D = 2$

Quasi-renormalizable QFT (preliminary)

We call the quantum field theory **quasi-renormalizable** if the generating function for the coefficients of leading logs of $2 \to 2$ scattering amplitude (defined from the recurrence relation) is a meromorphic function of the variable $z \equiv \log\left(\frac{\mu^2}{s}\right)$.

- Important simplifications in $D = 2$ case: no PW expansion and related mixing problem.
- O(N)-symmetric bi-quartic (Φ^4-type with 4 derivatives) theory:

$$S = \int d^2x \left(\frac{1}{2} \partial_\mu \Phi^a \partial^\mu \Phi^a - g_1 (\partial_\mu \Phi^a \partial^\mu \Phi^a)(\partial_\nu \Phi^b \partial^\nu \Phi^b) - g_2 (\partial_\mu \Phi^a \partial_\nu \Phi^a)(\partial^\mu \Phi^b \partial^\nu \Phi^b) \right).$$

- LL-approximation of the O(N)-symmetric bi-quartic theory is infrared finite in $D = 2$.

LLog coefficients in $D = 2$ bi-quartic theory I

- Transition "T" and reflection "R" $2 \to 2$ scattering amplitudes:

$$
\begin{aligned}
&\langle \Phi_c(p_3)\Phi_d(p_4)|S-1|\Phi_a(p_1)\Phi_b(p_2)\rangle = \\
&= i(2\pi)^2 \frac{1}{2}\left[\delta(\mathbf{p_1}-\mathbf{p_4})\delta(\mathbf{p_2}-\mathbf{p_3})\mathcal{M}^T_{abcd}(s) + \delta(\mathbf{p_1}-\mathbf{p_3})\delta(\mathbf{p_2}-\mathbf{p_4})\mathcal{M}^R_{abcd}(s)\right];
\end{aligned}
$$

  - Transition: $t = 0$, $u = -s$;
  - Reflection: $u = 0$, $t = -s$;

- Isotopic decomposition:

$$
\mathcal{M}^{T,\,R}_{abcd}(s) = \sum_{I=0}^{2} P^I_{abcd}\mathcal{M}^{I,\,T,R}(s).
$$

- LLog contribution into "T" and "R" amplitudes:

$$
\mathcal{M}^{I,\,\{T,\,R\}}\Big|_{\mathrm{LL}}(s) = s^2 \sum_{n=1}^{\infty} \omega_n^{I,\,\{T,\,R\}} \left[\frac{s}{4\pi}\log\left(\frac{\mu^2}{s}\right)\right]^{n-1}.
$$

LLog coefficients in $D = 2$ bi-quartic theory II

- Explicit form of recurrence relations in $D = 2$:

$$\omega_n^{I,\,T} = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \sum_{I'=0}^{2} \left(\delta^{II'} - (-1)^n C_{su}^{II'} \right) \left(\omega_k^{I',\,T} \omega_{n-k}^{I',\,T} + \omega_k^{I',\,R} \omega_{n-k}^{I',\,R} \right);$$

$$\omega_n^{I,\,R} = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \sum_{I'=0}^{2} \left(\delta^{II'} - (-1)^n C_{st}^{II'} \right) \left(\omega_k^{I',\,T} \omega_{n-k}^{I',\,R} + \omega_k^{I',\,R} \omega_{n-k}^{I',\,T} \right).$$

- Initial conditions ($n = 1$) from the tree-level calculation:

$$\omega_1^{I=0,\,T} = \omega_1^{I=0,\,R} = (2g_1(N+1) + g_2(N+3));$$
$$\omega_1^{I=1,\,T} = -\omega_1^{I=1,\,R} = (g_2 - 2g_1);$$
$$\omega_1^{I=2,\,T} = \omega_1^{I=2,\,R} = (2g_1 + 3g_2).$$

- For $n > 1$ only particular combination of couplings occurs in LL-coefficients:

$$1/F^2 = 2g_1 + g_2.$$

Diagonalization of the recurrence system I

- The system for $\omega_n^{I=0,\,T}$ is equivalent to

$$f_n = \frac{1}{n-1} \sum_{k=1}^{n-1} \Big(A_0 + (-1)^n A_1 + (-1)^k A_2 \Big) f_k f_{n-k}; \quad f_1 = 1.$$

The coefficients A_i read:

$$A_0 = 1 + \frac{1}{(N+2)(N-1)}; \; A_1 = -\frac{N+1}{(N+2)(N-1)}; \; A_2 = -\frac{2}{(N+2)(N-1)}.$$

- For $n \geq 2$:

$$\omega_n^{I=0,\,T} = f_n \times \left(\frac{(N+2)(N-1)}{NF^2} \right)^n.$$

Diagonalization of the recurrence system II

- Generating function:

$$\Omega(z) = \sum_{n=2}^{\infty} \omega_n^{I=0,\ T} z^{n-1}.$$

- $I = 0$ transition amplitude in LL-approximation:

$$\mathcal{M}^{I=0,\ T}\Big|_{\rm LL}(s) = s^2\left[2g_1(N+1) + g_2(N+3)\right] + \frac{s^2}{F^2}\,\Omega\left(\frac{s}{2\pi F^2}\log\left(\frac{\mu^2}{s}\right)\right).$$

- All other $\mathcal{M}^{I,\ T,\ R}$ are expressed through the same $\Omega(z)$.

How to solve the recurrence system I

- We introduce the generating function

$$f(z) = \sum_{n=1}^{\infty} f_n z^{n-1}.$$

- The recurrence system for f_n is equivalent to the differential equation:

$$\frac{d}{dz} f(z) = A_0 \; f(z)^2 + A_1 \; f(-z)^2 - A_2 \; f(z) f(-z), \quad f(0) = 1.$$

- **N.b.** $A_1 = A_2 = 0$ same form as the RG-equation in a renormalizable QFT:

$$\frac{d}{dz} f(z) = A_0 \; f^2(z); \;\; f(z) = \frac{1}{1 - A_0 z} \; : \text{ single Landau pole.}$$

How to solve the recurrence system II

- Even and odd parts of the generating function: $f(\pm z) = u(z) \pm v(z)$

$$u(z) = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} f_n z^{n-1}; \quad v(z) = \sum_{\substack{n=2\\ \text{even}}}^{\infty} f_n z^{n-1}.$$

- We need to address the system:

$$\begin{cases} v'(z) = (A_0 + A_1 - A_2)u^2(z) + (A_0 + A_1 + A_2)v^2(z); \\ u'(z) = 2(A_0 - A_1)u(z)v(z); \end{cases} \qquad u(0) = 1; \; v(0) = 0.$$

- The system possesses the following first integral A. Smirnov:

$$\left(u^2(z) + \frac{-A_0 + 3A_1 + A_2}{A_0 + A_1 - A_2} v^2(z) \right)^{A_0 - A_1} = (u(z))^{A_0 + A_1 + A_2}.$$

Equivalent mechanical system II

- New variables:
$$l(z) = \frac{1}{u(z)}$$
,
- Use of the first integral:
$$v(z) = -\frac{1}{2(A_0 - A_1)} \frac{1}{l(z)} \frac{d}{dz} l(z).$$
- Final form of the equation for $l(z)$:

$$\alpha_1 \left[l'(z)) \right]^2 = \alpha_0 l^{2\alpha_1}(z) - \alpha_0; \quad l(0) = 1.$$

$$\alpha_0 = -\frac{2N^3}{(N-1)^2(N+2)}; \quad \alpha_1 = \frac{N+2}{2N}.$$

Equivalent mechanical system III

- The problem is equivalent to a 1D motion of a mechanical system:
  - "time": $t = \frac{N}{(N-1)(N+2)} z$;
  - "coordinate": $q(t) = I\left(\frac{(N-1)(N+2)}{N}\; t\right)$.

$$\frac{m\;\dot{q}(t)^2}{2} + q(t)^{\gamma} = 1; \quad q(t=0) = 1; \quad \dot{q}(t=0) = 0;$$

  - "mass": $m = \frac{1}{2N^2}$;
  - exponent of the "potential": $\gamma = \frac{N+2}{N}$.

LL-amplitude through q

$$\Omega(z) = \frac{(N-1)(N+2)}{N}\;\left(\frac{1}{q(z)} - 1\right) - \frac{N-1}{2N}\;\frac{d}{dz}\log(q(z)).$$

Equivalent mechanical system II

- Dual "mechanical" system: $q(t) = r(t)^{\frac{2}{2-\gamma}}$:

$$\frac{M\ \dot{r}(t)^2}{2} + \left[2 - r(t)^{\delta}\right] = 1; \quad r(t=0) = 1; \quad \dot{r}(t=0) = 0;$$

  - "mass": $M = \frac{2}{(N-2)^2}$;
  - exponent of the "potential": $\delta = \frac{2\gamma}{\gamma-2}$.

Remarkable solutions I

- $N \to \infty$; $(\gamma = 0,\ m \to 0)$ – "motion under constant force":

$$q(t) = 1 - (Nt)^2; \qquad \boxed{\Omega(z) = \frac{N}{1 - Nz} - N.}$$

N.b. LL-amplitude possesses a single Landau pole; assuming $g_i \sim 1/N$, theory is equivalent to a renormalizable QFT.

- $N = 2$ $(\gamma = 2,\ m = \frac{1}{8})$ – "harmonic oscillator":

$$q(t) = \cos(4t); \qquad \boxed{\Omega(z) = \frac{2}{\cos(4z)} + \tan(4z) - 2}.$$

First example of a quasi-renormalizable theory!

- Poles and residues:

$$z_k^{(1)} = \frac{\pi}{8}(4k+1),\ k \in \mathbb{Z}, \qquad \operatorname*{Res}_{z=z_k^{(1)}} \Omega(z) = -\frac{3}{4};$$

$$z_k^{(2)} = \frac{\pi}{8}(4k+3),\ k \in \mathbb{Z}, \qquad \operatorname*{Res}_{z=z_k^{(2)}} \Omega(z) = \frac{1}{4}.$$

Remarkable solutions II

- $N \to 0$ ($\delta = 2,\ M = \frac{1}{2}$) – "inverted harmonic potential":

$$r(t) = \cosh(2t); \qquad \boxed{\Omega(z) = -\log[\cosh(2z)] - \tanh(2z)}.$$

Another example of a quasi-renormalizable theory!

  - Poles and residues:

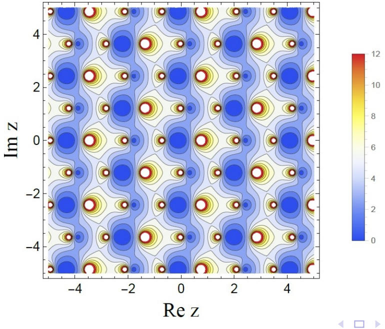
$$z_k = i\frac{\pi}{4}(2k+1)\,,\ k \in \mathbb{Z}, \qquad \operatorname*{Res}_{z=z_k} \Omega(z) = -\frac{1}{2}.$$

Remarkable solutions III

- Case $N \to 1$ ($\gamma = 3$, $m = \frac{1}{2}$) – (Bacher&Flajolet's pseudofactorial):

$$q(t) = \frac{3\wp\left(\sqrt{3}t; 0, -\frac{4}{27}\right) - 2}{3\wp\left(\sqrt{3}t; 0, -\frac{4}{27}\right) + 1}; \qquad \boxed{[\wp']^2 = 4\wp^3 - g_2\wp - g_3}$$

$$\boxed{\frac{\Omega(z)}{N-1} = \frac{1}{2}\,\frac{6\wp\left(\sqrt{3}z; 0, -\frac{4}{27}\right) - \sqrt{3}\wp'\left(\sqrt{3}z; 0, -\frac{4}{27}\right) + 2}{\wp\left(\sqrt{3}z; 0, -\frac{4}{27}\right)^2 - \frac{1}{3}\wp\left(\sqrt{3}z; 0, -\frac{4}{27}\right) - \frac{2}{9}}.}$$



Remarkable solutions IV

- $N = \frac{2}{3}$: $\gamma = 4$, $m = \frac{9}{8}$

$$\Omega(z) = \frac{4\wp\left(\frac{4}{3}z; -1, 0\right) - \wp'\left(\frac{4}{3}z; -1, 0\right) + 2}{-3\wp\left(\frac{4}{3}z; -1, 0\right)^2 + \frac{3}{4}}.$$

Qualitative general analysis

- Case $N > 0$ or $N < -2$: "particle" moves to the left from $q = 1$ to $q = 0$. This point is reached in finite "time" and with finite "velocity": pole singularity of $\Omega(z)$:

$$z_{\rm pole} = \frac{\sqrt{\pi}\Gamma\left(1 + \frac{N}{N+2}\right)}{2N\Gamma\left(\frac{1}{2} + \frac{N}{N+2}\right)}.$$

  - $N = \frac{2}{2p+1}$ ($p \in \mathbb{N}$) "particle" oscillates in the "potential" $q^{2(p+1)}$. $\Omega(z)$ has an infinite number of equidistant poles separated by:

$$\Delta z = \frac{(2p+1)\sqrt{\pi}\,\Gamma\left(1 + \frac{1}{2(p+1)}\right)}{2\,\Gamma\left(\frac{1}{2} + \frac{1}{2(p+1)}\right)}, \quad p \in \mathbb{N}.$$

  - Known elliptic solutions for $N = \frac{2}{3}, \frac{2}{5}$.

- Case $-2 < N < 0$: "particle" moves to the right from $q = 1$ and never reaches $q = 0$. No singularities on the real axis for $\Omega(z)$.

$$z_{\rm branch} = \pm i\frac{\sqrt{\pi}\Gamma\left(\frac{2}{N+2} - \frac{1}{2}\right)}{2N\,\Gamma\left(-\frac{N}{N+2}\right)}.$$

Conclusions and Outlook I

1. First examples of quasi-renormalizable theories:
   - All order summation of LLs in the O(2)-symmetric bi-quartic theory in $D = 2$:

$$\Omega^{N=2}(z) = \frac{2}{\cos(4z)} + \tan(4z) - 2.$$

   - $N \to 1$ limit case. LL-amplitude in terms of Dixon's functions.
2. Can one find further field theoretical models corresponding to (doubly)periodic solutions of the recurrence relations for LL coefficients?
3. Is it possible to have a non-trivial example of the exactly solvable quasi- renormalizable theory in $D = 2$?
4. A the connection between the LL-approximation of EFTs and the properties of general non-perturbative solution related to the underlying fundamental QFT. Can we learn something on the spectrum?
5. Can we understand how recurrence relations work at the diagram level? A. Connes and D. Kreimer linear space of graphs? D. Kazakov insight from SYM?

Outlook II

- Mixed O(N)-symmetric theory with renormalizable and non-renormalizable interactions: possible applications in solid state physics.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi \partial^\mu \Phi) + \frac{1}{F^2}(\Phi \partial_\mu \Phi)(\Phi \partial^\mu \Phi) + g_1(\partial_\mu \Phi \partial^\mu \Phi)(\partial_\nu \Phi \partial^\nu \Phi) + g_2(\partial_\mu \Phi \partial^\nu \Phi)(\partial_\nu \Phi \partial^\mu \Phi)$$

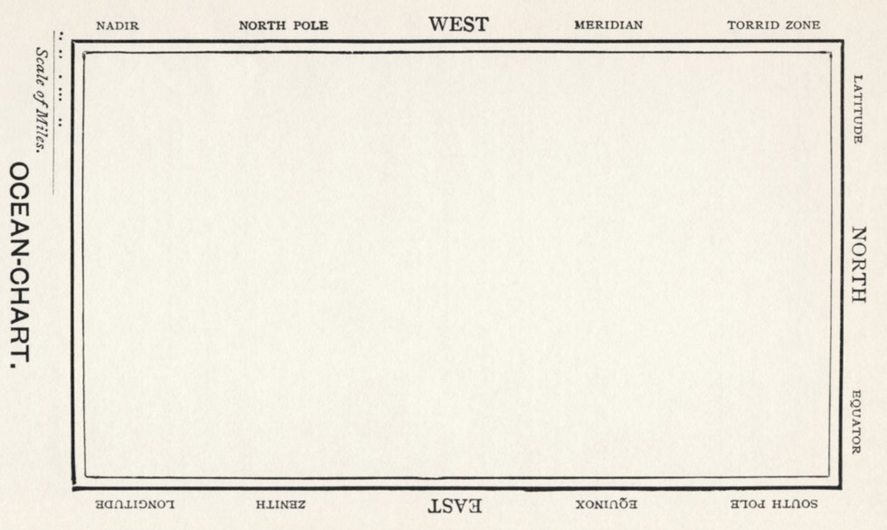
Weak-localization theory: role of higher derivatives in the nonlinear sigma-model

V. E. Kravtsov, I. V. Lerner, and V. I. Yudson

Spectroscopy Institute, USSR Academy of Sciences
(Submitted 8 December 1987)
Zh. Eksp. Teor. Fiz. **94**, 255–263 (July 1988)

It is shown that additional vertices containing higher powers $[(\partial Q)^{2n}, n \geqslant 2]$ of the gradients of the field Q, which appear in the microscopic derivation of the Q-functional of the nonlinear sigma model, have a positive anomalous dimensionality proportional to $n^2 - n$. By the same token, these vertices turn out to be substantial for sufficiently large n, notwithstanding their negative normal dimensionality $-2n + 2$. It turns out that it is precisely these vertices which determine the asymptotic behavior of the distribution function of the mesoscopic fluctuations, as well as the long-time asymptotic behavior of the relaxation currents in disordered conductors. In particular, allowance for these vertices leads to a change of the variance in the logarithmic normal asymptote of the distribution function of the conductivity-fluctuations.

Where we are?



NADIR
NORTH POLE
WEST
MERIDIAN
TORRID ZONE
OCEAN-CHART.
Scale of Miles.
LATITUDE
NORTH
EQUATOR
LONGITUDE
ZENITH
EAST
EQUINOX
SOUTH POLE