Bootstrapping Ising,O(N) and related models

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Overview

- Review
- Why multi-operators? What are the challenges?
- New techniques
- O(2) result
- Other ongoing and future works

Bootstrap idea

Bootstrap: making predictions for physical observables from general principles of symmetry and quantum mechanics.

Basic ingredients for conformal field theory

Assumption 1, Conformal Symmetry

Fields should form representation of conformal group. Our fields are labelled by two quantum numbers: l (spin), Δ (conformal weight).

Conformal symmetry fixes the form of 2pt and 3pt function up to a constant.

For example for scalar:

$\langle \phi(x)\,\phi(y)\rangle = \frac{1}{|x-y|^{2\Delta_\phi}}$

$\langle \phi_1(x)\,\phi_2(y)\,\phi_3(z)\rangle = \lambda_{123}\,\frac{1}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}\,|y-z|^{\Delta_2+\Delta_3-\Delta_1}\,|z-x|^{\Delta_3+\Delta_1-\Delta_2}}$

Basic ingredients for conformal field theory

Assumption 2: Operator Product Expansion (OPE)

$\phi_i(x)\,\phi_j(y) = \sum_k \lambda_{i\,j\,k}$ (sth. depends on $x - y$ and ∂_y) $O_k(y)$ where O_k is primary field

The key point is that the structure in (...) is fixed by conformal symmetry. Let's write

$\phi_i(x)\,\phi_j(y) = \sum_a \lambda_{i\,j\,k}\, C_k(x - y, \partial_y)\, O_k(y)$ (summation over primary field)

Conformal partial wave decomposition:

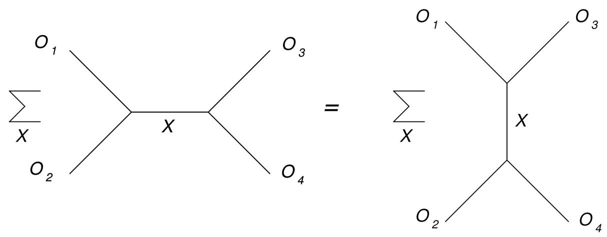
$\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle$
$= \sum \lambda_{12\,O}{}^2\, C_a(x_1 - x_2, \partial_2)\, C_b(x_3 - x_4, \partial_4) \left\langle O^a(x_2)\, O^b(x_4)\right\rangle$
$\equiv \sum_O \lambda_{12\,O}{}^2\, x_{12}{}^{-\Delta_\phi}\, x_{34}{}^{-\Delta_\phi}\, g_{\Delta,l}(u, v)$
where
$u = \frac{x_{12}{}^2\, x_{34}{}^2}{x_{13}{}^2\, x_{24}{}^2}$, $v = \frac{x_{14}{}^2\, x_{23}{}^2}{x_{13}{}^2\, x_{24}{}^2}$

$g_{\Delta,l}(u, v) \equiv x_{12}{}^{\Delta_\phi}\, x_{34}{}^{\Delta_\phi}\, C_a(x_1 - x_2, \partial_2)\, C_b(x_3 - x_4, \partial_4) \left\langle O^a(x_2)\, O^b(x_4)\right\rangle$ is the conformal block for operator O with conformal weight Δ and spin l.

Basic ingredients of conformal field theory

Assumption 3: Crossing symmetry
(the OPE is associative)

$$\langle \overline{\phi(x_1)\,\phi(x_2)}\;\overline{\phi(x_3)\,\phi(x_4)}\rangle = \langle \overline{\phi(x_1)\,\phi(x_3)}\;\overline{\phi(x_2)\,\phi(x_4)}\rangle$$



$$u^{-\Delta_\phi} \sum_{O\in\phi\times\phi} \lambda_{\phi\phi O}{}^2\, g_{\Delta,l}(u,\, v) = v^{-\Delta_\phi} \sum_{O\in\phi\times\phi} \lambda_{\phi\phi O}{}^2\, g_{\Delta,l}(v,\, u)$$

Define convolved conformal block $F_{\pm,\Delta,l} = u^{\Delta_\phi}\, g_{\Delta,l}(u,\, v) \pm v^{\Delta_\phi}\, g_{\Delta,l}(v,\, u)$.

We have the bootstrap equation $\sum_{O\in\phi\times\phi} \lambda_{\phi\phi O}{}^2\, F_{-,\Delta,l}(u,\, v) = 0$

Unitarity in CFTs

Unitarity: all the states in Hilbert space have positive norm.

Fact 1: Unitarity bound
For $l = 0$, $\Delta \geq \frac{\text{dim}}{2} - 1$ or $\Delta = 0$
For $l > 0$, $\Delta \geq \text{dim} + l - 2$

Fact 2: For non-unitary CFT, some operators has negative norm. If we insist to normalize $\langle \phi(x)\, \phi(y) \rangle = \frac{1}{|x-y|^{2\Delta_\phi}}$ (we did this in defining the conformal block), then some OPE coefficient λ^2 is negative.

For unitary CFT, $\lambda^2 \geq 0$.

Basic logic of conformal bootstrap

$\sum_{O\in\phi\times\phi} \lambda_{\phi\phi O}{}^{2} F_{\Delta,l}(u, v) = 0$

Let a linear functional α satisfy

$\alpha(F_{0,0}(u, v)) = 1$
$\alpha(F_{\Delta,l=0}(u, v)) \geq 0$ for $\Delta \geq \Delta_0$
$\alpha(F_{\Delta,l}(u, v)) \geq 0$ for $l > 0$, $\Delta \geq \Delta_{\text{unitary}}$

Note: this problem depends on two numbers that we choose: Δ_0, Δ_ϕ (and spacetime dimension)

If I found such α ...

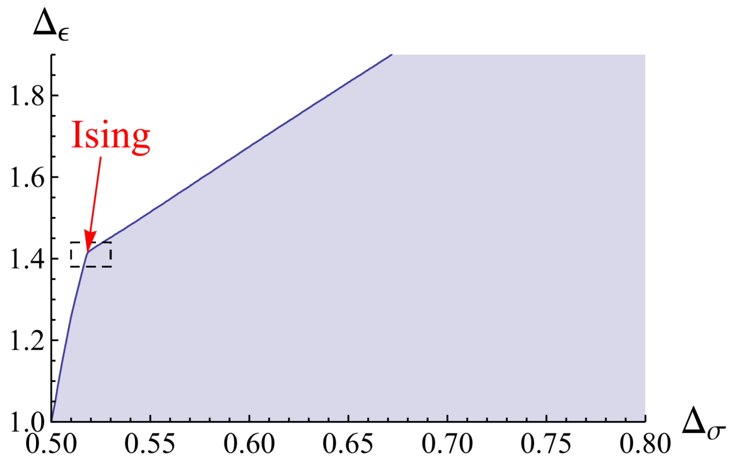
Assume there is CFT with first scalar in $\phi\times\phi$ OPE has dimension $\Delta > \Delta_0$. Applying α to both side of the bootstrap equation, we will found LHS > 0 but RHS = 0.

Therefore the lowest operator must have dimension lower then Δ_0 . We call this method Feasibility Test / Positivity Test .

This problem can be approximated by semidefinite program (SDP). We can use the software SDPB to solve it.

3D Ising model

Let's rename Δ_0 to be Δ_ϵ and Δ_ϕ to be Δ_σ . Do this feasibility test in 3D:



(From El Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi arXiv:1203.6064)

Working with mixed correlator system

Consider CFT with $\mathbb{Z}_2$ symmetry (parity). Assuming σ is $\mathbb{Z}_2$ odd, ϵ is $\mathbb{Z}_2$ even.

Exchanging 1st and 3rd operators in the correlator, we have following equations:

(1), $\langle\sigma\sigma\sigma\sigma\rangle$:
$\sum_O \lambda_{\sigma\sigma O}{}^2 F^{\sigma\sigma,\sigma\sigma}_{-,\Delta,l} = 0$

(2), $\langle\epsilon\epsilon\epsilon\epsilon\rangle$:
$\sum_O \lambda_{\epsilon\epsilon O}{}^2 F^{\epsilon\epsilon,\epsilon\epsilon}_{-,\Delta,l} = 0$

(3), $\langle\sigma\epsilon\sigma\epsilon\rangle$:
$\sum_O \lambda_{\sigma\epsilon O}{}^2 F^{\sigma\epsilon,\sigma\epsilon}_{-,\Delta,l} = 0$

(4), $\langle\epsilon\sigma\sigma\epsilon\rangle$:
$\sum_O \lambda_{\epsilon\sigma O} F^{\epsilon\sigma,\sigma\epsilon}_{\pm,\Delta,l}\, \lambda_{\sigma\epsilon O} \mp \lambda_{\sigma\sigma O} F^{\sigma\sigma,\epsilon\epsilon}_{\pm,\Delta,l}\, \lambda_{\epsilon\epsilon O} = 0$

In matrix form:

$$(\lambda_{\sigma\sigma O}\ \lambda_{\epsilon\epsilon O})\begin{pmatrix} F^{\sigma\sigma,\sigma\sigma}_{-,\Delta,l} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix}\lambda_{\sigma\sigma O}\\ \lambda_{\epsilon\epsilon O}\end{pmatrix} + \lambda_{\sigma\epsilon O}{}^2 (0) = 0$$
$$(\lambda_{\sigma\sigma O}\ \lambda_{\epsilon\epsilon O})\begin{pmatrix} 0 & 0 \\ 0 & F^{\epsilon\epsilon,\epsilon\epsilon}_{-,\Delta,l} \end{pmatrix}\begin{pmatrix}\lambda_{\sigma\sigma O}\\ \lambda_{\epsilon\epsilon O}\end{pmatrix} + \lambda_{\sigma\epsilon O}{}^2 (0) = 0$$
$$(\lambda_{\sigma\sigma O}\ \lambda_{\epsilon\epsilon O})\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix}\lambda_{\sigma\sigma O}\\ \lambda_{\epsilon\epsilon O}\end{pmatrix} + \lambda_{\sigma\epsilon O}{}^2 \left(F^{\sigma\epsilon,\sigma\epsilon}_{-,\Delta,l}\right) = 0$$
$$(\lambda_{\sigma\sigma O}\ \lambda_{\epsilon\epsilon O})\begin{pmatrix} 0 & \frac{1}{2}F^{\sigma\sigma,\epsilon\epsilon}_{\pm,\Delta,l} \\ \frac{1}{2}F^{\sigma\sigma,\epsilon\epsilon}_{\pm,\Delta,l} & 0 \end{pmatrix}\begin{pmatrix}\lambda_{\sigma\sigma O}\\ \lambda_{\epsilon\epsilon O}\end{pmatrix} \mp (-1)^l \lambda_{\sigma\epsilon O}{}^2 \left(F^{\epsilon\sigma,\sigma\epsilon}_{\pm,\Delta,l}\right) = 0$$

Let's define

$$\vec{V}_{+,\Delta,l} = \begin{pmatrix} \begin{pmatrix} F^{\sigma\sigma,\sigma\sigma}_{-,\Delta,l} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & F^{\epsilon\epsilon,\epsilon\epsilon}_{-,\Delta,l} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2}F^{\sigma\sigma,\epsilon\epsilon}_{-,\Delta,l} \\ \frac{1}{2}F^{\sigma\sigma,\epsilon\epsilon}_{-,\Delta,l} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2}F^{\sigma\sigma,\epsilon\epsilon}_{+,\Delta,l} \\ \frac{1}{2}F^{\sigma\sigma,\epsilon\epsilon}_{+,\Delta,l} & 0 \end{pmatrix} \end{pmatrix}, \qquad \vec{V}_{-,\Delta,l} = \begin{pmatrix} 0 \\ 0 \\ F^{\sigma\epsilon,\sigma\epsilon}_{-,\Delta,l} \\ (-1)^l F^{\epsilon\sigma,\sigma\epsilon}_{-,\Delta,l} \\ -(-1)^l F^{\epsilon\sigma,\sigma\epsilon}_{+,\Delta,l} \end{pmatrix}$$

$$\sum_{O^+} (\lambda_{\sigma\sigma O}\ \lambda_{\epsilon\epsilon O})\, \vec{V}_{+,\Delta,l} \begin{pmatrix}\lambda_{\sigma\sigma O}\\ \lambda_{\epsilon\epsilon O}\end{pmatrix} + \sum_{O^-} \lambda^2_{\sigma\epsilon O}\, \vec{V}_{-,\Delta,l} = 0$$

Then positivity constrains can state as: find a linear functional $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ such that
$\alpha \cdot V_+ \geq 0$
$\alpha \cdot V_- \geq 0$

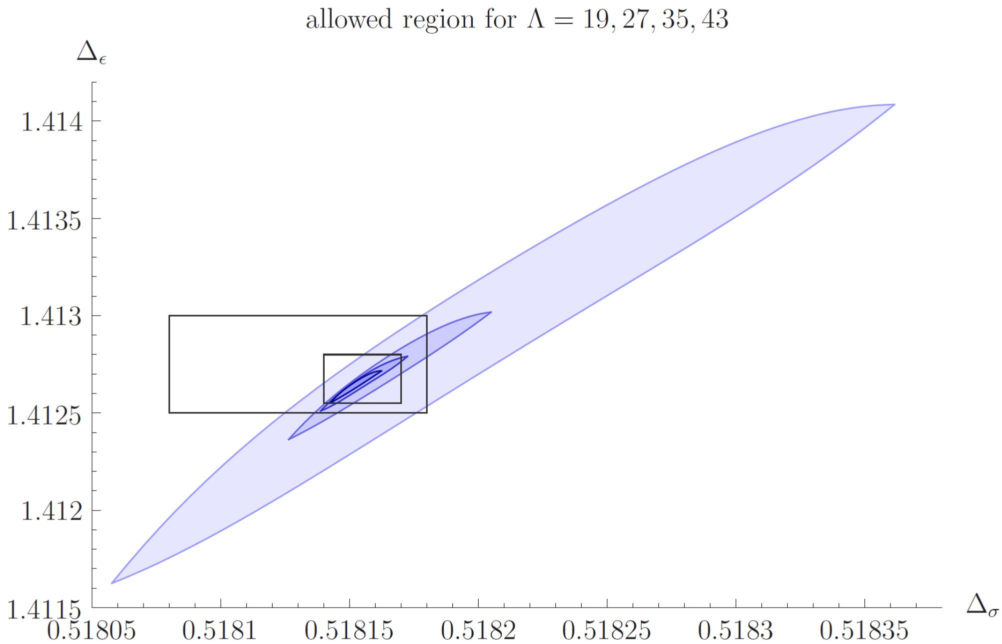
(with respect to certain assumption on spectrum)

Note: $\alpha \cdot V_+$ is a 2×2 matrix. $\alpha \cdot V_+ \geq 0$ means the matrix is semi-positive, i.e.
$(\lambda_{\sigma\sigma O} \; \lambda_{\epsilon\epsilon O}) \, \alpha \cdot V_+ \begin{pmatrix} \lambda_{\sigma\sigma O} \\ \lambda_{\epsilon\epsilon O} \end{pmatrix} \geq 0$ for arbitrary $\lambda_{\sigma\sigma O}, \lambda_{\epsilon\epsilon O}$

Again this problem can be solved by SDPB.

3D Ising island from mixed correlators

Assume there is only one relevant parity even scalar (ϵ), and only relevant parity odd scalar (σ).



(Kos, Poland, Simmons-Duffin arXiv:1406.4858)
(Simmons-Duffin, arXiv:1502.02033)

The OPE scan

The term involves external operator looks like

$$V_{\text{ext}} = \begin{pmatrix} \lambda_{\sigma\sigma\epsilon} & \lambda_{\epsilon\epsilon\epsilon} \end{pmatrix} V_{+}(\Delta = \Delta_{\epsilon}) \begin{pmatrix} \lambda_{\sigma\sigma\epsilon} \\ \lambda_{\epsilon\epsilon\epsilon} \end{pmatrix} + \lambda_{\sigma\epsilon\sigma}{}^{2}\, V_{-}(\Delta = \Delta_{\sigma})$$

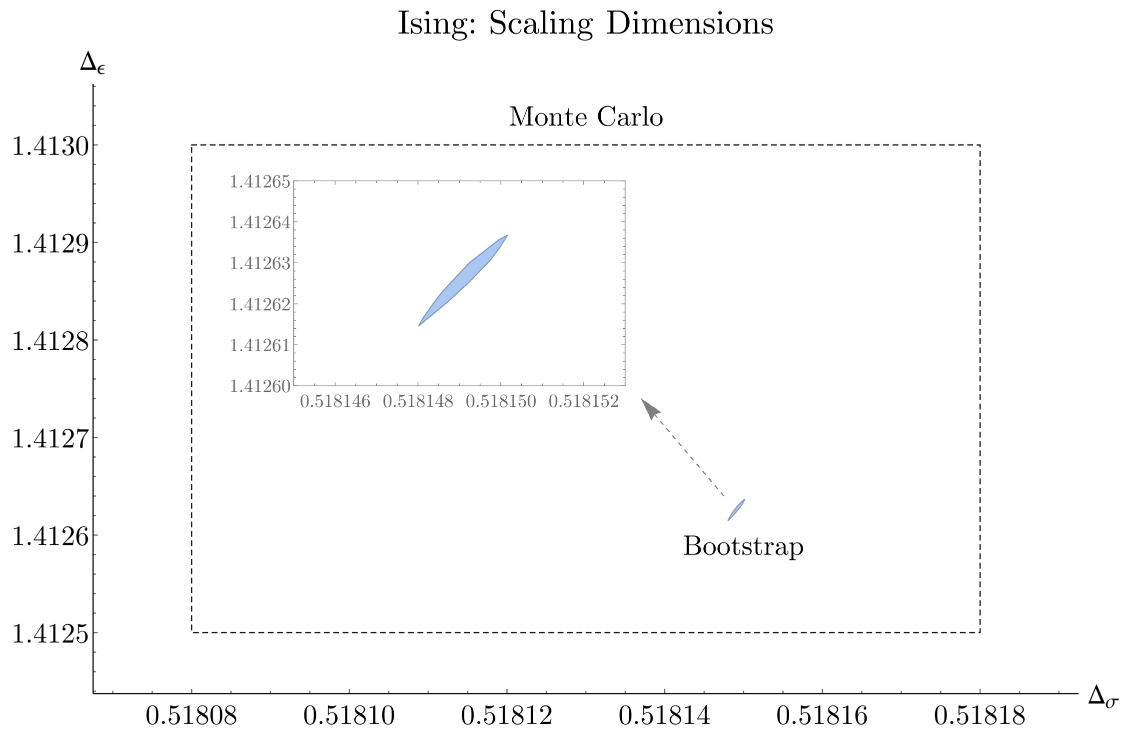
This is a 2*2 matrix spanned by $\lambda_{\sigma\sigma\epsilon}, \ \lambda_{\epsilon\epsilon\epsilon}$.

As Slava suggested in arXiv:1406.4858, we may not have an uniform α that $\alpha.V_{\text{ext}} > 0$ for any $\theta = \lambda_{\sigma\sigma\epsilon} / \lambda_{\epsilon\epsilon\epsilon}$. But there could be a situation that given a specific $\theta = \lambda_{\sigma\sigma\epsilon} / \lambda_{\epsilon\epsilon\epsilon}$ there is always an α_{θ} such that $\alpha_{\theta}.V_{\text{ext}} > 0$.

Ising island with OPE scan

Let's define $\theta = \tan^{-1}(\lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon})$. For a given $(\Delta_\sigma, \Delta_\epsilon)$, there might not exist a functional α that satisfy all positivity condition. But it is possible for any given θ, there exist a corresponding α_θ satisfy all positivity conditions.

At $\Lambda = 43$, scan over θ space, then project down to $(\Delta_\sigma, \Delta_\epsilon)$ plane:

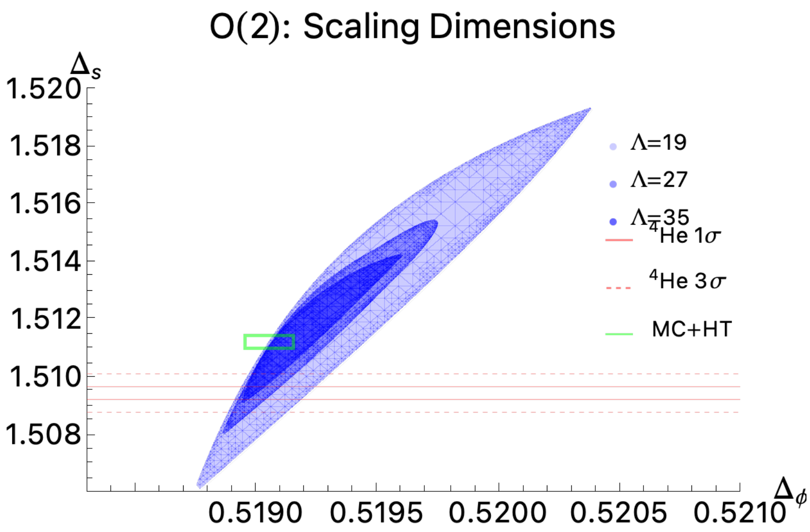


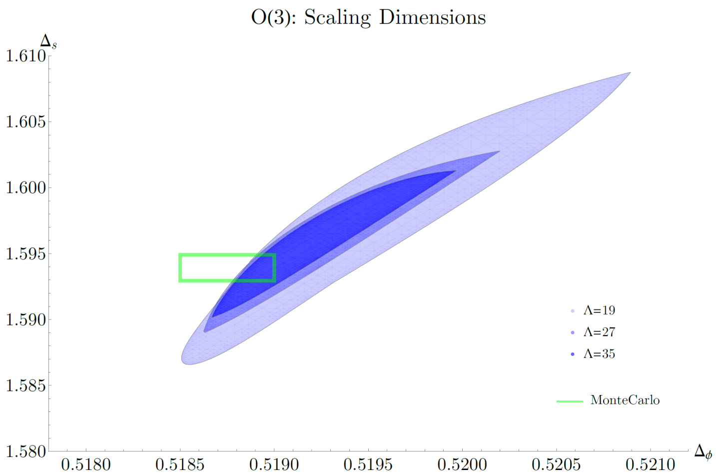
(Kos, Poland, Simmons-Duffin, Vichi arXiv:1603.04436)

OPE scanning is more than "shrinking the island". In many cases, it is necessary for non-trivial mixed correlator bootstrap.

O(N) island with OPE scan

$\theta = \tan^{-1}(\lambda_{\epsilon\epsilon\epsilon}/\lambda_{\phi\phi\epsilon})$





(Kos, Poland , Simmons-Duffin, Vichi arXiv:1603.04436)

What do we want to do next?

1, We want to improve precision.

In O(N), dimension of T (traceless symmetric tensor) Δ_T is smaller than Δ_s , thus probably has an important contribution to the bootstrap equation.

We need to mix v, s, t .

What do we want to do next?

2, We want to find island for more complicated critical point.

How do we find island? Why we have island for Ising/O(N) in the first place?

The island is a reflection of state decoupling.

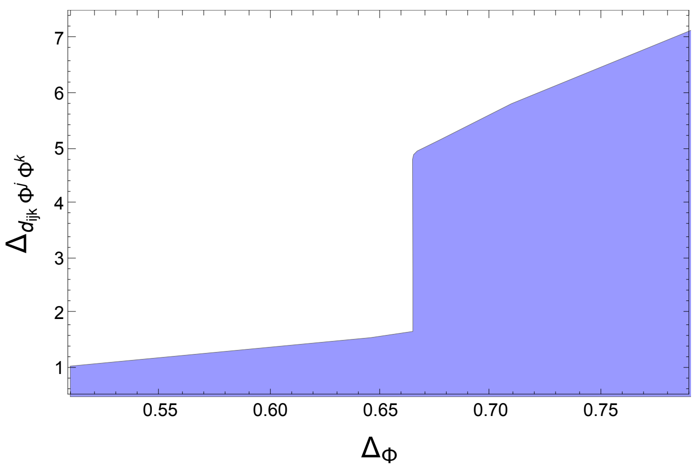
Let’s take 3D $\mathcal{N} = 1$ SCFT with S_N symmetry as an example.

$\mathcal{L} = \int d\theta^2 \left(-\frac{1}{2} D_\alpha \Phi_i D^\alpha \Phi_i + \mathcal{W}(\Phi_i)\right)$ with $\mathcal{W} = \frac{1}{6} d_{ijk} \Phi_i \Phi_j \Phi_k$

The Φ_i is parity odd boson and fundamental representation of S_N.

EOM tells us $d_{ijk} \Phi_i \Phi_j$ (in $\mathcal{B}_{+,l=0}$), become descendant of Φ_k.

For S_3 symmetry, let’s bound dimension of $d_{ijk} \Phi_i \Phi_j$ v.s. Φ_i :



(Junchen Rong, Ning Su, to appear)

Take 3D Ising/O(N) as an example:

EOM in ϕ^4 theory : $\Box\phi = \phi^3$

For generalized free theory, if $\Delta\phi \sim 0.5$, we expect $\Delta\phi' \approx 3\Delta\phi \approx 1.5$.

But for Ising/O(N), this operator decouple from the primary spectrum, thus we expect a large gap between $\Delta\phi$ and $\Delta\phi$.

How about more complicated CFTs?

3D QED, cubic models ($\mathcal{L} = \partial\phi_i \partial\phi_i + \lambda_1 \phi^2 \phi^2 + \lambda_2 \sum_i \phi_i \phi_i \phi_i \phi_i$).

We need to mix more operators.

Generic challenges for multi-operators bootstrap

If we do not scan OPE :
(1), the island is large.
(2), the mixing system might be trivial (in the sense the functional $\vec{\alpha}$ scale away some of the equations).

Generic challenges for multi-operators bootstrap

If we do OPE scan :

In 3D Ising, assume we want to mix σ, ϵ, ϵ' . The term involve external operators are spanned by

$\lambda_{\sigma\sigma\epsilon}$, $\lambda_{\sigma\sigma\epsilon}$, $\lambda_{\epsilon\epsilon\epsilon}$, $\lambda_{\epsilon\epsilon\epsilon'}$, $\lambda_{\epsilon\epsilon'\epsilon'}$, $\lambda_{\epsilon'\epsilon'\epsilon'}$

Includes Δ_σ, Δ_ϵ, $\Delta_{\epsilon'}$, totally it's 9 dimensional space.

Assume to sample a 2D island, we need about 200 points, i.e. about $\sqrt{200} \approx 14$ points for one dimension. Then estimation for 9D space would be

$14^9 = 20\,000\,000\,000$

In the old method, each point took 10*32 cpu hours. Each cpu hour=0.01 usd.

Total cost : 6.4×10^{10} = 6 billion USD ! And we still do not know if there is an allowed point between two disallowed points!

The new techniques

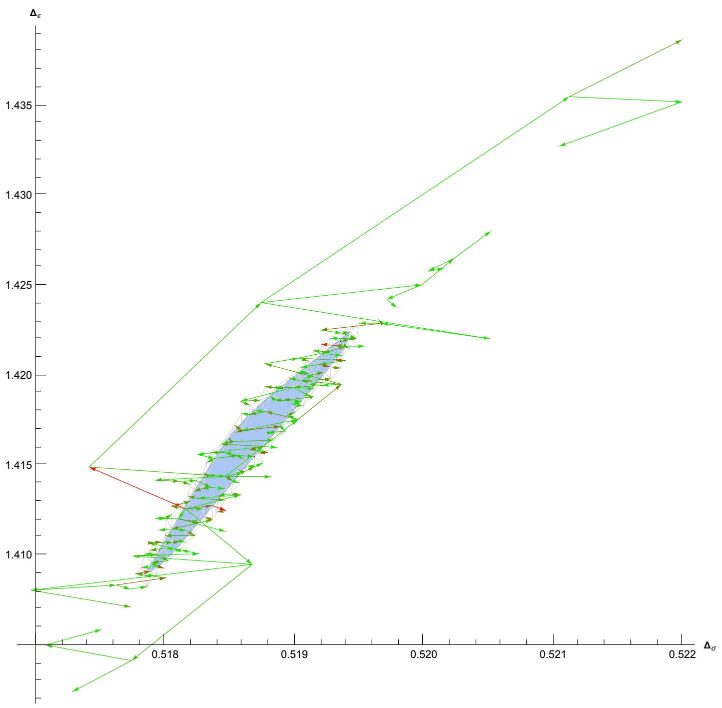
The new OPE scan algorithm :
1, Computation time is linear to number of OPEs.
2, It is deterministic : if there is a feasible point in OPE scan, we have (almost) 100% certainty to find it. The time about log(volume of OPE space).

Hot-start SDPB :
Save 70% computation time without OPE scan.
Save much much more when combined with OPE scan algorithm.

New version of SDPB:
Capable of running huge SDP problem. The $\vec{\alpha}$ can have 5566 components!

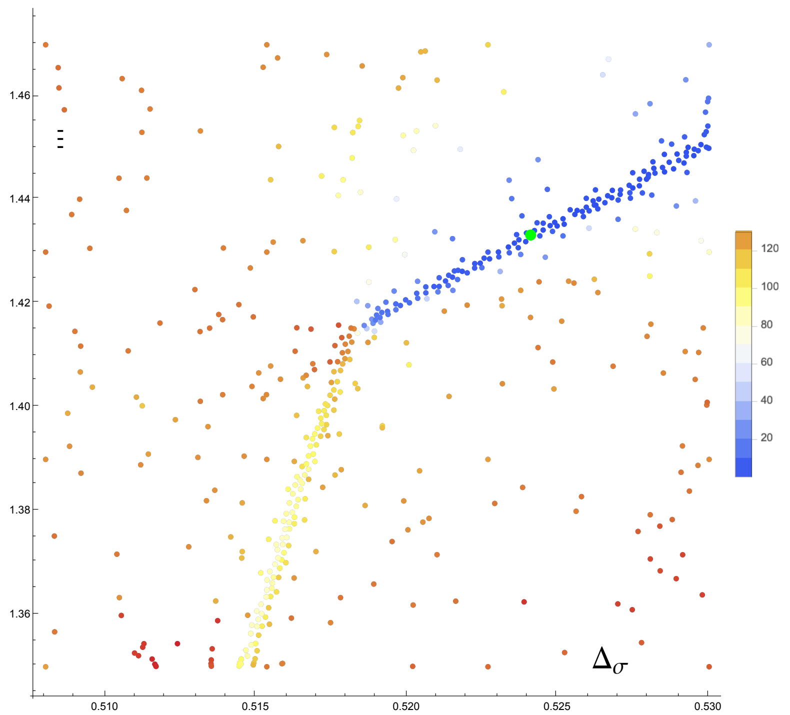
Hot-start

In 3D Ising σ ϵ mixed correlator bootstrap :



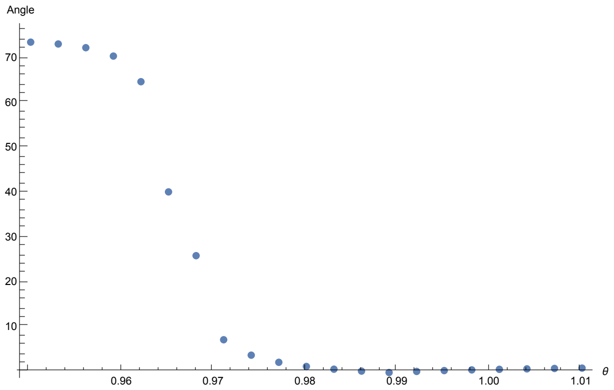
How to effective do hot-start ? How different are the functionals when we vary points?

3D Ising single correlator $\langle\sigma\sigma\sigma\sigma\rangle$:



Same thing happen in OPE space.

For Ising σ,ϵ mix, fix a point (Δ_σ, Δ_ϵ) on the island, plot “angle between functionals” v.s. $\theta = \lambda_{\sigma\sigma\epsilon} / \lambda_{\epsilon\epsilon\epsilon}$:



Each functional $\vec{\alpha}$ actually does not only exclude that specific point, but also a region around it !

The OPE scan algorithm

1, Use SDPB to do feasibility test on a point $\vec{\lambda}_1$ in OPE space. Assume this point is disallowed, we get a functional $\vec{\alpha}_1$.

2, From $\vec{\lambda}_1.\left(\vec{\alpha}_1 \cdot V_{ext}\right).\vec{\lambda}_1 < 0$, we found a quadratic inequality that constrains OPE space.

3, We choose a good point $\vec{\lambda}_2$ in the region that is not excluded, and repeat step 1&2. Until : (1) we found a feasible point. (2) the entire region is excluded.

For N number of disallowed points, we have N number of quadratic inequalities. This is quadratically constrained feasibility problem. In general it is very hard, but for conformal bootstrap, we can solve it.

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1 - -------------
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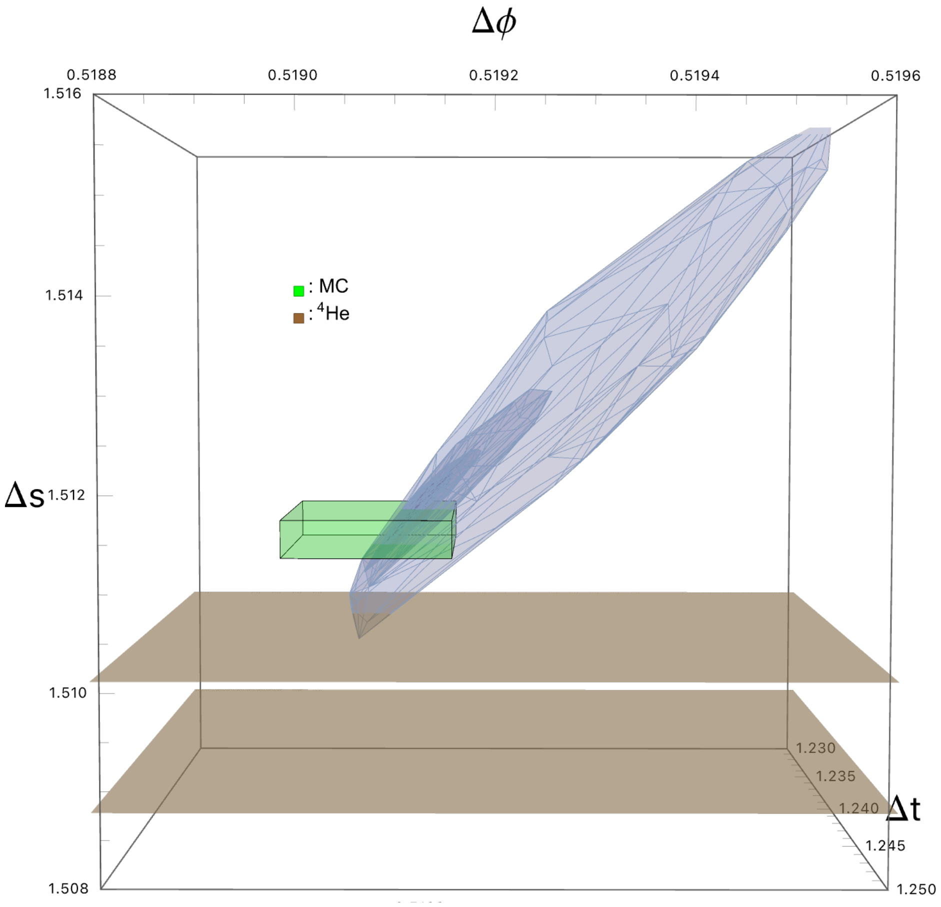
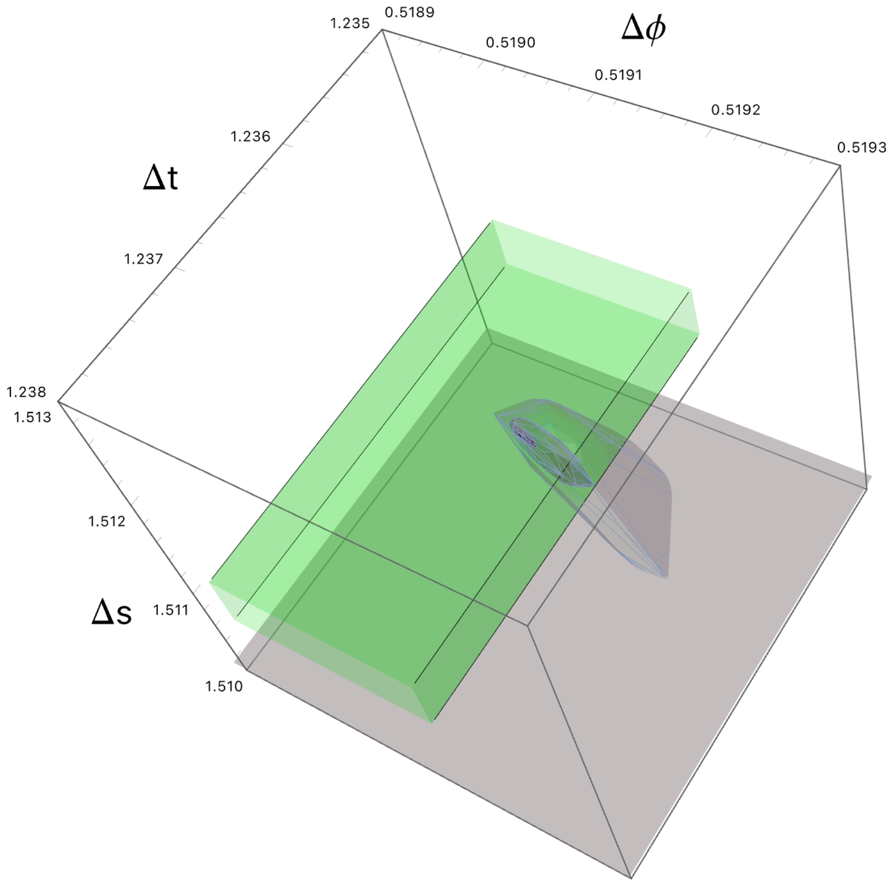
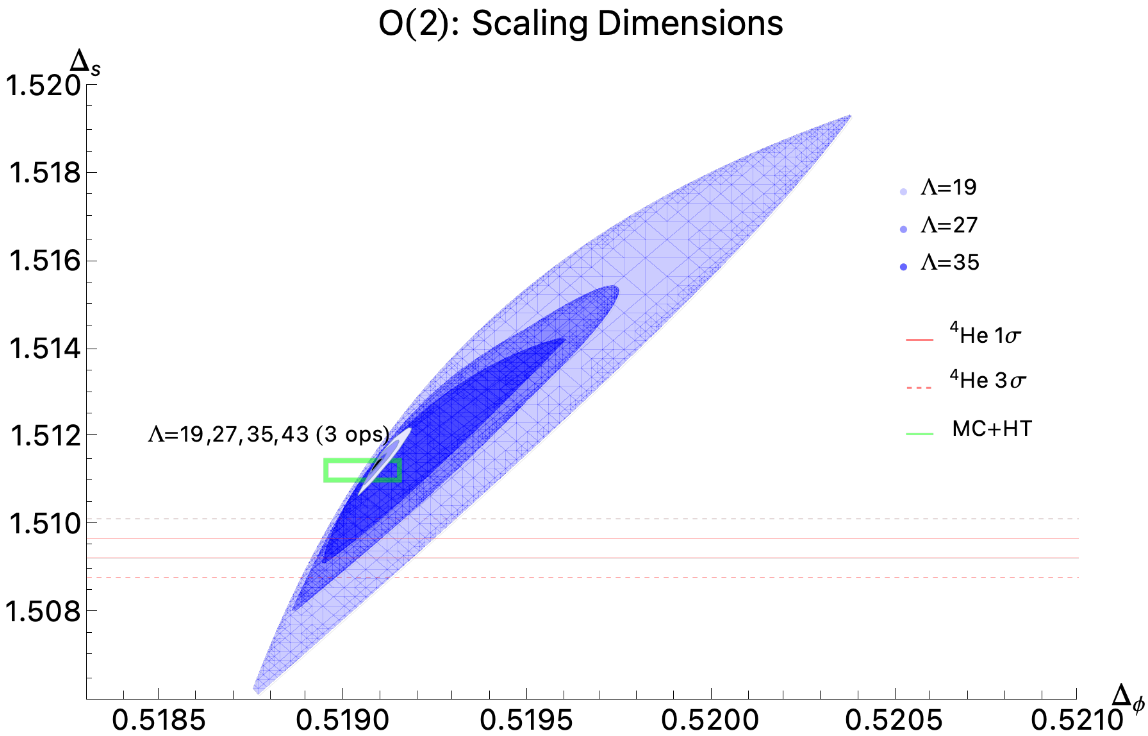
Result : 3D O(2)

Chester, Simmons-Duffin, Liu, Poland, Su, Vichi, Walter, to appear.

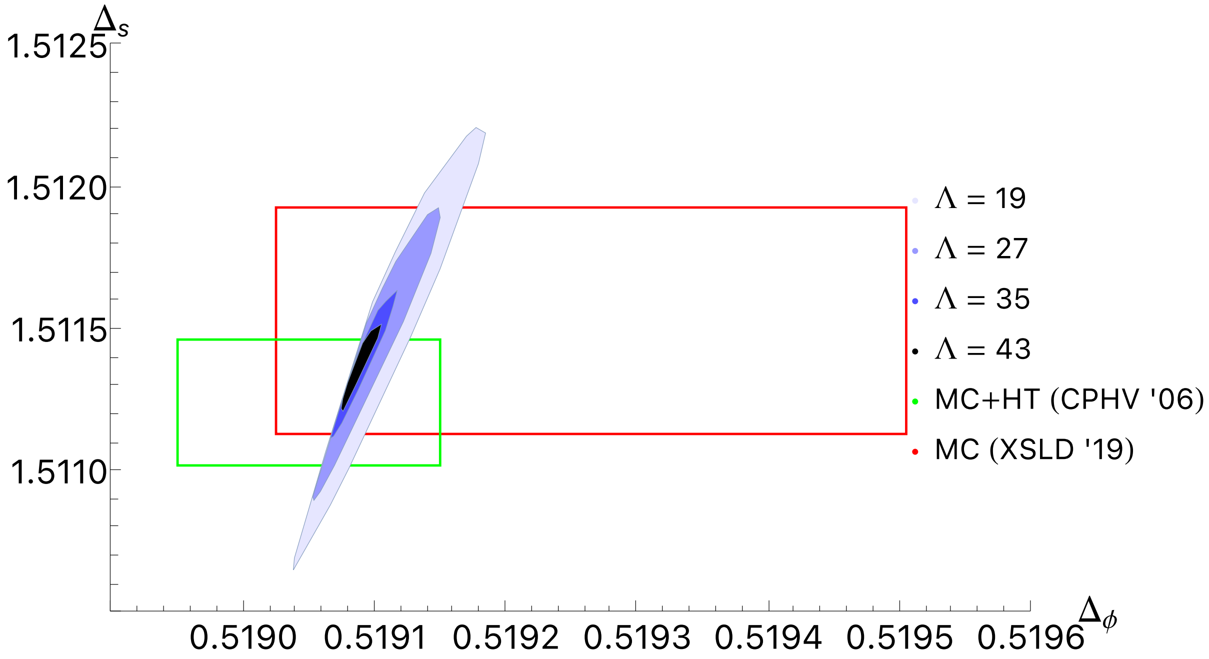






Green box : Campostrini, Hasenbusch, Pelissetto, Vicari 2006

Experiment : Lipa, Nissen, Stricker, Swanson, Chui, Phys.Rev. B68 (2003) 174518.



Red box : Xu, Sun, Lv, Deng 2019

The convergence continue to $\Lambda = 43$.

The software “Bootstrapper”

The entire process is automatized !

Autoboot → Bootstrapper → SDPB

You can easily reproduce our result. You just need to specify the representation.

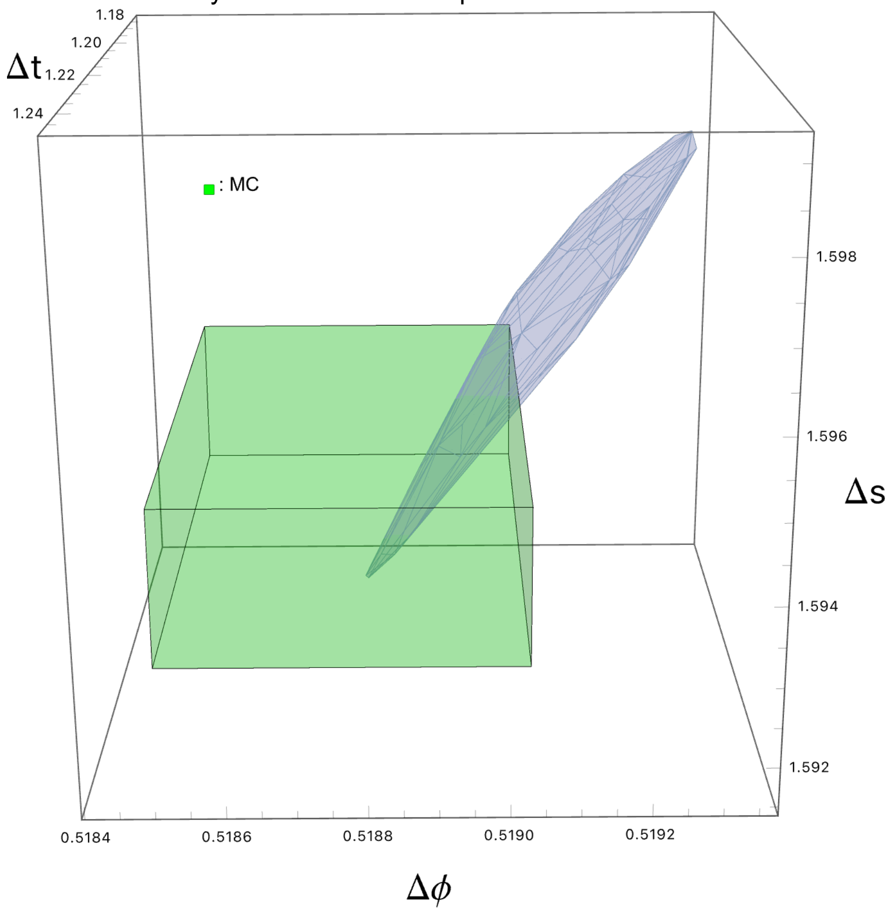
Other ongoing and future works

Ongoing work with Chester, Simmons-Duffin, Liu, Poland, Vichi, Walter.

Cubic model v.s. O(3), which one is stable ?

O(3) : $\mathcal{L} = \partial\phi_i \partial\phi_i + \lambda_1 \phi^2 \phi^2$
Cubic : $\mathcal{L} = \partial\phi_i \partial\phi_i + \lambda_1 \phi^2 \phi^2 + \lambda_2 \sum_i \phi_i \phi_i \phi_i \phi_i$

If the rank 4 tensor operator in O(3) is relevant, it will drive O(3) to flow to cubic. We can bootstrap S, V, T mixed correlator system to detect this operator.



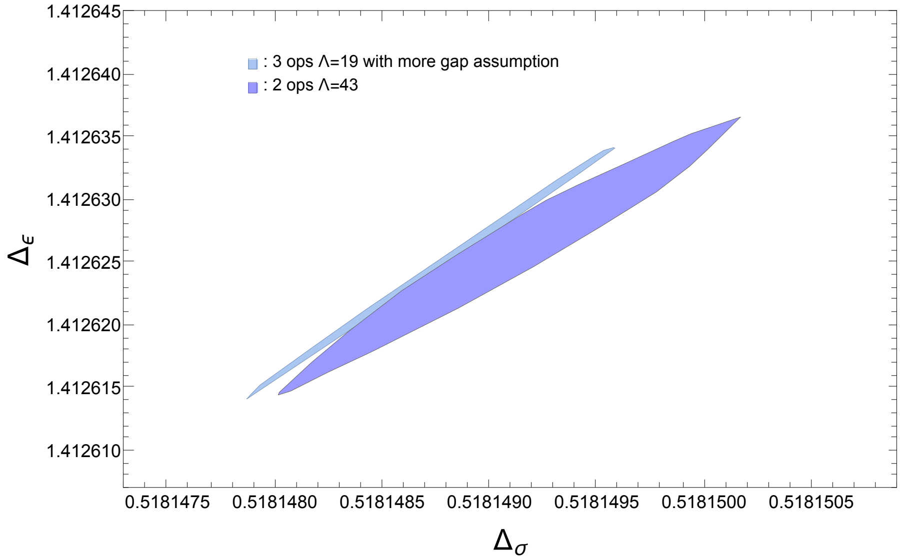
MC : (Δ_ϕ, Δ_s) : Calabrese, Pelissetto, Vicari 2002 .
Δ_T : Campostrini, Hasenbusch, Pelissetto, Rossi, Vicari 2002 (arXiv:cond-mat/0611353)

The rank 4 tensor is about 2.999 (not certain yet) .

Other ongoing and future works

Ongoing work with Simmons-Duffin, Rong, Vichi.

Ising model $\sigma,\epsilon,\epsilon'$ mixed correlator bootstrap:



Viral current appear in $\epsilon \times \epsilon'$: 10.5

Other ongoing and future works

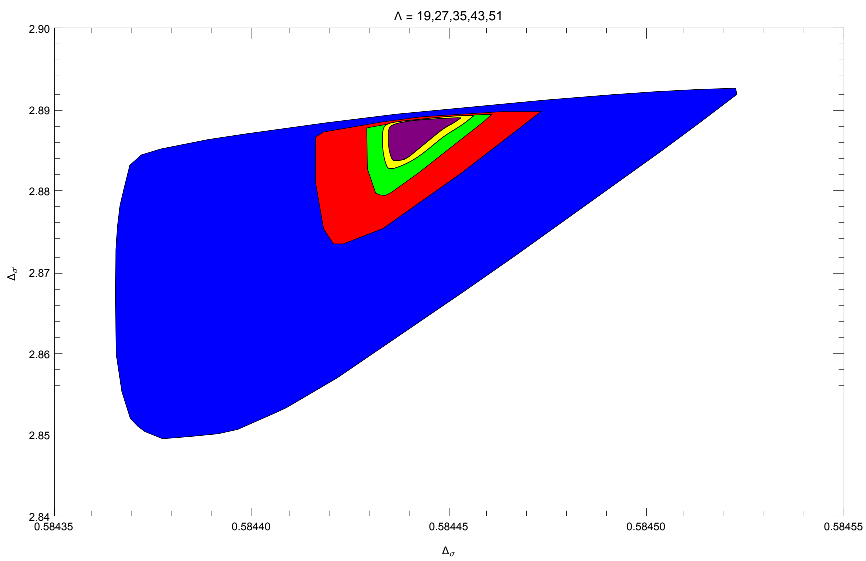
3D $\mathcal{N}$=1 super-Ising model:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\overline{\Psi}\partial\!\!\!/\,\psi + \frac{\lambda}{2}\phi\overline{\Psi}\partial\!\!\!/\,\psi + \frac{\lambda^2}{8}\phi^4$$

ψ : 2-component Majorana fermion, labeled by $\alpha \in \{1, 2\}$

This action is invariant under $e^{i\overline{Q}\epsilon}$ with $Q^\alpha\phi = -i\,\psi^\alpha$, $\overline{Q}_\alpha\psi^\alpha = 4\,i\,F$

Bootstrapping $\langle\phi\phi\phi\phi\rangle$:



(Atanasov, Hillman, Poland, Rong, Su, to appear)

We would do σ,σ' mix coorelator bootstrap.

Other ongoing and future works

Can we find an effective algorithm in Δ space? There must be an effective algorithm in Δ space.

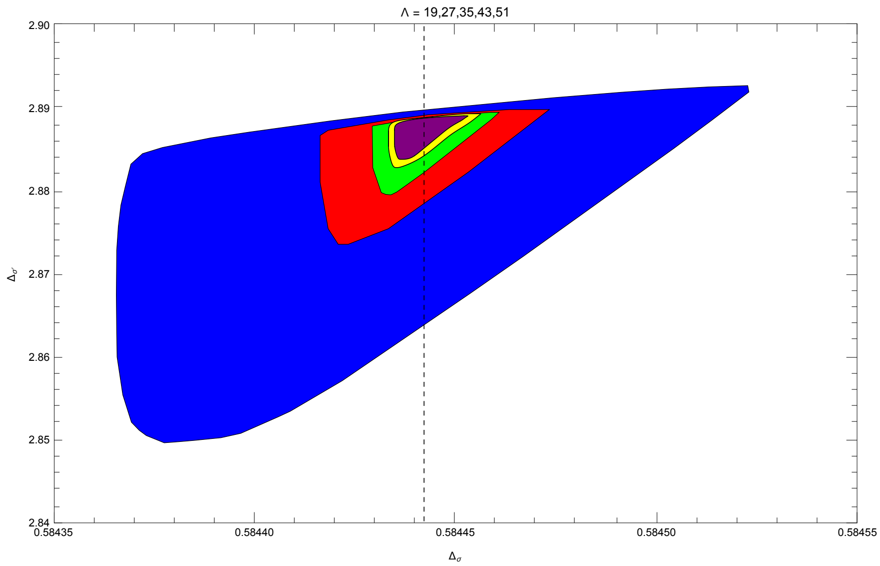
A motivation for better numerics

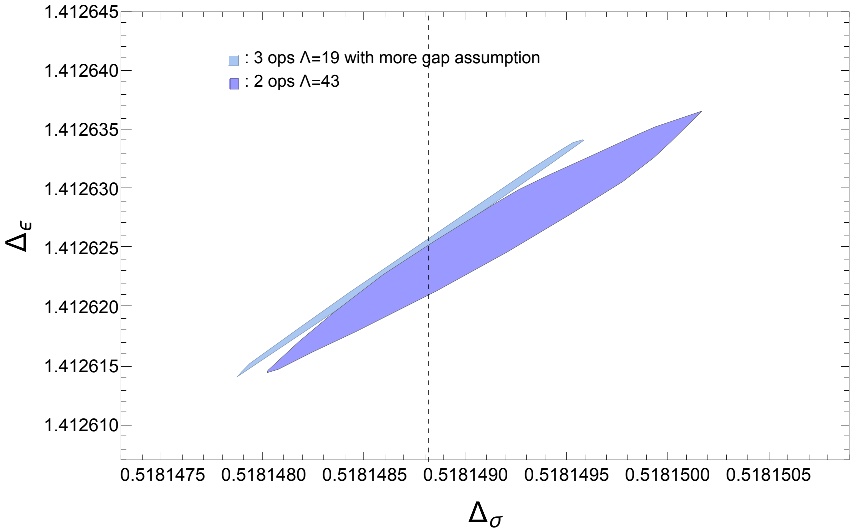
For 3D Ising and super-Ising model, there are about 10 analytic expressions compatible with current precision for Δ_σ .

My favorite choice:

Ising: $\Delta_\sigma = \frac{\Gamma\left(\frac{1}{6}\right)+4}{\Gamma\left(\frac{1}{24}\right)-5} = 0.5181488\,085\,872\,788\,360$

Super-Ising: $\Delta_\sigma = \frac{\Gamma\left(\frac{5}{24}\right)-4}{\Gamma\left(\frac{1}{3}\right)-2} = 0.5844418\,599\,263\,290\,08$





There is a beautiful mathematical structure behind $\Gamma\left(\frac{\square}{24}\right)$: all $\Gamma\left(\frac{\square}{24}\right)$ can be expressed in terms of $\Gamma\left(\frac{1}{3}\right)$, $\Gamma\left(\frac{1}{4}\right)$, $\Gamma\left(\frac{1}{8}\right)$, $\Gamma\left(\frac{1}{24}\right)$ by reflection formula and Gauss multiplicative formula. They are also related to elliptic integral (arXiv:0403510).

Thank you!

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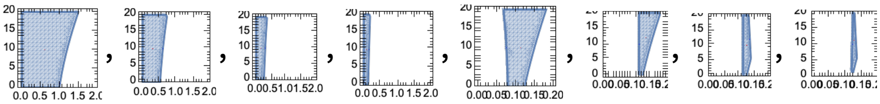

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  };
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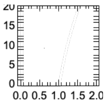
arealist = {400, 23.61329547581043`, 11.600326905499319`,
   5.623841730221956`, 2.6882727344280077`, 1.433533220394531`,
   0.703822207729859`, 0.36562213080118644`, 0.12624419421018349`};
```

emptyplot =



;

```
ani1 = Animate[Show[emptyplot, plotlist⟦i⟧, Epilog → {
      Text["Step " <> ToString[i], {1.416, 3.7}],
      Text["Cut ratio : " <> ToString[NumberForm[1 - arealist⟦i + 1⟧ / arealist⟦i⟧, {2, 2}]],
       {1.559, 2.513}]}, PlotRange → {{0, 2.1}, {0, 20}},
    AxesOrigin → {0, 0}, ImageSize → 300], {i, 1, 8, 1}]
```

```
i

Show[emptyplot, plotlist⟦3⟧, Epilog → {Text[Step 3, {1.416, 3.7}], Text[Cut ratio :
1 - -------------
    arealist[[3]], {1.559, 2.513}]}, PlotRange → {{0, 2.1}, {0, 20}}, AxesOrigin → {0, 0
ImageSize → 300]
```