APS η-invariant, path integrals, and mock modularity

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Based on work with Atish Dabholkar and Arnab Rudra

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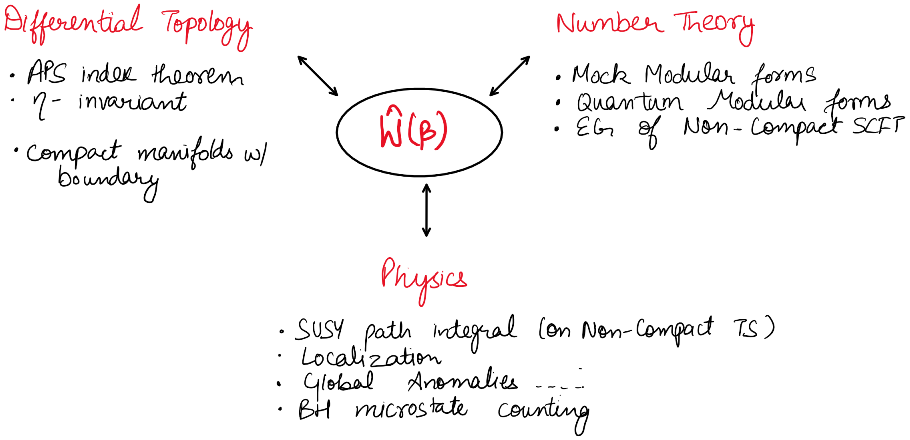
Aim


Differential Topology
• APS index theorem
• η- invariant
• Compact manifolds w/ boundary
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Number Theory
• Mock Modular forms
• Quantum Modular forms
• EG of Non-Compact SCFT
Physics
• SUSY path integral (on Non-Compact TS)
• Localization
• Global Anomalies ....
• BH microstate counting

Index Theorem

Index of a differential operator $\mathcal{D}$ on a manifold $\mathcal{M}$ is defined as

$$\text{index}(\mathcal{D}) = \text{dim Ker}(\mathcal{D}) - \text{dim Co-Ker}(\mathcal{D})$$

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The manifold $\mathcal{M}$ can be:

Compact with no boundary e.g. sphere $\longrightarrow$ Atiyah-Singer index theorem

Proof using SUSY QM

[Witten, Alvarez-Gaume, Friedan, Windey]

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The manifold $\mathcal{M}$ can be:

Non-compact e.g. $\mathbb{R}^n \longrightarrow$ Callais index theorem
Proof using SUSY QM for some cases known.

[Imbimbo, Mukhi '84]

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The manifold $\mathcal{M}$ can be:

Compact with boundary e.g. Disc $\longrightarrow$ Atiyah-Patodi-Singer index theorem.

SUSY QM ???

Motivation

- Index theorems have a lot of physics applications e.g. Chiral Anomaly.
- Proof of APS index theorem using SUSY QM was a long standing problem.
- η-invariant has a number of interesting physics applications
    - Parity Anomaly
    - Phases of topological insulators
- Mock modular forms and their cousins play an important role in the black hole microstate counting, more generally in the context of elliptic genera of noncompact SCFTs.

Outline

- **Part I**
  - Review of Atiyah-Singer (AS) and Atiyah-Patodi-Singer (APS) index theorems.
  - Re-derive APS index theroem using scattering theory in supersymmetric quantum mechanics and give a path integral prescription to compute APS index.
- **Part II**
  - Mock Modular forms and non-compact SCFT
  - Relation to Quantum Modular Forms.

Part I

Atiyah- Patodi- Singer Index Theorem

Atiyah-Singer index theorem

On a compact manifold $\mathcal{M}$ without a boundary

$$index(\not{D}) \;\; = \;\; n_+ - n_- = \int_{\mathcal{M}} \alpha(x)$$

Atiyah-Singer index theorem

On a compact manifold $\mathcal{M}$ without a boundary

$$\begin{aligned} index(\not{D}) &= n_+ - n_- = \int_{\mathcal{M}} \alpha(x) \\ &= \frac{1}{32\pi^2}\int_{\mathcal{M}} tr\, F \wedge F \end{aligned}$$

Atiyah-Singer index theorem

On a compact manifold $\mathcal{M}$ without a boundary

$$
\begin{aligned}
index(\not{D}) &= n_+ - n_- = \int_{\mathcal{M}} \alpha(x) \\
&= -\frac{1}{24}\int_{\mathcal{M}} \frac{tr\, R \wedge R}{16\pi^2}
\end{aligned}
$$

Atiyah-Singer index theorem

On a compact manifold $\mathcal{M}$ without a boundary

$$index(\not{D}) \;=\; n_+ - n_- = \int_{\mathcal{M}} \alpha(x)$$

Topological quantity
Independent of any regulator.

Supersymmetric Quantum Mechanics

Consider SUSY QM with one real supercharge

$$I = \frac{1}{2}\int dt \left[g_{ij}(x)\frac{dx^i}{dt}\frac{dx^j}{dt} + i\,\delta_{ab}\psi^a\left(\frac{d\psi^b}{dt} + \omega_{akb}\frac{dx^k}{dt}\psi^b\right)\right]$$

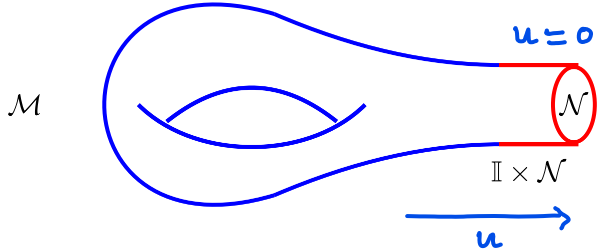
$$\begin{aligned} \not{D} &\leftrightarrow Q \\ \not{D}^2 &\leftrightarrow H \\ \gamma^{2n+1} &\leftrightarrow (-1)^F \\ index(\not{D}) &\leftrightarrow W(\infty) = W(0) \end{aligned}$$

where

$$W(\beta) = Tr_{\mathcal{H}}\,(-1)^F e^{-\beta H}$$

Atiyah-Patodi-Singer index theorem

Consider a compact manifold $\mathcal{M}$ with boundary $\partial\mathcal{M} = \mathcal{N}$ where $\mathcal{N}$ is a compact, connected, oriented manifold with no boundary.



The variation of the Dirac action is (roughly) of the form

$$\delta S = \int_{\partial M} [\psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_-]$$

One can impose local boundary conditions

$$\psi_+|_{\partial M} = \pm\psi_-|_{\partial M}$$

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These do not preserve chirality.
Not good for index problem.

APS boundary conditions

The Dirac operator near the boundary becomes

$$\begin{aligned} \not{D} &= \gamma^u \partial_u + \gamma^m D_m \\ &= \gamma^u (\partial_u \; + \bar{\gamma} \mathcal{B}) \end{aligned}$$

where $\mathcal{B} = \widehat{\gamma}^m D_m$ is the boundary Dirac operator. The eigenfunctions can be written as

$$\Psi_-(u, y) = \sum_\lambda \Psi_-^\lambda(u)\, e_\lambda(y)$$

$$\Psi_+(u, y) = \sum_\lambda \Psi_+^\lambda(u)\, e_\lambda(y)$$

where $\{e_\lambda(y)\}$ are the complete set of eigenmodes of $\mathcal{B}$.

APS boundary conditions amounts to Dirichlet boundary condition for half the modes:

$$\begin{aligned} \Psi_{+}^{\lambda}(0) &= 0 \qquad \forall \quad \lambda < 0 \\ \Psi_{-}^{\lambda}(0) &= 0 \qquad \forall \quad \lambda > 0\,. \end{aligned}$$

APS boundary conditions amounts to Dirichlet boundary condition for half the modes:

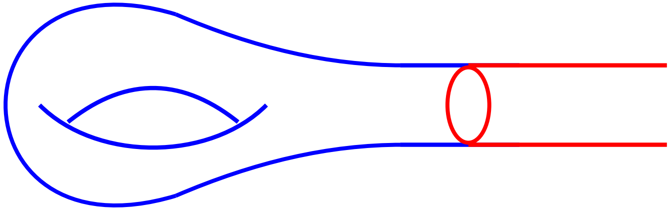
$$\begin{aligned}\Psi_+^\lambda(0) &= 0 \qquad \forall \quad \lambda < 0\\ \Psi_-^\lambda(0) &= 0 \qquad \forall \quad \lambda > 0\,.\end{aligned}$$

For the remaining half, one uses Robin boundary conditions

$$\begin{aligned}\frac{d\Psi_+^\lambda}{du}(0) + \lambda\Psi_+^\lambda(0) &= 0 \qquad \forall \quad \lambda > 0\\ -\frac{d\Psi_-^\lambda}{du}(0) + \lambda\Psi_-^\lambda(0) &= 0 \qquad \forall \quad \lambda < 0\,.\end{aligned}$$

APS boundary condtion $\leftrightarrow$ L_2 normalizability on $\widehat{\mathcal{M}}$

$\widehat{\mathcal{M}}$



The APS index theorem states that the index of Dirac operator with APS boundary conditions on the compact Riemannian manifold $\mathcal{M}$ with boundary $\mathcal{N}$ is given by

$$\mathcal{I} = \int_{\mathcal{M}} \alpha(x) - \frac{1}{2}\eta$$

Integer ← $\mathcal{I}$; $\int_{\mathcal{M}} \alpha(x)$: Not integer; $\frac{1}{2}\eta$: Not integer

where

$$\eta = \sum_{\lambda} \mathrm{sgn}(\lambda)$$

It measures the spectral asymmetry of the boundary operator $\mathcal{B}$ on $\mathcal{N}$.

η- invariant

- Ambiguity for $\lambda = 0$.
- Different regularizations

$$\widehat{\eta}(\beta) = \sum_{\lambda} \operatorname{sgn}(\lambda)\, \operatorname{erfc}\left(|\lambda|\sqrt{\beta}\right)$$

$$\eta(s) = \sum_{\lambda} \frac{\lambda}{|\lambda|^{s+1}} = \sum_{\lambda} \frac{\operatorname{sgn}(\lambda)}{|\lambda|^{s}}$$

- The index is semi -topological i.e. it can change under smooth deformations of the boundary.

Can one find a path integral derivation of APS index theorem??

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Putting a field space boundary condition in path integral formalism is very difficult.

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Putting a field space boundary condition in path integral formalism is very difficult.

We relate the APS index to Witten index with non-compact target space.

Non-Compact Witten Index

- We will express the Dirac index on $\mathcal{M}$ in terms of the noncompact Witten index $\widehat{W}$ on $\widehat{\mathcal{M}}$.
- The spectrum will contain a continuum of scattering states as well as L_2-normalizable states.
- The operator $\widehat{W} = \operatorname*{Tr}_{\mathcal{H}} (-1)^F e^{-\beta \widehat{H}}$ is not well-defined.
- A natural formalism for this purpose is provided by 'Rigged Hilbert space' or 'Gel'fand triplet' which generalizes the Von Neumann formulation of quantum mechanics based on a Hilbert space.

- The Gel'fand triplet consists of a Hilbert space $\mathcal{H}$, the Schwartz space $\mathcal{S}$, and the conjugate Schwartz space $\mathcal{S}^\times$.
- The Hilbert space $\mathcal{H}$ is isomorphic to the space $L_2(dx, \mathbb{R})$ of square-integrable wave functions on $\mathbb{R}$:

$$\mathcal{H} = \{|\psi\rangle\} \qquad \text{with} \qquad \langle\psi|\psi\rangle := \int dx\, \psi^*(x)\psi(x) < \infty\,;$$

- The Schwartz space is the space of infinitely differentiable 'test functions' with exponential fall off. The conjugate Schwartz space $\mathcal{S}^\times$ is the set $\{|\phi\rangle\}$ such that

$$|\phi\rangle \in \mathcal{S}^\times \qquad \text{if} \qquad \langle\psi|\phi\rangle < \infty \qquad \forall \quad |\psi\rangle \in \mathcal{S}\,.$$

$$\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}^\times$$

- The conjugate Schwartz space $\mathcal{S}^{\times}$ is where objects like the Dirac delta distribution $\delta(x)$ and plane waves e^{ipx} reside. They have finite overlap with 'test functions' belonging to $\mathcal{S}$.
- One can define the noncompact Witten index by

$$\widehat{W}(\beta) := \operatorname*{Tr}_{SP} \left[(-1)^F e^{-\beta \widehat{H}}\right]$$

- One might worry that the noncompact Witten index is divergent.
- It is a supertrace so if there is a gap between the ground states and the scattering states, then the bosonic and fermionic Hamiltonians differ from each other only over a region with compact support in $\mathbb{R}$. Hence it yields a finite answer.

Derivation of APS index theorem

States in $\mathcal{M}$ satisfying APS boundary condition $\leftrightarrow$ L_2 normalizable states on $\widehat{\mathcal{M}}$.

Assuming that the continuum states of $\mathcal{M}$ are separated from the ground states by a gap,

$$\mathcal{I}_{\mathcal{M}} \quad = \quad \widehat{W}(\infty)$$

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Consider

$$\widehat{W}(0) = \int_{\widehat{\mathcal{M}}} \alpha = \int_{\mathcal{M}} \alpha$$

Derivation of APS index theorem

States in $\mathcal{M}$ satisfying APS boundary condition $\leftrightarrow$ L_2 normalizable states on $\widehat{\mathcal{M}}$.

Assuming that the continuum states of $\mathcal{M}$ are separated from the ground states by a gap,

$$\mathcal{I}_{\mathcal{M}} = \widehat{W}(\infty)$$

$$\mathcal{I}_{\mathcal{M}} = \widehat{W}(0) + (\widehat{W}(\infty) - \widehat{W}(0))$$

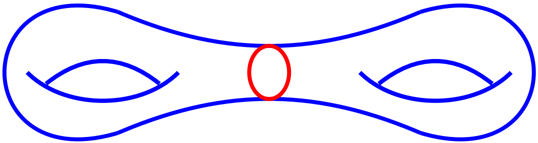
We identify

$$\widehat{\eta}(\beta) := 2(\widehat{W}(\beta) - \widehat{W}(\infty))$$

which in the limit $\beta \to 0$ reduces APS η-invariant.

AS piece

There is a simpler way to compute $\widehat{W}(0)$. One can simply double the manifold to $\overline{\mathcal{M}}$ by gluing its copy as below

$\overline{\mathcal{M}}$ 

Since $\overline{\mathcal{M}}$ is a manifold without a boundary, there is no contribution from the η-invariant. We get,

$$\widehat{W}(0) = \frac{1}{2}\overline{W}(0) = \int_{\mathcal{M}} \alpha$$

Boundary piece

$$\begin{aligned}
\widehat{\eta}(\beta) &:= 2(\widehat{W}(\beta) - \widehat{W}(\infty)) \\
\widehat{W}(\beta) &= \operatorname*{Tr}_{SP_b}\left[(-1)^F e^{-\beta\widehat{H}}\right] + \operatorname*{Tr}_{SP_s}\left[(-1)^F e^{-\beta\widehat{H}}\right]
\end{aligned}$$

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$$2\operatorname*{Tr}_{SP_s}\left[(-1)^F e^{-\beta \widehat{H}}\right] = \widehat{\eta}(\beta)$$

- This supertrace is equal to the difference in the density of bosonic and fermionic scattering states.
- This can be related to the difference in phase shifts of fermionic and bosonic wavefunction.

Boundary piece

$$\widehat{\eta}(\beta) := 2(\widehat{W}(\beta) - \widehat{W}(\infty))$$

$$\widehat{W}(\beta) = \operatorname*{Tr}_{SP_b}\left[(-1)^F e^{-\beta\widehat{H}}\right] + \operatorname*{Tr}_{SP_s}\left[(-1)^F e^{-\beta\widehat{H}}\right]$$

$$2\operatorname*{Tr}_{SP_s}\left[(-1)^F e^{-\beta\widehat{H}}\right] = \widehat{\eta}(\beta)$$

$$\widehat{\eta}(\beta) = 2\sum_{\lambda}\int dk\left[\rho_{+}^{\lambda}(k) - \rho_{-}^{\lambda}(k)\right]e^{-\beta E(k)}$$

The asymptotic form of the scattering wave functions is then

$$\begin{aligned}\psi_{+}^{\lambda k}(u) &\sim c_{+}^{\lambda}\left[e^{iku}+e^{i\delta_{+}^{\lambda}(k)-iku}\right]\\ \psi_{-}^{\lambda k}(u) &\sim c_{-}^{\lambda}\left[e^{iku}+e^{i\delta_{-}^{\lambda}(k)-iku}\right]\end{aligned}$$

where $\delta_{\pm}^{\lambda}(k)$ are the phase shifts.

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where $\delta_\pm^\lambda(k)$ are the phase shifts.

$$\rho_+^\lambda(k) - \rho_-^\lambda(k) = \frac{1}{\pi}\frac{d}{dk}\left[\delta_+^\lambda(k) - \delta_-^\lambda(k)\right] .$$

SUSY determines this difference in terms of asymptotic data.

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where $\delta_{\pm}^{\lambda}(k)$ are the phase shifts.
SUSY determines this difference in terms of asymptotic data.

$$2\delta_{+}^{\lambda}(k) - 2\delta_{-}^{\lambda}(k) = -i\ln\left(\frac{ik+\lambda}{ik-\lambda}\right) + \pi$$

in each eigensubspace with eigenvalue λ.

$$\widehat{\eta(\beta)} = 2\mathop{\mathrm{Tr}}_{SP_s}\left[(-1)^{F} e^{-\beta\widehat{H}}\right] = \sum_{\lambda} \mathrm{sgn}(\lambda)\,\mathrm{erfc}\left(|\lambda|\sqrt{\frac{\beta}{2}}\right)$$

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We have thus proven

$$\mathcal{I} = \widehat{W}(0) + (\widehat{W}(\infty) - \widehat{W}(0)) = \int_{\mathcal{M}} \alpha - \frac{1}{2}\eta$$

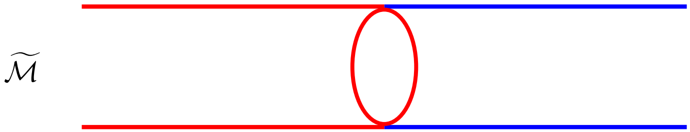
which is the Atiyah-Patodi-Singer index theorem.

η-invariant from Path integral

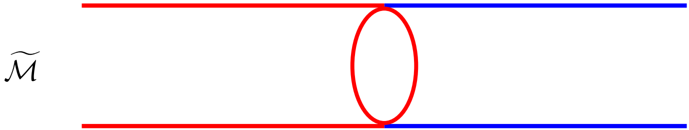
- η-invariant gets contribution only from difference in densities of the scattering states of $\widehat{\mathcal{M}}$ which depends only on asymptotic data.
- One can consider path integral on $\mathbb{R}^+ \times \mathcal{N}$ with APS boundary condition at the origin.
- This half line problem can be mapped to a computation on $\mathbb{R}$ by extending it in a manner consistent with the APS boundary conditions.

$$P: \qquad u \to -u \quad , \quad \mathcal{B} \to -\mathcal{B}$$

[Troost '17]



- Decompose the boundary operator into eigenvalues ($\{\lambda\}$); each eigenvalue can be modelled by adding a super-potential in the worldline QM.



- Decompose the boundary operator into eigenvalues ($\{\lambda\}$); each eigenvalue can be modelled by adding a super-potential in the worldline QM.
- The worldline lagrangian is given by:

$$L = \frac{1}{2}\dot{u}^2 + \frac{1}{2}\psi_-\dot{\psi}_- + \frac{1}{2}\psi_+\dot{\psi}_+ + \frac{1}{2}F^2 + ih'(u)F + ih''(u)\psi_-\psi_+$$

with $h'(u) = \lambda \tanh u$

APS index theorem and SQM

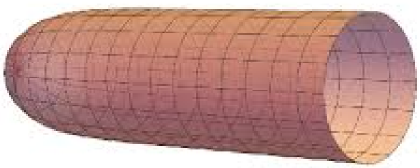
$$\mathcal{I} = \widehat{W}(0) + (\widehat{W}(\infty) - \widehat{W}(0)) = \int_{\mathcal{M}} \alpha - \frac{1}{2}\eta$$

η- invariant of finite Cigar

Cigar geometry is given by

$$ds^2 = k\,(d\rho^2 + \tanh^2\rho\, d\psi^2)$$

where $\rho = \rho_0$ is the boundary and ψ is a periodic direction with period 2π.



The Dirac operator near the boundary takes the form

$$\begin{aligned} i\not{D} &= \gamma^r(i\partial_r - w\,K_r) + \gamma^\theta(i\partial_\theta - w\,K_\theta) \\ &= i\gamma^r\left[\partial_r - \frac{1}{\tanh r}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}(i\partial_\theta - w\,k\tanh^2 r)\right] \end{aligned}$$

We find the boundary operator

$$\mathcal{B} = -(i\partial_\theta - w\,k)\,.$$

The η-invariant is then given by:

$$\eta = \sum_{n\in\mathbb{Z}} \mathrm{sgn}(w\,k - n) = \sum_{n\in\mathbb{Z}} \mathrm{sgn}\left(w - \frac{n}{k}\right)$$

The η- invariant can also be computed from the Witten index on non-compact cigar.

$$\widehat{W}(\beta) = \sum_n e^{-\beta n w}\Bigg[-\frac{1}{2}\text{sgn}\left(\frac{n}{k} - w\right)\text{erfc}\left(\sqrt{\frac{k\beta}{2}}\left|\frac{n}{k} - w\right|\right) + \text{sgn}(\beta n)\,\Theta\left[w\left(\frac{n}{k} - w\right)\right] + \text{sgn}(w\beta)\Theta(n\beta\,\text{sgn}(w\beta))\Bigg]$$

We obtain

$$\eta(0) = 2(\widehat{W}(0) - \widehat{W}(\infty)) = \sum_n \text{sgn}\left(w - \frac{n}{k}\right)$$

Part II

Mock-Modularity

Elliptic Genus

- EG is a generalization of Witten Index when the base manifold is $2D$
- Consider a $(1+1)D$ SUSY sigma model living on a torus of modular parameter τ, then the Elliptic Genus is defined as:

$$\chi(q,z) = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{2\pi i \tau H_L} e^{-2\pi i \bar{\tau} H_R} e^{2\pi i z J_L} \tag{1}$$

[Witten '87]

where
$F = F_L + F_R$ is the total fermion number
J_L is the R charge of the left movers

- It is a twisted partition function i.e. the boundary conditions of various fields are twisted due to J_0

Holomorphy and Elliptic genus

- The Elliptic genus for a SCFT with compact target space is independent of $\bar{\tau}$.
- Elliptic genus, in this case, is a Jacobi form.

Holomorphy and Elliptic genus

- The Elliptic genus for a SCFT with compact target space is independent of $\bar{\tau}$.
- Elliptic genus, in this case, is a Jacobi form.
- For an SCFT with non-compact target space, EG is not independent of $\bar{\tau}$.
- It is expected to be a Mock Jacobi form.

[Dabholkar, Murthy, Zagier; Pioline; Bringmann...]

Mock Modular Form

A Mock Modular form of weight k is the first member of a pair of functions (h, g) such that

- h is a holomorphic functions in τ but it is not modular.
- g is a holomorphic modular form of weight $2 - k$.
- The sum

$$\hat{h}(\tau, \bar{\tau}) = h(\tau) + g^*(\tau, \bar{\tau})$$

  is modular where the function g^* is given by:

$$(4\pi\tau_2)^k \frac{\partial g^*}{\partial \bar{\tau}} = -2\pi i \overline{g(\tau)}$$

- $g(\tau)$ is called the shadow of the mock modular form h.
- Above equation is called the Holomorphic anomaly equation.

Let us compare two formulae

$$\text{Index}(\not{D}) = \int_{\mathcal{M}} \alpha(x) - \frac{1}{2}\eta$$

$$\hat{h}(\tau, \bar{\tau}) = h(\tau) + g^*(\tau, \bar{\tau})$$

Cigar Elliptic Genus

- We do not know if there is some connection in general.
- We noticed the connection in case of Cigar target space.
- Lots of work on Cigar Elliptic Genus by various people

  [Ashok, Doroud, Troost; Eguchi, Sugarwa; Harvey, Lee; Murthy...]
- We computed it using path integral in SUSY QM.

The full elliptic genus is given by

$$\begin{aligned}\widehat{\chi}(\tau,\bar{\tau}|z) &= \operatorname*{Tr}_{\mathcal{H}} (-1)^{\tilde{J}} e^{-2\pi\tau_2(L_0+\tilde{L}_0)} e^{2\pi i\tau_1(L_0-\tilde{L}_0)} e^{2\pi i z J} \\ &= \widehat{W}(2\pi\tau_2)\cdot \mathcal{Z}_{\text{oscill}}\; e^{2\pi i\tau_1 m w}\; e^{2\pi i z J}\end{aligned}$$

We find that the elliptic genus for the cigar is given by

$$\begin{aligned}\widehat{\chi}(\tau,\bar{\tau}|z) &= -i\frac{\vartheta_1(\tau,z)}{\eta^3(\tau)}\sum_w\sum_n\Bigg[\frac{1}{2}\mathrm{sgn}\left(\frac{n}{k}-w\right)\mathrm{erfc}\left(\sqrt{k\pi\tau_2}\left|w-\frac{n}{k}\right|\right) \\ &\quad -\mathrm{sgn}(\beta n)\,\Theta\left[w\left(\frac{n}{k}-w\right)\right]\Bigg] q^{-(n-wk)^2/4k} q^{(n+wk)^2/4k} y^{J_L}\end{aligned}$$

It is proportional to the completion of Appell-Lerch sum $\mathcal{A}_{1,k}\left(\tau,\frac{z}{k}\right)$.

- The EG of Cigar SCFT is a Mock Jacobi form of weight $1/2$.
- Notice that the g^* in this case is proportional to the regulated η-invariant of Cigar.
- Without the phases, the second piece vanishes, which is consistent with the fact that AS piece for Cigar vanishes.

- By looking at the definition of η-invariant in terms of non-compact Witten index, we can define 'character-valued' η-invariant of the elliptic genus

$$\eta(\tau_1|z) = \mathcal{Z}_{\text{oscill}} \sum_{w} 2 \left(\widehat{W}(0) - \widehat{W}(\infty) \right) e^{2\pi i \tau_1 (L_0 - \bar{L}_0)} y^J \,.$$

- The 'character-valued η-genus' takes the form

$$\eta(\tau_1|z) = -2i \frac{\vartheta_1(\tau_1, z)}{\eta(\tau_1)^3} \left[\sum_{l \in \mathbb{Z}/2k\mathbb{Z}} f_\ell(\tau_1) \vartheta_{k,l} \left(\tau_1, \frac{z}{k} \right) + \mathcal{A}_{1,k} \left(\tau_1, \frac{z}{k} \right) \right] .$$

where f_ℓ is a Quantum Modular form.

Future Directions

- We would like to generalize the proof of APS theorem for the manifolds which have non-product metric near the boundary.
- We would like to see if the relation between Mock modularity and APS index can be made more precise.

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- We would like to generalize the proof of APS theorem for the manifolds which have non-product metric near the boundary.
- We would like to see if the relation between Mock modularity and APS index can be made more precise.
- For the cases when the Witten index is zero, one can define more refined indices which tell about SUSY breaking.
  We would like to explore the relations between scattering theory and these refined indices.

Thank You!