Integrable coupled sigma-models from affine Gaudin models

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Collaboration with François Delduc (CNRS, ENS de Lyon), Marc Magro (ENS de Lyon) and Benoit Vicedo (University of York)
[1811.12316], [1903.00368] and [1907.04836]

Introduction

Introduction: integrable σ-models

- Integrability: realm of highly symmetric systems
  $\rightarrow$ exact methods for the computation of physical quantities

- σ-models: two-dimensional field theories on target space M
- In high energy physics, condensed matter, conformal field theory, ...

- Well chosen M: integrable σ-models
  $\rightarrow$ rare but growing list of examples over the years

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- Simplest example: principal chiral model, with M a Lie group
- Symmetric space σ-model, with M the quotient of a Lie group ($O(3)$-model on $\mathbb{S}^2 = SO(3)/SO(2)$ in condensed matter, ...)
- Superstring on $\text{AdS}_5 \times \text{S}^5$ (AdS/CFT correspondence, dual to $\mathcal{N} = 4$ Super-Yang-Mills in 4d, ...)

Introduction: deformed integrable σ-models

- Continuous integrable deformations of integrable σ-models (β-, γ-, η-, λ- deformations, ...)
- Applied to principal chiral model, symmetric space σ-model, superstring on $AdS_5 \times S^5$, ...

- Various questions related to deformations
  $\rightarrow$ links with quantum groups, double field theory, generalised supergravity, non-commutative geometry, ...

- Whole family of integrable fields theories
- Common structure behind integrability of this family
  $\rightarrow$ affine Gaudin models

Introduction: affine Gaudin models

- Gaudin models: integrable systems associated with Lie algebras $\mathfrak{g}$
- Finite Gaudin models $\leftrightarrow$ $\mathfrak{g}$ semi-simple finite dimensional
  $\rightarrow$ integrable spin chains
- Affine Gaudin models (AGM) $\leftrightarrow$ $\mathfrak{g}$ affine Kac-Moody algebra
  $\rightarrow$ integrable 2D field theories
  $\rightarrow$ contain integrable σ-models

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Introduction: applications of affine Gaudin models

- Applications of seeing integrable σ-models as AGM ?
- Quantisation:
  - problem of non-ultralocality
  - quantisation of finite Gaudin models well-understood (relations to the Geometric Langlands Correspondence)
  - progress on the quantisation of affine ones by analogy
- Previously known integrable σ-models: small class among AGM
  $\rightarrow$ construction of new classical integrable σ-models
  (by coupling together known ones)

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Contents

Finite Gaudin models

Integrable spin chains and mechanical systems

Integrability for mechanical systems

- Mechanical system with canonical variables $(q_1, \cdots, q_n, p_1, \cdots, p_n)$:

Liouville integrability $\Longleftrightarrow$ n conserved quantities $\mathcal{Q}_i$ in involution
$$\{\mathcal{H}, \mathcal{Q}_i\} = \{\mathcal{Q}_i, \mathcal{Q}_j\} = 0$$

- Noether theorem: conserved charges $\Leftrightarrow$ symmetries
  $\rightarrow$ highly symmetric systems

- Liouville theorem: model solvable by quadratures (algebraic manipulations and integration)

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- Quantum integrability: $[\mathcal{H}, \mathcal{Q}_i] = [\mathcal{Q}_i, \mathcal{Q}_j] = 0$

The Gaudin model as a spin chain

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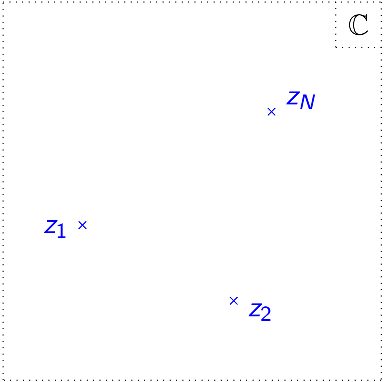
  $\rightarrow$ historically introduced as a spin chain [Gaudin 76']



- Positions $z_1, z_2, \cdots, z_N$ in $\mathbb{C}$

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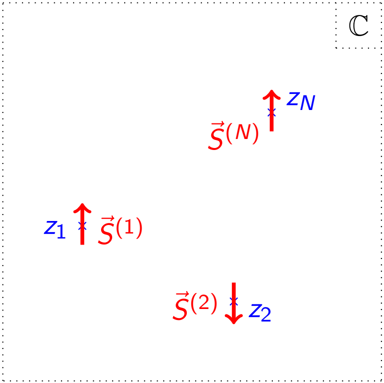

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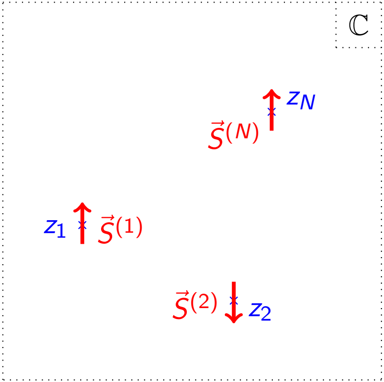

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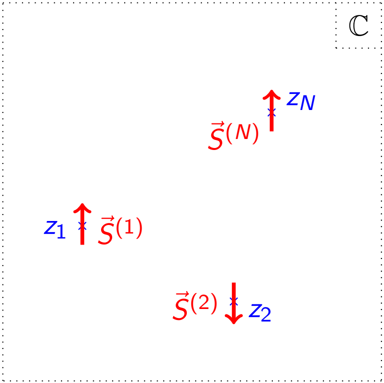

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$$\mathcal{H}_r = \sum_{s \neq r} \frac{\vec{S}^{(r)} \cdot \vec{S}^{(s)}}{z_r - z_s}$$

- Total energy:

$$\mathcal{H} = \sum_{r=1}^{N} a_r \mathcal{H}_r$$

Integrability of the Gaudin spin chain

- Spin operators: $\vec{S} = (S_x, S_y, S_z)$

$$\left[S_\mu , S_\nu \right] = \epsilon_{\mu\nu\rho} S_\rho$$

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$\rightarrow$ integrability

- Classical Gaudin model: $[\cdot,\cdot] \rightarrow \{\cdot,\cdot\}$

$$\Big\{S_\mu^{(r)}, S_\nu^{(s)}\Big\} = \delta_{rs}\, \epsilon_{\mu\nu\rho}\, S_\rho^{(r)}, \qquad\qquad \{\mathcal{H}_r, \mathcal{H}_s\} = 0$$

Gaudin model with arbitrary Lie algebra [Gaudin 83']

- Spin operators: $\mathfrak{su}(2)$ algebra

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- Mixed product identity:

$$\left(\vec{X} \times \vec{Y}\right) \cdot \vec{Z} = \vec{X} \cdot \left(\vec{Y} \times \vec{Z}\right)$$

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- Integrability: $\{\mathcal{H}_r, \mathcal{H}_s\} = 0$?
- Invariant scalar product

$$\kappa([X,Y],Z) = \kappa(X,[Y,Z])$$

Finite Gaudin models

- Gaudin models: associated with Lie algebras with invariant scalar product (non-degenerate bilinear form $\kappa(\cdot,\cdot)$)
- Classical "spin" operators $J^{(r)} = J^{(r)}_a I^a$ in $\mathfrak{g}$

$$\left\{ J^{(r)}_a, J^{(s)}_b \right\} = \delta_{rs}\, f_{ab}{}^{c}\, J^{(r)}_a$$

$\rightarrow$ N independent copies of the Kirillov-Kostant bracket of $\mathfrak{g}$

- Finite Gaudin models: finite dimensional semi-simple algebras (matrix algebras $\mathfrak{su}(N)$, $\mathfrak{so}(N)$, $\mathfrak{sl}(N)$, $\mathfrak{sp}(2N)$, ...)
- Invariant scalar-product: Killing form ($\kappa(X,Y) = \text{Tr}(XY)$)
- Includes $\mathfrak{su}(2)$

Lax matrix

- Lax matrix: ($\mathfrak{g}$-valued)

$$\mathcal{L}(z) = \sum_{r=1}^{n} \frac{J^{(r)}}{z - z_r}$$

Spectral parameter z: auxiliary complex parameter

- Spectral parameter dependent Hamiltonian:

$$\mathcal{H}(z) = \frac{1}{2}\kappa(\mathcal{L}(z), \mathcal{L}(z)) = \frac{1}{2}\text{Tr}(\mathcal{L}(z)^2), \qquad \{\mathcal{H}(z), \mathcal{H}(w)\} = 0$$

- Gaudin Hamiltonians: $\mathcal{H}_r = \sum_{s \neq r} \frac{\kappa\left(J^{(r)}, J^{(s)}\right)}{z_r - z_s} = \operatorname*{res}_{z=z_r} \mathcal{H}(z)$

Lax equation and higher degree Hamiltonians

- Hamiltonian $\mathcal{H} = \sum_r a_r \mathcal{H}_r \to$ dynamics $\frac{\mathrm{d}}{\mathrm{d}t} = \{\mathcal{H}, \cdot\}$
- Dynamics of the Lax matrix: Lax equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(z) = [\mathcal{L}(z), \mathcal{M}(z)]$$

- Higher degree Hamiltonians:

$$\mathcal{H}^p(z) = \mathrm{Tr}\big(\mathcal{L}(z)^p\big), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^p(z) = 0$$

- Poisson brackets $\{\mathcal{L}_a(z), \mathcal{L}_b(w)\}$ between components of Lax matrix
  - $\to$ take the so-called r-matrix form
    - $\to$ Involution: $\{\mathcal{H}^p(z), \mathcal{H}^q(w)\} = 0$

Realisations of finite Gaudin models

- Formal finite Gaudin model:
  - Hamiltonian model with dynamical variables $J_a^{(r)}$
  - $J_a^{(r)}$ satisfy Kirillov-Kostant bracket
  - Restricted
  - No possible Lagrangian formulation

- Realisation of finite Gaudin models:
  - Phase space $\mathcal{P}$ (typically generated by canonical coordinates q_i, p_i)
  - $\mathcal{P}$ contains combinations $\mathcal{J}_a^{(r)}$ satisfying KK brackets
  - Transfer all the integrable structure

    $\rightarrow$ integrable mechanical model on $\mathcal{P}$

Example of realisation: Neumann model

- Lie algebra $\mathfrak{sl}(2)$: basis $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
- Commutation relation: $[H,E] = 2E$, $[H,F] = -2F$ and $[E,F] = H$

- Phase space $\mathbb{R}^{2n}$ with canonical variables $(q_1, \cdots, q_n, p_1, \cdots, p_n)$:
- Canonical bracket: $\{p_r, q_s\} = \delta_{rs}, \ \ \{p_r, p_s\} = \{q_r, q_s\} = 0$

- Realisation: $\mathcal{J}_H^{(r)} = q_r p_r, \ \ \mathcal{J}_E^{(r)} = -\frac{1}{2} p_r^2, \ \ \mathcal{J}_F^{(r)} = \frac{1}{2} q_r^2$

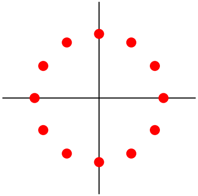
$$\left\{ \mathcal{J}_H^{(r)}, \mathcal{J}_E^{(s)} \right\} = 2\,\delta_{rs}\,\mathcal{J}_E^{(r)}, \ \ \left\{ \mathcal{J}_H^{(r)}, \mathcal{J}_F^{(s)} \right\} = -2\,\delta_{rs}\,\mathcal{J}_F^{(r)}$$
$$\left\{ \mathcal{J}_E^{(r)}, \mathcal{J}_F^{(s)} \right\} = \delta_{rs}\,\mathcal{J}_H^{(r)}$$

  Kirillov-Kostant bracket of $\mathfrak{sl}(2)$

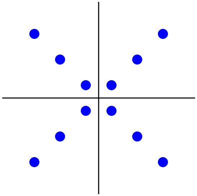
- Associated integrable Gaudin model: (unreduced) Neumann model
  $\rightarrow$ particle on a sphere in anisotropic harmonic potential

Coupling realisations of Gaudin models

$\mathcal{H}_r = \sum_{s \neq r} \frac{\kappa(J^{(r)}, J^{(s)})}{z_r - z_s}$ → interaction of sites r and s controlled by distance $z_r - z_s$



Model (1): sites $z_1^{(1)}, \cdots, z_{N_1}^{(1)}$



Model (2): sites $z_{N_1+1}^{(2)}, \cdots, z_{N_1+N_2}^{(2)}$

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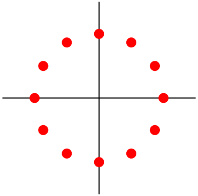

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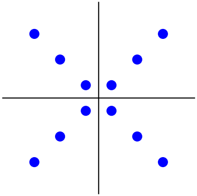
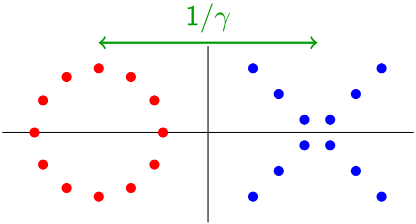


Model (2): sites $z_{N_1+1}^{(2)}, \cdots, z_{N_1+N_2}^{(2)}$

Coupled model ($N_1 + N_2$ sites)

$$z_r = z_r^{(1)} - \frac{1}{2\gamma}, \qquad z_r = z_r^{(2)} + \frac{1}{2\gamma}$$

$$\mathcal{H} \xrightarrow{\gamma \to 0} \mathcal{H}^{(1)} + \mathcal{H}^{(2)}$$



Affine Gaudin models

Integrable field theories

[Feigin Frenkel '07, Vicedo '17, Delduc SL Magro Vicedo '19]

Loop algebra

- Finite dimensional semi-simple Lie algebra $\mathfrak{g}$: $[I_a, I_b] = f_{ab}{}^c \, I_c$
- Kirillov-Kostant bracket: $\{J_a, J_b\} = f_{ab}{}^c \, J_c$

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- Loop algebra $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$: basis $I_a^n = I_a \otimes t^n$, $n \in \mathbb{Z}$

$$[I_a^n, I_b^m] = [I_a, I_b] \otimes t^n t^m = f_{ab}{}^c\, I_a^{n+m}$$

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- Kirillov-Kostant bracket: $\{J_a^n, J_b^m\} = f_{ab}{}^c\, J_a^{n+m}$

- Field on $\mathbb{S}^1 = [0, 2\pi]$: $J_a(x) = \sum_{n\in\mathbb{Z}} J_a^n\, e^{inx}$

$$\{J_a(x), J_b(y)\} = f_{ab}{}^c \sum_{n,m\in\mathbb{Z}} J_c^{n+m}\, e^{inx+imy} \overset{p=n+m}{=} f_{ab}{}^c \left(\sum_{p\in\mathbb{Z}} J_c^p\, e^{ipx}\right)\left(\sum_{n\in\mathbb{Z}} e^{in(x-y)}\right)$$

Loop algebra

- Finite dimensional semi-simple Lie algebra $\mathfrak{g}$: $[I_a, I_b] = f_{ab}{}^c\, I_c$
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- Poisson bracket of an Hamiltonian field theory:

$$\{J_a(x), J_b(y)\} = f_{ab}{}^c\, J_c(x)\, \delta(x-y)$$

Affine algebra

- Affine algebra: central extension $\widehat{g} = \mathcal{L}\mathfrak{g} \oplus \mathbb{C}K$ of loop algebra

$$[I_a^n, I_b^m] = f_{ab}{}^c \, I_c^{n+m} \qquad\qquad [K, \cdot] = 0$$

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$$[I_a^n, I_b^m] = f_{ab}{}^c\, I_c^{n+m} + \kappa_{ab}\, n\, \delta_{n+m,0}\, K, \qquad [K, \cdot] = 0$$

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- Send I_a^n to J_a^n forming the field $J_a(x) = \sum_{n\in\mathbb{Z}} J_a^n\, e^{inx}$
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$$\{J_a(x), J_b(y)\} = f_{ab}{}^c\, J_c(x)\, \delta(x-y) - \ell\, \kappa_{ab}\, \partial_x \delta(x-y)$$

- Lie algebra valued field $J(x) = J_a(x) I^a$
  $\rightarrow$ Kac-Moody current

Affine Gaudin models

- Affine Gaudin model: associated with Kac-Moody affine algebra (affine algebra with a derivation element, has an invariant scalar product)
- Attach to each site a Kirillov-Kostant bracket of affine algebra

- Defining data:
  - sites: points $z_1, \cdots, z_N$ in $\mathbb{C}$
  - levels: numbers $\ell^{(1)}, \cdots, \ell^{(N)}$

- Phase space: Kac-Moody currents $J^{(r)}(x)$

$$\left\{ J_a^{(r)}(x), J_b^{(s)}(y) \right\} = \delta_{rs} \left(f_{ab}{}^{c}\, J_c^{(r)}(x)\, \delta(x-y) - \ell^{(r)}\, \kappa_{ab}\, \partial_x \delta(x-y) \right)$$

- Gaudin Lax matrix and twist function:

$$\Gamma(z,x) = \sum_{r=1}^{N} \frac{J^{(r)}(x)}{z - z_r} \qquad \text{and} \qquad \varphi(z) = \sum_{r=1}^{N} \frac{\ell^{(r)}}{z - z_r} - \ell^{\infty}$$

Hamiltonian

- Hamiltonians $\mathcal{H}_r$ at site (r) from scalar product on affine algebra
- Other basis more convenient

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$$\varphi(z) = \sum_{r=1}^{N} \frac{\ell^{(r)}}{z - z_r} - \ell^\infty = -\ell^\infty \frac{\prod_{i=1}^{N}(z - \zeta_i)}{\prod_{r=1}^{N}(z - z_r)}$$

- Zeroes $\zeta_1, \cdots, \zeta_N$ of the twist function

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- Quadratic charges associated with zeroes:

$$\mathcal{Q}_i = \underset{z=\zeta_i}{\text{res}}\, \mathcal{Q}(z)\, \mathrm{d}z, \qquad \mathcal{Q}(z) = -\frac{1}{2\varphi(z)} \int_{\mathbb{D}} \mathrm{d}x\; \kappa(\Gamma(z,x), \Gamma(z,x))$$

- Involution: $\{\mathcal{Q}_i, \mathcal{Q}_j\} = 0$

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- Involution: $\{\mathcal{Q}_i, \mathcal{Q}_j\} = 0$

- Hamiltonian: $\mathcal{H} = \sum_{i=1}^{N} \epsilon_i\, \mathcal{Q}_i$ ($\mathcal{Q}_i$ conserved and in involution)

Integrable field theories

- Affine Gaudin model integrable ?
  $\rightarrow$ integrability for field theories ?

- Mechanical system with canonical variables $(q_1, \cdots, q_n, p_1, \cdots, p_n)$:

  Liouville integrability $\Longleftrightarrow$ n conserved quantities $\mathcal{Q}_i$ in involution $\{\mathcal{H}, \mathcal{Q}_i\} = \{\mathcal{Q}_i, \mathcal{Q}_j\} = 0$

- Integrability for classical field theory
  $\rightarrow$ infinite number of conserved charges in involution
- Ambiguous notion

- For two-dimensional field theory
  $\rightarrow$ Lax formalism

Lax pair and monodromy matrix

- 2D field theory: time t, space x (in $\mathbb{D} =]-\infty, +\infty[$ or $\mathbb{D} = [0, 2\pi]$)

- Lax connection: $\mathfrak{g}$-valued fields $\big(\mathcal{L}(z; x, t), \mathcal{M}(z; x, t)\big)$
- Spectral parameter: auxiliary parameter $z \in \mathbb{C}$
- Equations of motion $\Leftrightarrow$ zero curvature equation

$$\partial_t \mathcal{L}(z; x, t) - \partial_x \mathcal{M}(z; x, t) + \big[\mathcal{L}(z; x, t), \mathcal{M}(z; x, t)\big] = 0, \quad \forall\, z$$

Similar to $\partial_t \mathcal{L}(z) = [\mathcal{M}(z), \mathcal{L}(z)]$ for mechanical and finite Gaudin models

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- Monodromy matrix: $T(z, t) = \mathrm{P}\overleftarrow{\exp}\left(- \int_{\mathbb{D}} \mathrm{d}x \; \mathcal{L}(z; x, t) \right)$

- Infinite number of conserved quantities: $\mathcal{Q}_n(z) = \mathrm{Tr}\big(T(z)^n \big)$

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- Poisson bracket of $\mathcal{L}(z)$ of Maillet form
  $\rightarrow \mathcal{Q}_n(z)$ and $\mathcal{Q}_m(w)$ in involution

Integrability of affine Gaudin models

- Back to the case of affine Gaudin models
- Dynamics $\partial_t = \{\mathcal{H}, \cdot\}$, with $\mathcal{H} = \sum_{i=1}^{N} \epsilon_i \mathcal{Q}_i$
- Lax matrix: $\mathcal{L}(z,x) = \dfrac{\Gamma(z,x)}{\varphi(z)}$
- Zero curvature equation: their exists $\mathcal{M}(z,x)$ such that

$$\{\mathcal{H}, \mathcal{L}(z,x)\} - \partial_x \mathcal{M}(z,x) + \big[\mathcal{M}(z,x), \mathcal{L}(z,x)\big] = 0$$

- Poisson brackets of the Kac-Moody currents
  $\rightarrow$ Maillet bracket for $\mathcal{L}(z,x)$
- Integrability automatic !

A few more results and concepts

Affine Gaudin models with multiplicities:

- Gaudin Lax matrix and twist function

$$\Gamma(z,x) = \sum_{r=1}^{N} \sum_{p=0}^{m_r-1} \frac{J^{(r)}_{[p]}(x)}{(z-z_r)^{p+1}} \quad \text{and} \quad \varphi(z) = \sum_{r=1}^{N} \sum_{p=0}^{m_r-1} \frac{\ell^{(r)}_p}{(z-z_r)^{p+1}} - \ell^{\infty}$$

- Takiff currents $J^{(r)}_{[p]}(x)$ (Poisson bracket generalising Kac-Moody currents)
- Quadratic charges $\mathcal{Q}_i$ associated with zeroes $\zeta_1, \cdots, \zeta_M$ of $\varphi(z)$
- Hamiltonian $\mathcal{H} = \sum_{i=1}^{M} \epsilon_i \, \mathcal{Q}_i \rightarrow$ integrability

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- Lorentz invariance $\Leftrightarrow \epsilon_i = \pm 1$
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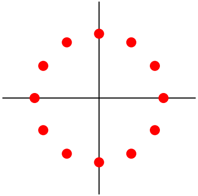
Realisation:

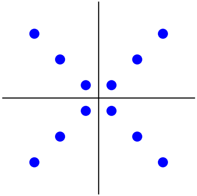
- Phase space $\mathcal{P}$ with currents $\mathcal{J}^{(r)}_{[p]}(x)$ satisfying Takiff brackets

  $\rightarrow$ integrable field theory on $\mathcal{P}$

Coupling affine Gaudin models



Model (1): sites $z_1^{(1)}, \cdots, z_{N_1}^{(1)}$



Model (2): sites $z_{N_1+1}^{(2)}, \cdots, z_{N_1+N_2}^{(2)}$

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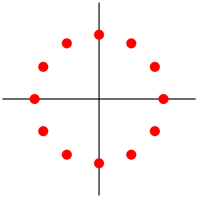

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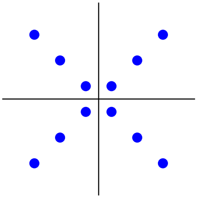


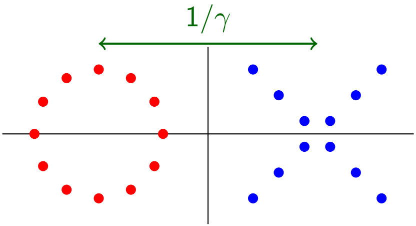
Model (2): sites $z_{N_1+1}^{(2)}, \cdots, z_{N_1+N_2}^{(2)}$

Coupled model ($N_1 + N_2$ sites)

$$z_r = z_r^{(1)} - \frac{1}{2\gamma}, \qquad z_r = z_r^{(2)} + \frac{1}{2\gamma}$$

$$\mathcal{H} \xrightarrow{\gamma \to 0} \mathcal{H}^{(1)} + \mathcal{H}^{(2)}$$

Preserves relativistic invariance



Integrable σ-models from affine Gaudin models

Principal chiral field realisation

- Semi-simple Lie group G
- Canonical fields on T^*G: $i = 1, \cdots, \dim G$

  coordinate fields $\phi_i(x)$ and conjugate momenta fields $\pi_i(x)$
- Canonical bracket:

$$\{\phi_i(x), \pi_j(y)\} = \delta_{ij}\, \delta(x-y), \quad \{\phi_i(x), \phi_j(y)\} = \{\pi_i(x), \pi_j(y)\} = 0$$

- Well chosen combinations of $\phi_i(x)$, $\partial_x \phi_i(x)$ and $\pi_i(x)$

  $\rightarrow$ Takiff currents $\mathcal{J}_{[0]}(x)$ and $\mathcal{J}_{[1]}(x)$ of multiplicity 2

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- Affine Gaudin model with one site, with Takiff currents $\mathcal{J}_{[0]}$ and $\mathcal{J}_{[1]}$
- Hamiltonian field theory on $T^*G \Leftrightarrow$ Lagrangian field theory on G
- Inverse Legendre transform

  $\rightarrow$ action of an integrable 2d field theory on G

Principal chiral model as affine Gaudin model

- Integrable 2d field theory on a G-valued field $g(x,t)$
- Light-cone coordinates $x^{\pm} = \frac{1}{2}(t \pm x)$ and currents $j_{\pm} = g^{-1}\partial_{\pm} g \in \mathfrak{g}$
- Action: integrable σ-model on G

$$S[g] = K \iint \mathrm{d}t\, \mathrm{d}x \; \kappa(j_+, j_-)$$

$\rightarrow$ principal chiral model

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$\rightarrow$ principal chiral model

- Change the Kac-Moody current $\mathcal{J}_{[0]}$
- Adding a Wess-Zumino term:

$$S[g] = K \iint \mathrm{d}t\,\mathrm{d}x\;\kappa(j_+, j_-) + \Bbbk\, I_{\text{WZ}}[g]$$

- PCM with WZ term: realisation of affine Gaudin model [Vicedo '17]

Integrable coupled σ-model from affine Gaudin model

- PCM with Wess-Zumino term $\leftrightarrow$ model with 1 site of multiplicity 2
- Application of the general coupling procedure
  $\rightarrow$ model with N sites of multiplicity 2

- Parameters:
  - positions $z_1, \cdots, z_N$ of the sites
  - levels $\ell_0^{(r)}$ and $\ell_1^{(r)}$

- Twist function: $\varphi(z) = \sum_{r=1}^{N} \left(\frac{\ell_0^{(r)}}{z - z_r} + \frac{\ell_1^{(r)}}{(z - z_r)^2} \right) - 1$

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- Twist function: $\varphi(z) = -\dfrac{\prod_{i=1}^{2N}(z-\zeta_i)}{\prod_{r=1}^{N}(z-z_r)^2}$
- Lorentz invariance: two types of zeroes $\zeta_i^\pm$ (parameter $\epsilon_i = \pm 1$ in $\mathcal{H}$)
- Factorisation of the twist function

$$\varphi(z) = -\varphi_+(z)\varphi_-(z), \qquad \varphi_\pm(z) = \prod_{r=1}^{N} \frac{z-\zeta_r^\pm}{z-z_r}$$

Integrable coupled σ-model

- Inverse Legendre transform $\rightarrow$ action in terms of $j_\pm^{(r)} = g^{(r)\,-1}\partial_\pm g^{(r)}$

$$S = \sum_{r,s=1}^{N} \int \mathrm{d}x\,\mathrm{d}t\ \rho_{rs}\,\kappa\left(j_+^{(r)}, j_-^{(s)}\right) + \sum_{r=1}^{N} k_r\, I_{\mathrm{WZ}}\left[g^{(r)}\right]$$

$$k_r = \frac{1}{2}\,\underset{z=z_r}{\mathrm{res}}\ \varphi_+(z)\varphi_-(z)\,\mathrm{d}z, \qquad \rho_{rs} = \frac{1}{2}\,\underset{z=z_r}{\mathrm{res}}\left(\underset{w=z_s}{\mathrm{res}}\ \frac{\varphi_+(z)\varphi_-(w)}{z-w}\,\mathrm{d}z\,\mathrm{d}w\right) - \delta_{rs}\frac{k_r}{2}$$

- Action directly related to the Hamiltonian integrable structure

- ρ_{rs} and k_r invariant under $z_r \mapsto z_r + a$ and $\zeta_r^\pm \mapsto \zeta_r^\pm + a$
  $\rightarrow 3N-1$ free parameters

- Lax pair:

$$\mathcal{L}_\pm(z) = \sum_{r=1}^{N} \frac{\varphi_{\pm,r}(z_r)}{\varphi_{\pm,r}(z)} j_\pm^{(r)}, \qquad \varphi_{\pm,r}(z) = (z-z_r)\varphi_\pm(z)$$

Conclusion

Conclusion: new classical integrable σ-models

- Integrable η and λ deformation of the PCM as AGM?
  - $\rightarrow$ split double pole into two simple poles
  - $\rightarrow$ change realisation
- Integrable deformation of the coupled σ-model with N copies
  - $\rightarrow$ whole panorama of models to explore
    (first results in [Delduc SL Magro Vicedo '19] and [SL '19])
  - $\rightarrow$ contains and extends the models of [Georgiou Sfetsos '18]?
- Integrable σ-model on symmetric space G/H (or $\mathbb{Z}_T$-coset)
  - $\rightarrow$ cyclotomic affine Gaudin model with 1 site
  - $\rightarrow$ gauge symmetry
- Model with N sites:
  - $\rightarrow$ conjecture: integrable σ-model on $G \times \cdots \times G/H_{\text{diag}}$

Conclusion: quantum level

- Quantisation of integrable coupled σ-models
- Renormalisation group flow
  $\rightarrow$ preserves integrability ?
  $\rightarrow$ CFT fixed points ? rich structure in [Georgiou Sfetsos '18] for deformed models
- S-matrix ?

- Quantum integrability ?
- Non-ultralocality $\rightarrow$ QISM does not apply
- Quantisation via Gaudin models
  - finite case well understood (description of the spectrum via opers, relation to Hitchin systems and geometric Langlands correspondence)
  - progresses in the affine case by analogy (using affine opers) [Feigin Frenkel '07, SL Vicedo Young '18 '18]
  - relation with ODE/IM correspondence ? [Dorey Tateo '99, Bazhanov Lukyanov Zamolodchikov '01]

4d Chern-Simons theory

- 4d Chern-Simons theory [Costello '13]
- Generate integrable systems:
  - Spin chains [Costello, Witten, Yamazaki '17 '18]
  - 2d field theories [Costello, Yamazaki '19]

- Order defects

  $\rightarrow$ Gross-Neuveu and Thirring models, ...
- Disorder defects

  $\rightarrow$ PCM, symmetric space models, coupled PCMs, ...

- Hamiltonian analysis of the models with disorder defects [Vicedo '19]

  $\rightarrow$ affine Gaudin models

Thank you !