Sphere Packing, Modular Bootstrap and Extremal Functionals

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Simons Center & YITP, Stony Brook

Based on work with T. Hartman and L. Rastelli: 1905.01319

and earlier work D.M.: 1611.10060

D.M., M. Paulos: 1803.10233

Institut Henri Poincaré
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Does pure gravity exist as a fully consistent quantum theory?

← only gravitons and black holes in the spectrum

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Along the way will uncover a very close connection to the recent solution of the sphere packing problem in dimensions 8 and 24.

[Cohn, Elkies '01; Viazovska '16; Cohn, Kumar, Miller, Radchenko, Viazovska '16]

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Stronger bound at large central charge

$$\Delta < \frac{2c}{17} + O(1)$$

$$\sim \frac{c}{8.5}$$

Outline

1. Virasoro Modular Bootstrap
    - AdS_3/CFT_2 and the modular bootstrap
    - Analytic functionals review
    - Proof of the main theorem

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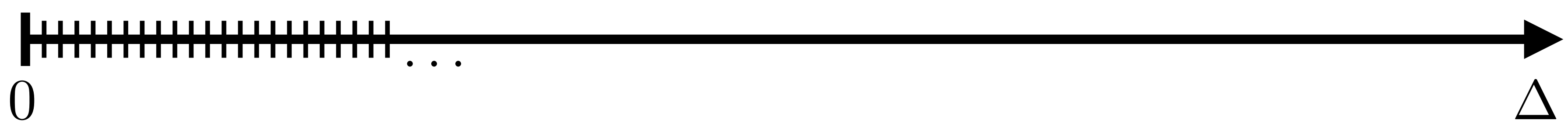
gravitons

$|\Omega\rangle,\ L_{-2}|\Omega\rangle,\ L_{-2}L_{-2}|\Omega\rangle\ ,\ldots$



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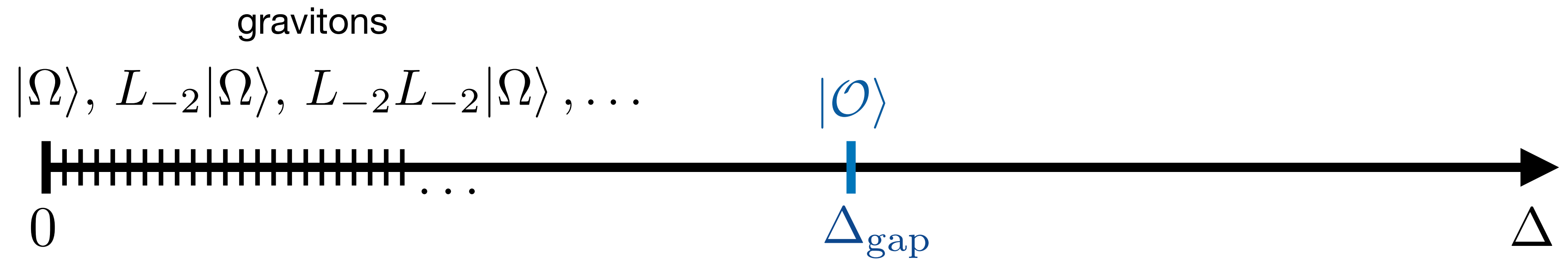

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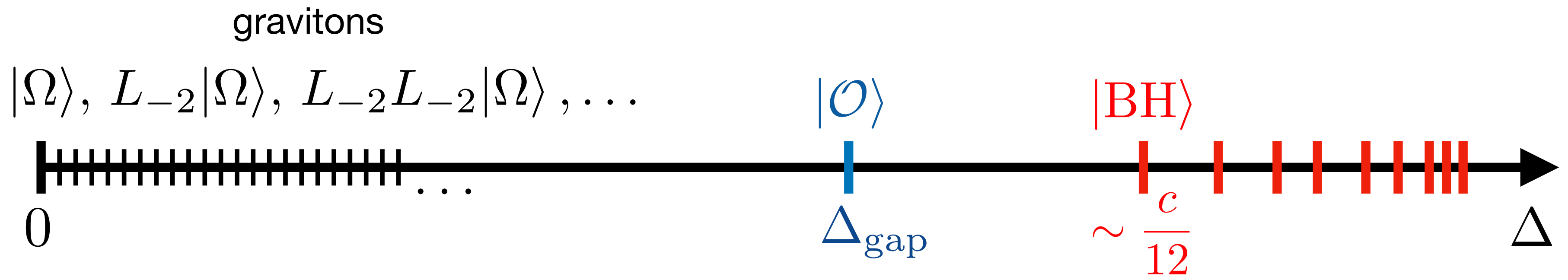

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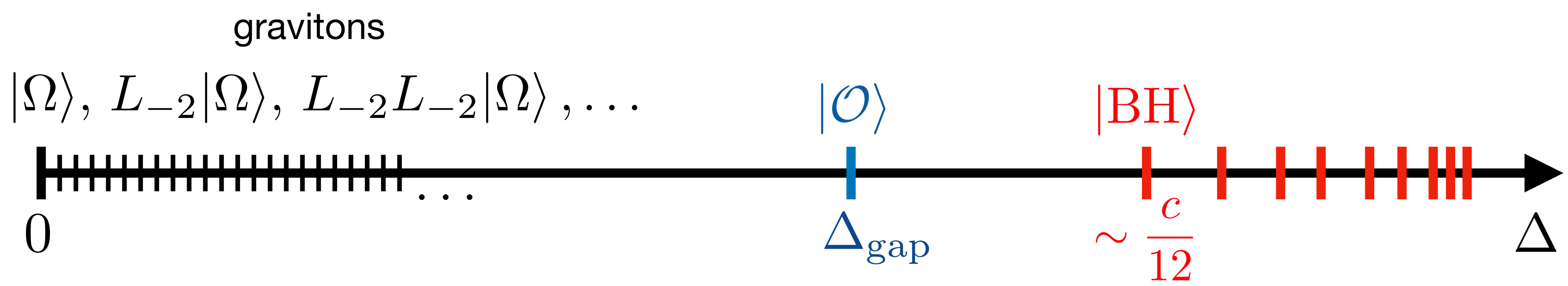

Torus partition function at zero angular potential

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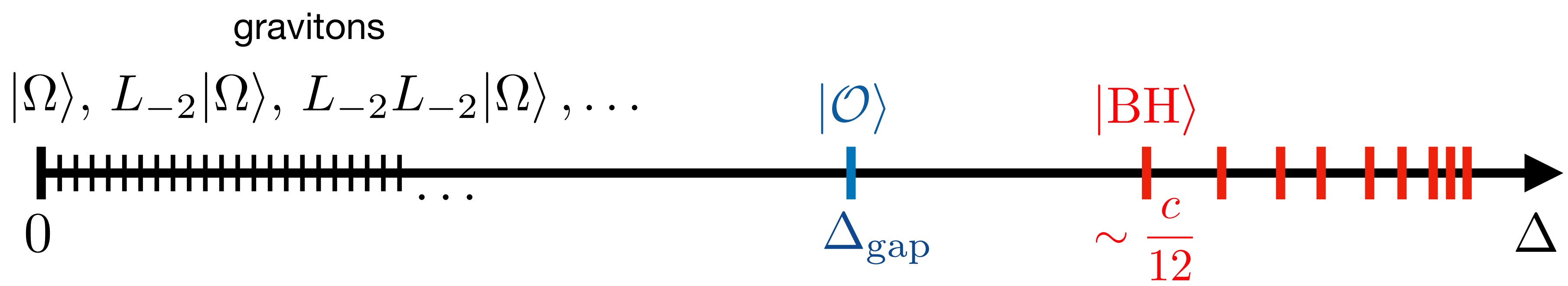

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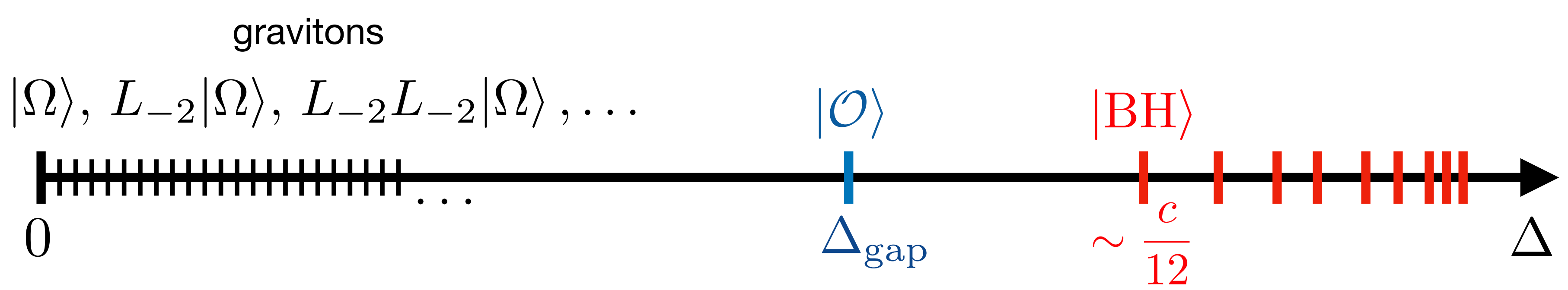

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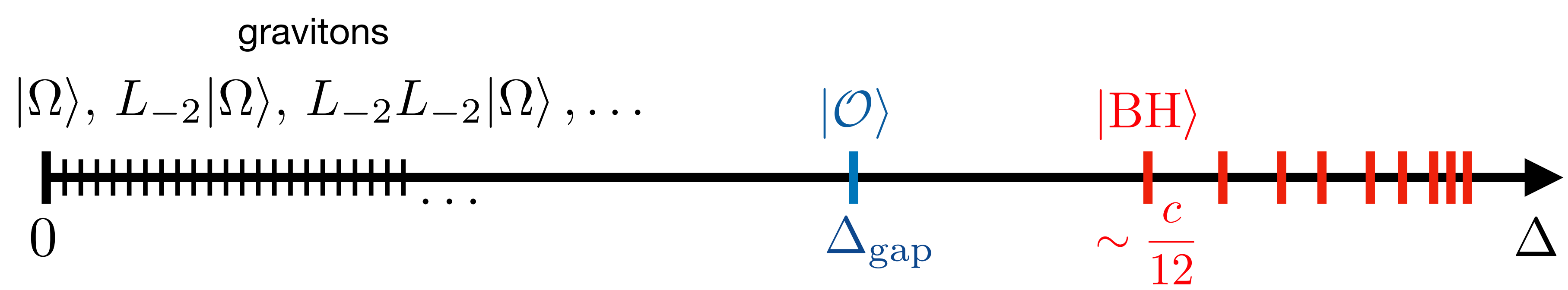

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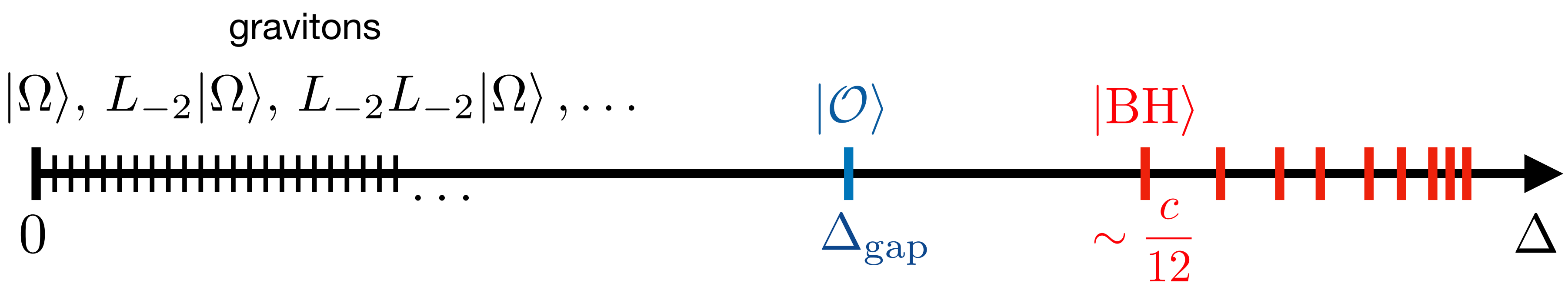

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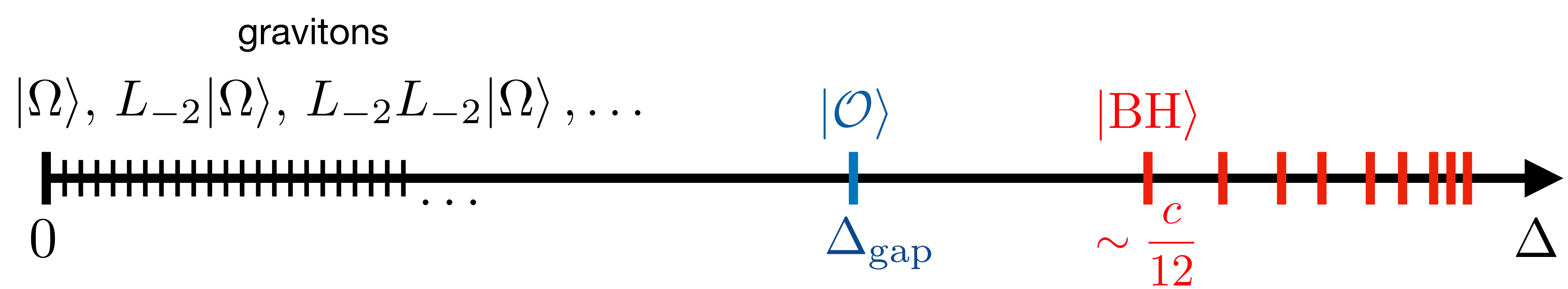

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Working with full-fledged CFTs, not chiral CFTs!

$$\Delta \notin \mathbb{Z}$$

$$Z(\tau) \neq Z(\tau+1)$$

in general

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Upper bounds on Δ_{gap} can be found as follows:

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$\mu < \frac{1}{12}$ would prove that semi-classical pure gravity is not consistent as a quantum theory.

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Conjectures based on finite-c numerics:

$\Delta_V(c) < \frac{c}{9} + O(1)$ as $c \to \infty$ **[Collier, Lin, Yin '16]**

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A different construction of ω is needed to make analytic progress.

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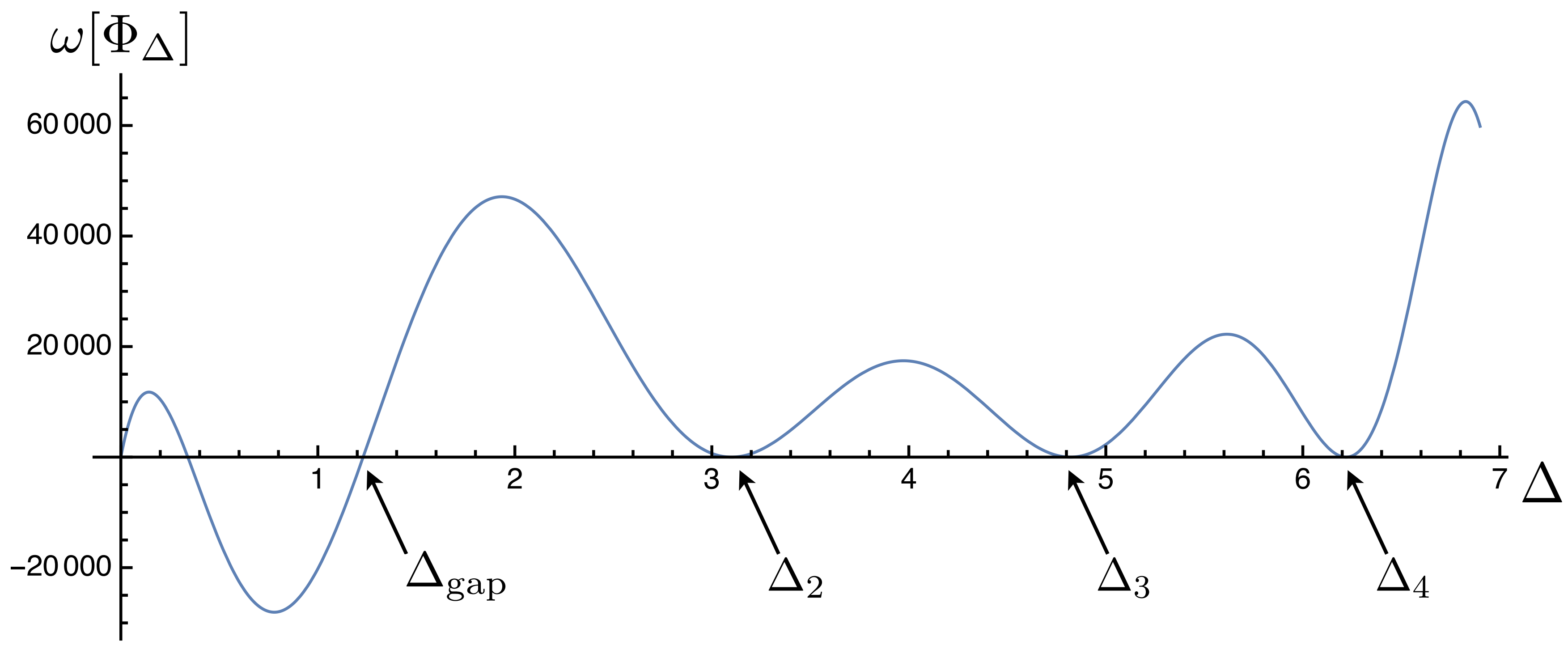

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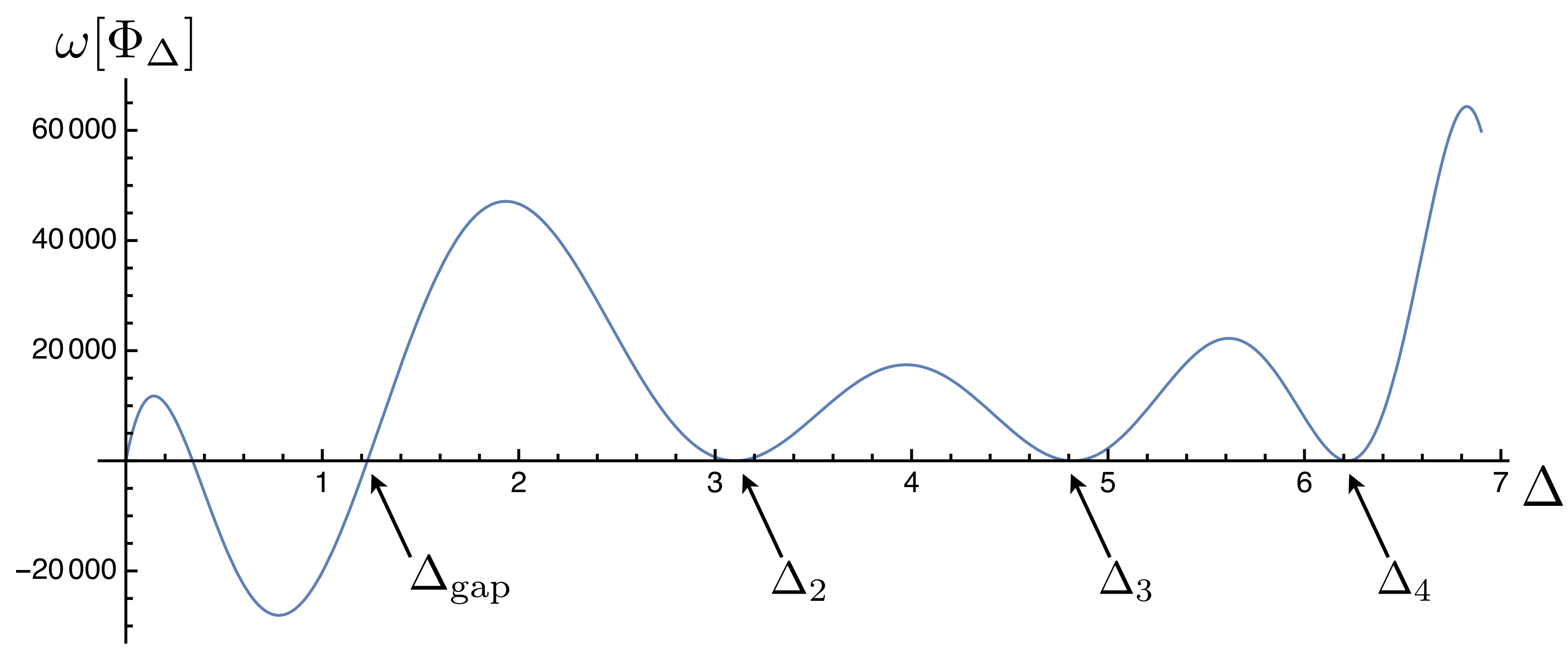

The only analytic construction of the optimal functional known so far is for the four-point function bootstrap on a line.

Nevertheless, this will be enough to prove our main theorem.

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Theorem: The OPE of two identical primaries σ in a unitary CFT always contains a non-identity conformal primary of dimensions

$$\Delta \leq 2\Delta_\sigma + 1$$

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The solution with maximal gap is the fermionic mean-field theory.

Spectrum: $2\Delta_\sigma + 1,\ 2\Delta_\sigma + 3, \ldots$

Theorem: The OPE of two identical primaries σ in a unitary CFT always contains a non-identity conformal primary of dimensions

$$\Delta \leq 2\Delta_\sigma + 1$$

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Optimal Bound for the 1D Bootstrap [DM '16; DM, Paulos '18]

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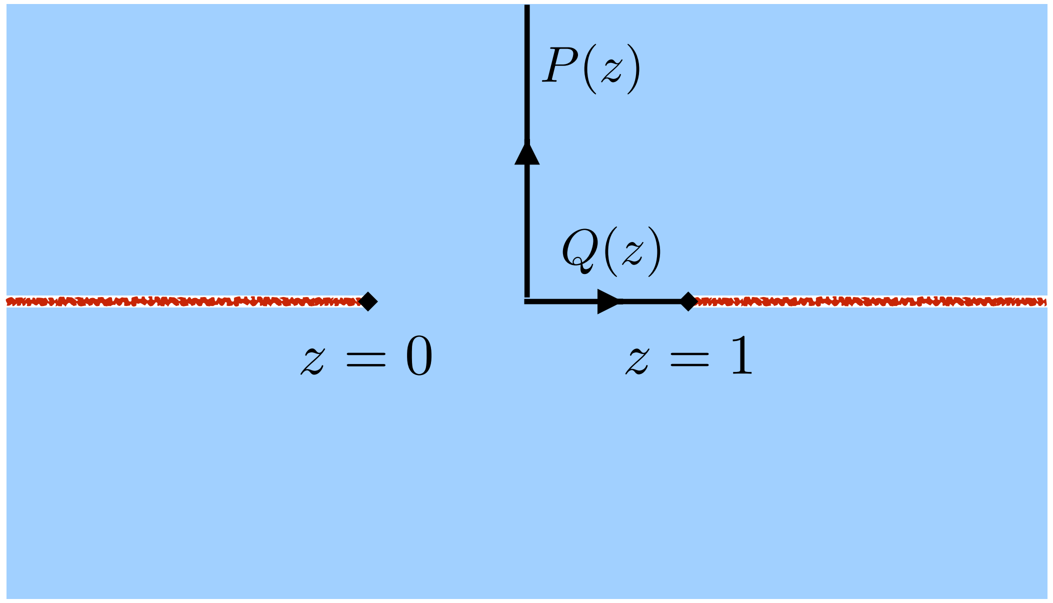

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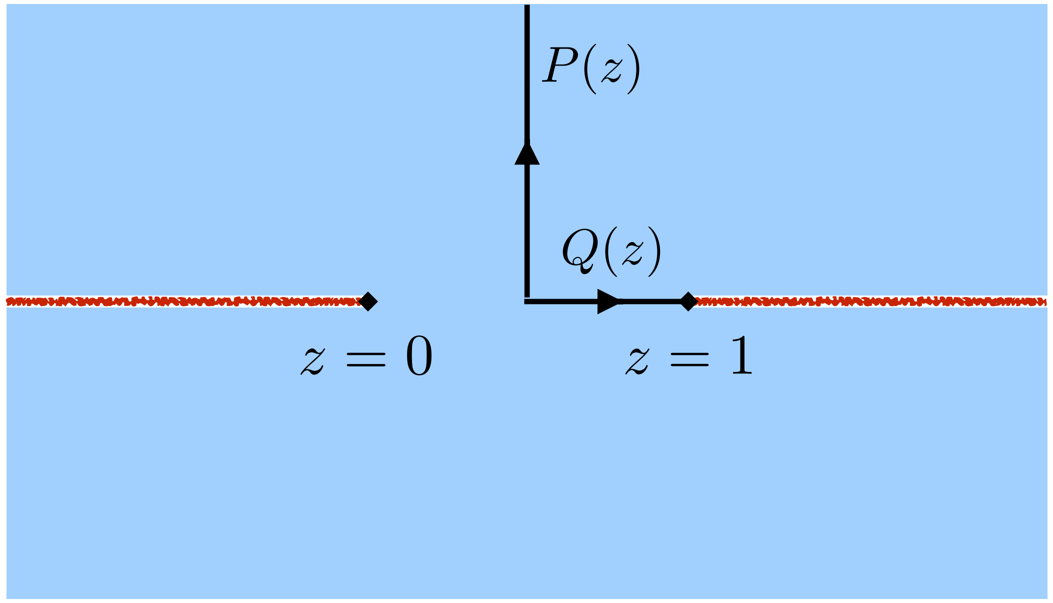

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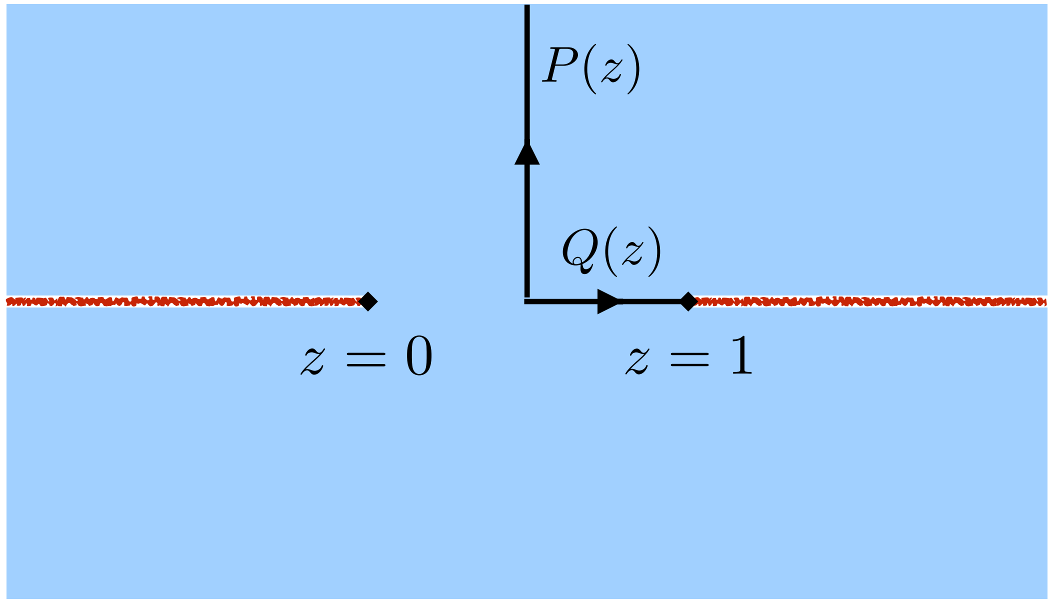

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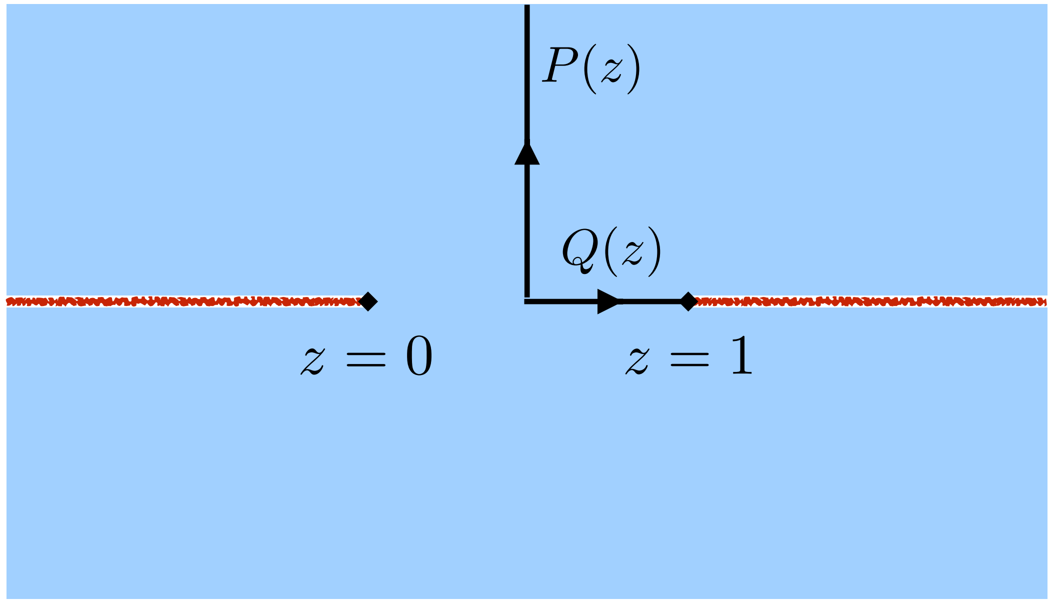

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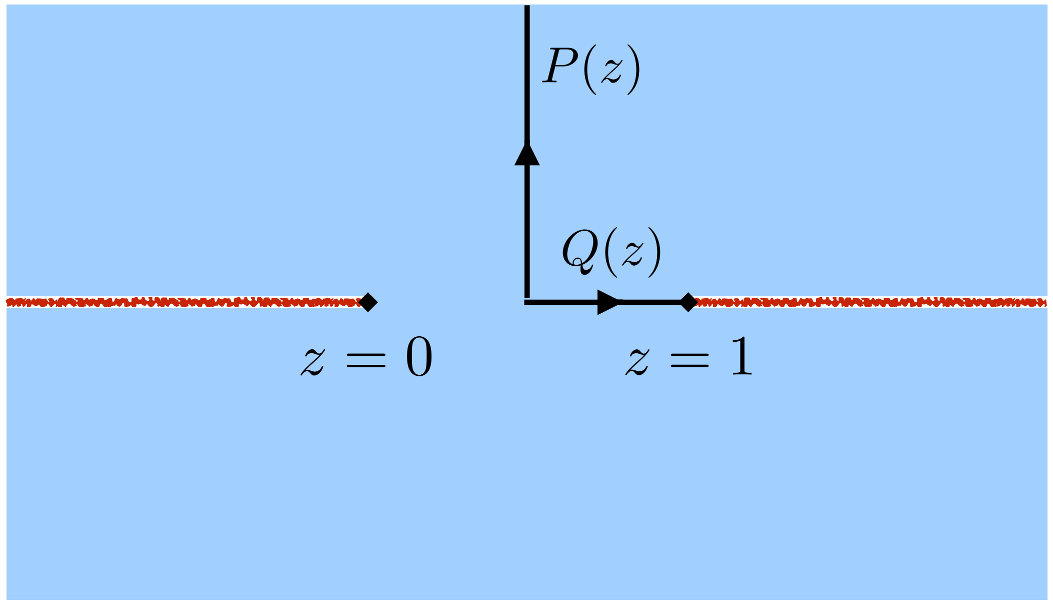

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General solution for $\Delta_\sigma \in \mathbb{R}_+$: **[DM, Paulos ’18]**

$$P(z) = \frac{2z-1}{[z(z-1)]^{3/2}}\left[{}_3\tilde{F}_2\left(-\frac{1}{2}, \frac{3}{2}, 2\Delta_\sigma + \frac{3}{2}; \Delta_\sigma + 1, \Delta_\sigma + 2; -\frac{1}{4z(z-1)}\right) + \right.$$
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Back to the Torus: The Pillow Map

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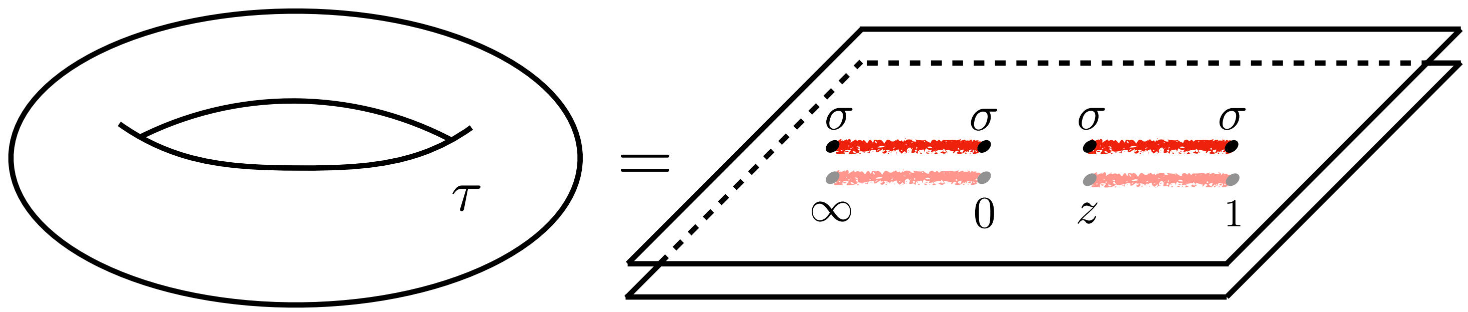

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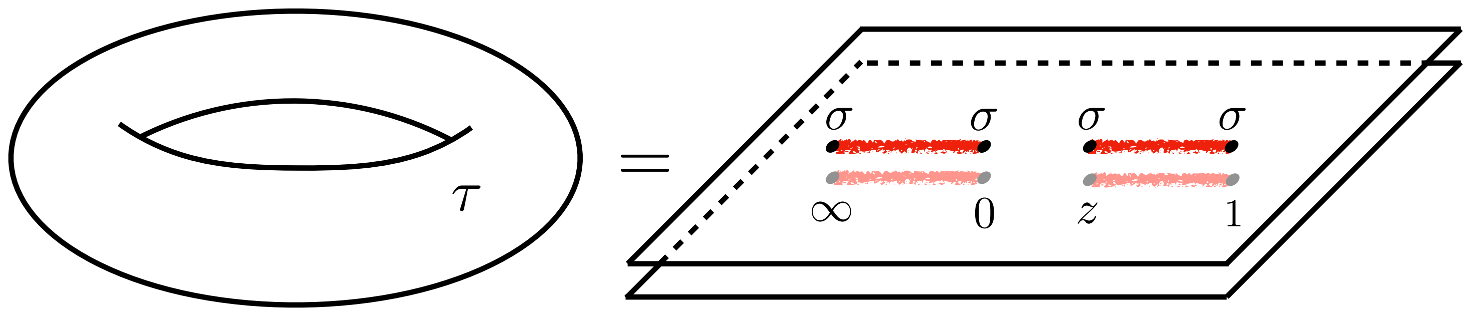

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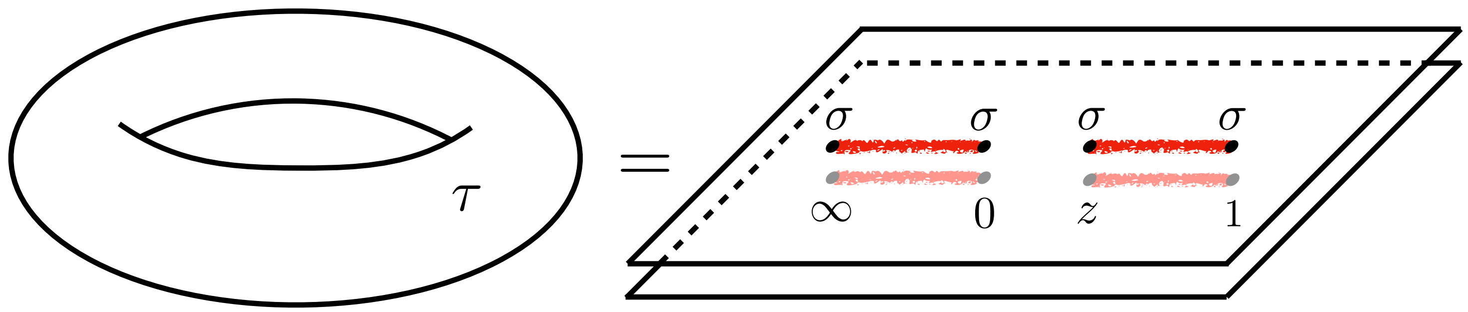

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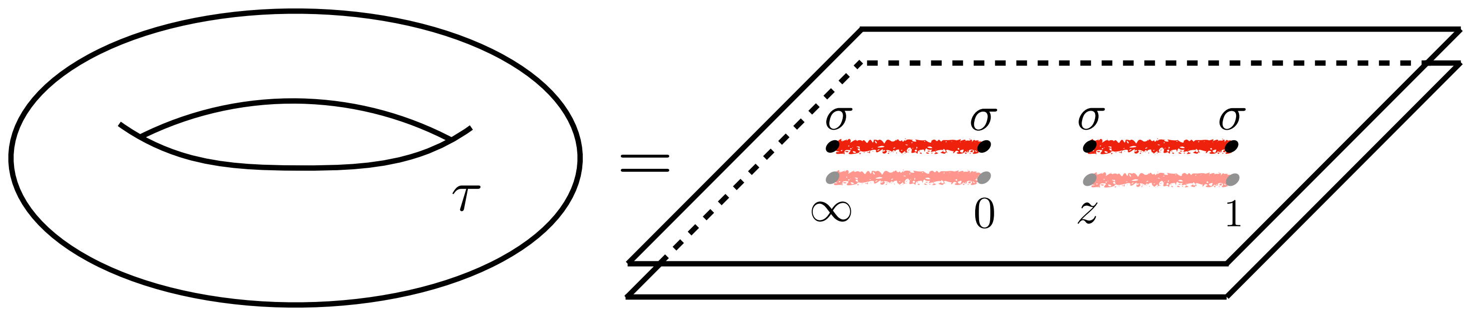

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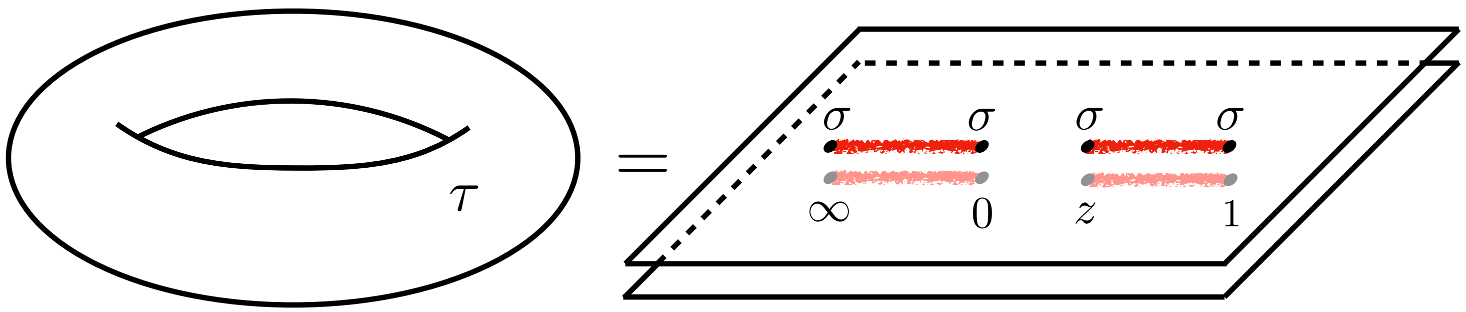

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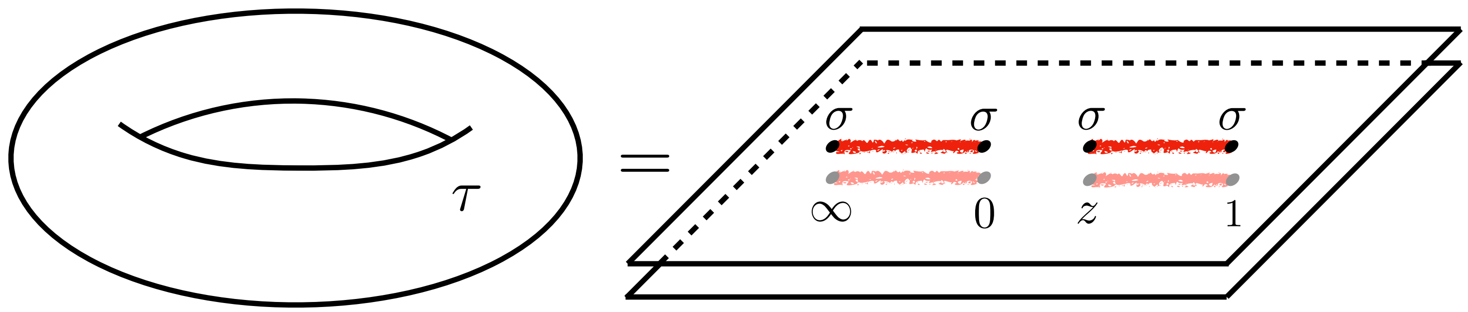

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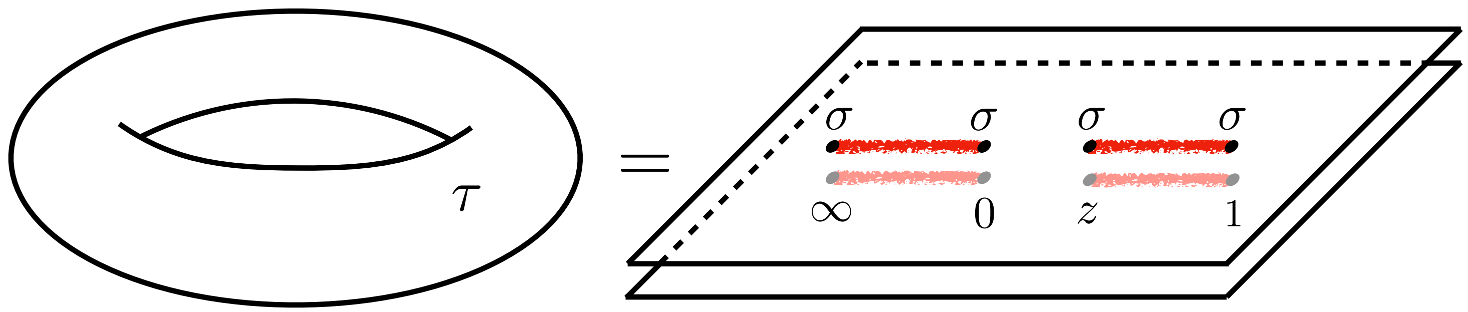

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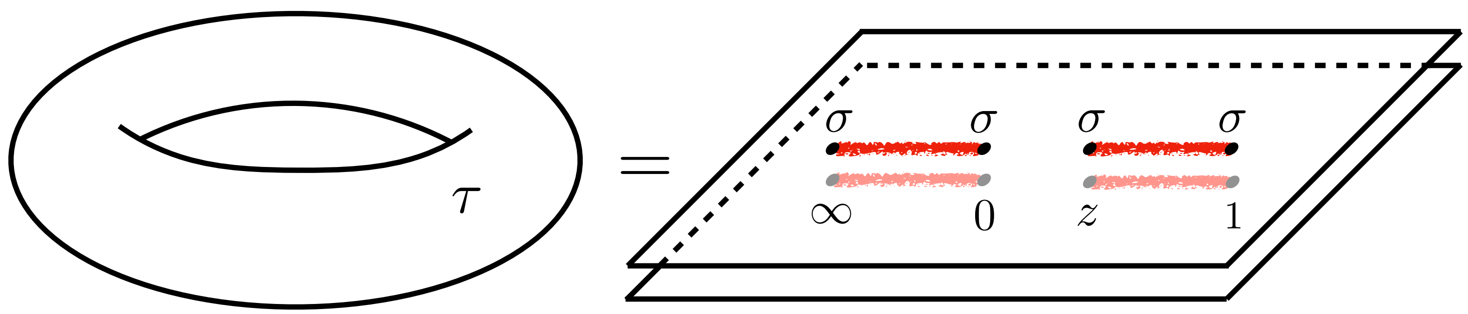

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Subtlety: Virasoro characters $\neq$ $sl(2,\mathbb{R})$ conformal blocks.

Need to check $\omega[\Phi_{\mathrm{vac}}] \geq 0$

Modular Bootstrap Conclusions [Hartman, DM, Rastelli ’19]

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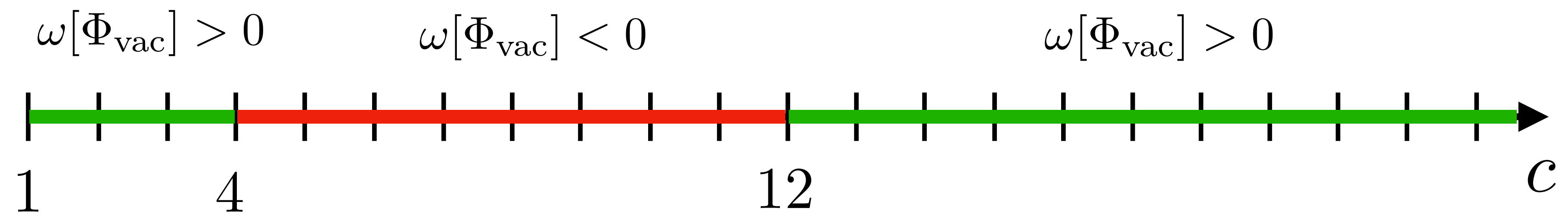

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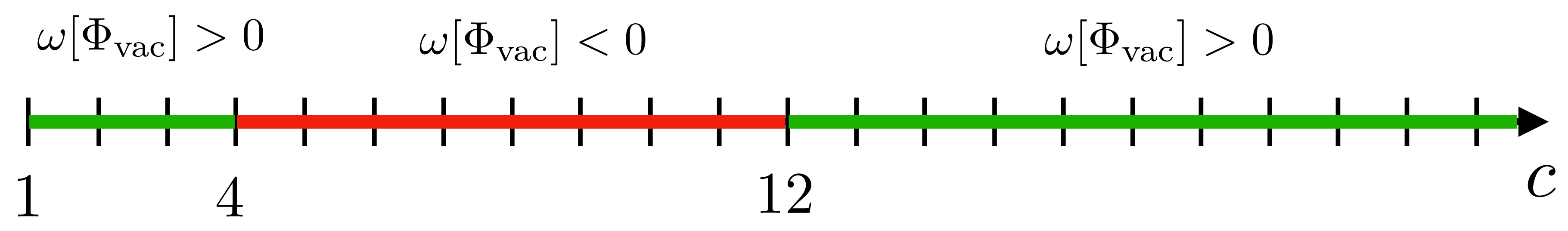

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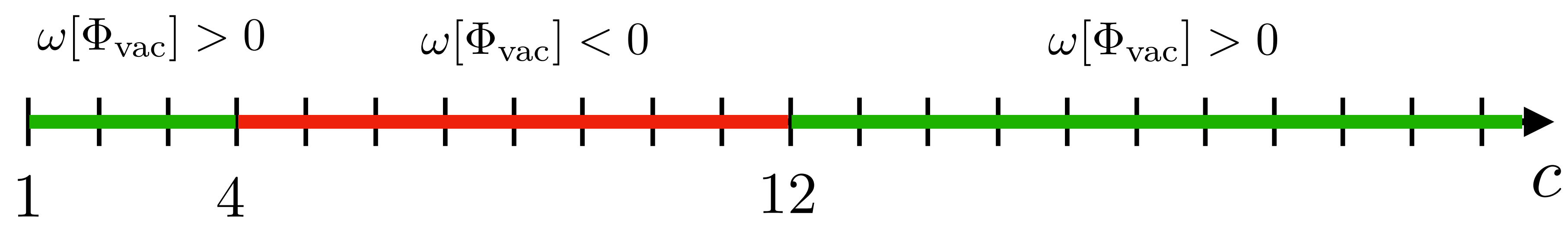

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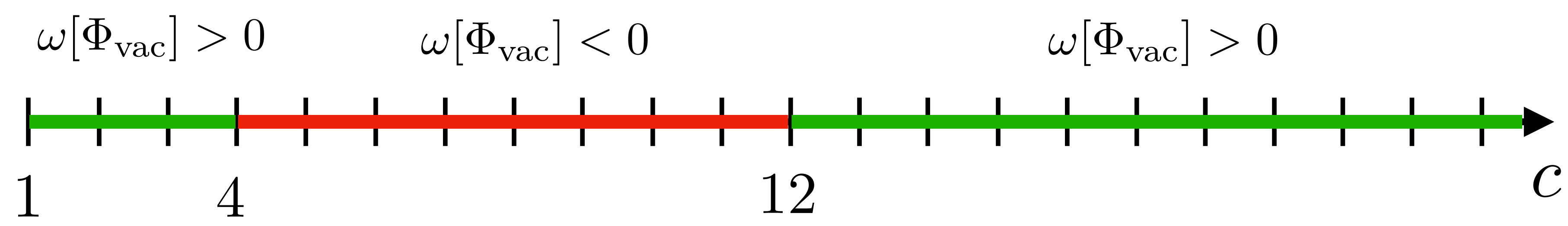

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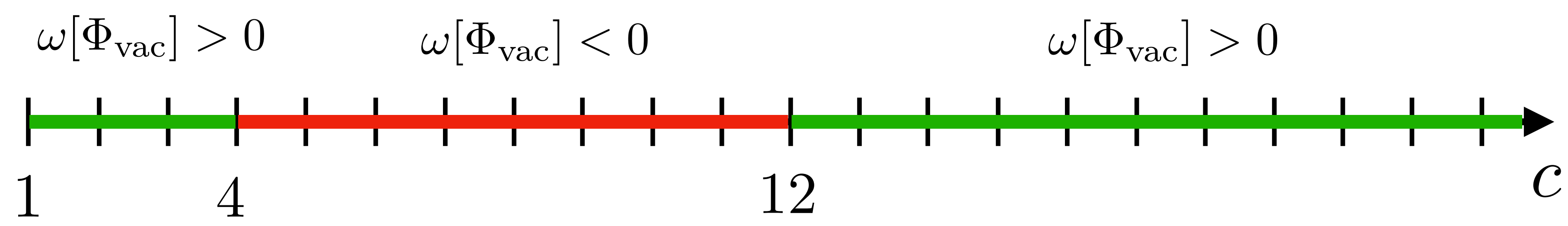

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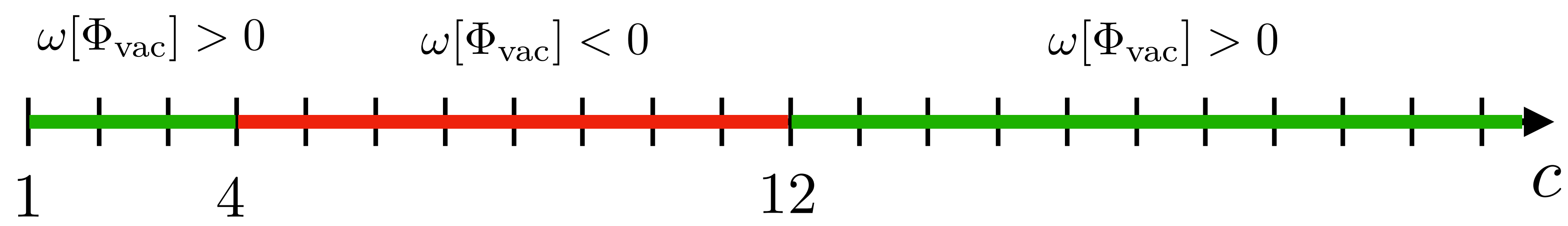

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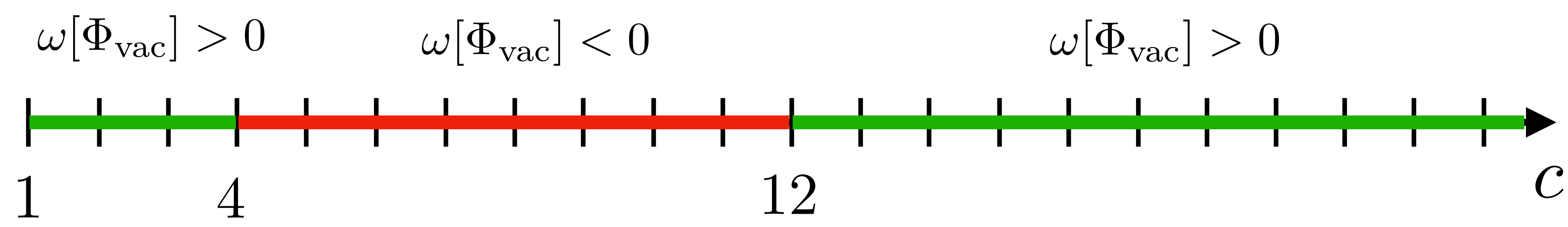

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These two cases will map to the solution of the sphere packing problem in $d = 8$ and $d = 24$.

Road Map

1. Virasoro Modular Bootstrap
    - AdS_3/CFT_2 and the modular bootstrap
    - Analytic functionals review
    - Proof of the main theorem

2. Sphere Packing Problem
    - Sphere packing review
    - Bounds from linear programming
    - The solution in 8 and 24 dimensions from the analytic bootstrap

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Statement: Find the densest arrangement of identical non-overlapping spheres in $\mathbb{R}^d$.

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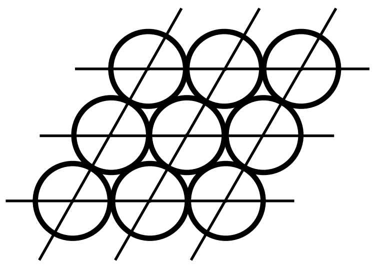

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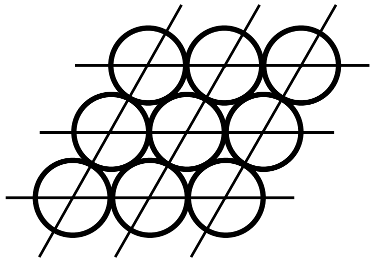

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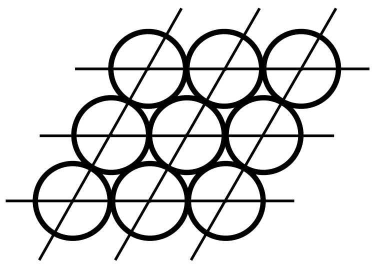

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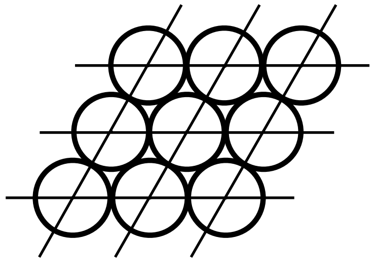

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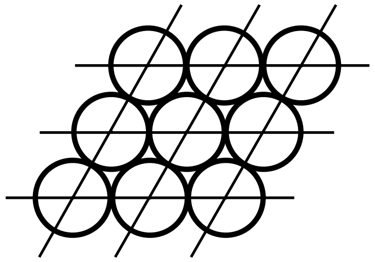

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Sphere Packing Problem

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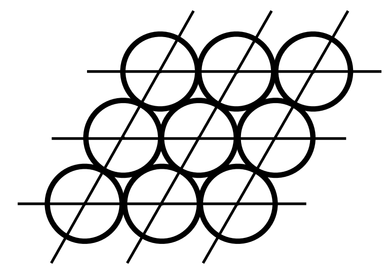
Deep problem, connections to number theory, cryptography, etc.

$d = 1$ trivial

$d = 2$ the honeycomb lattice [Toth '40]



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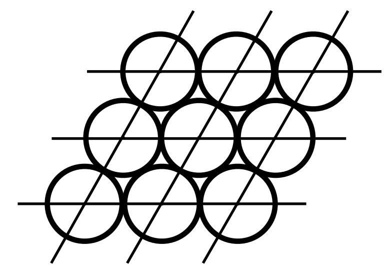

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No requirement to be a lattice in general! Efficient packings in large d highly irregular.
[Torquato, Stillinger '05]

The Sphere Packing Bootstrap

[Cohn, Elkies '01]
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Idea: Prove a universal upper bound on the density of any packing in $\mathbb{R}^d$ and show that this bound is saturated by the E_8 and Leech lattice in $d = 8, 24$.

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- Use functional bootstrap to derive an upper bound on Δ_{gap}

$\Longrightarrow$ upper bound on the sphere packing density

Conclusion: Modular bootstrap in the presence of $U(1)^c$ symmetry constrains the sphere-packing density in $d = 2c$ dimensions!

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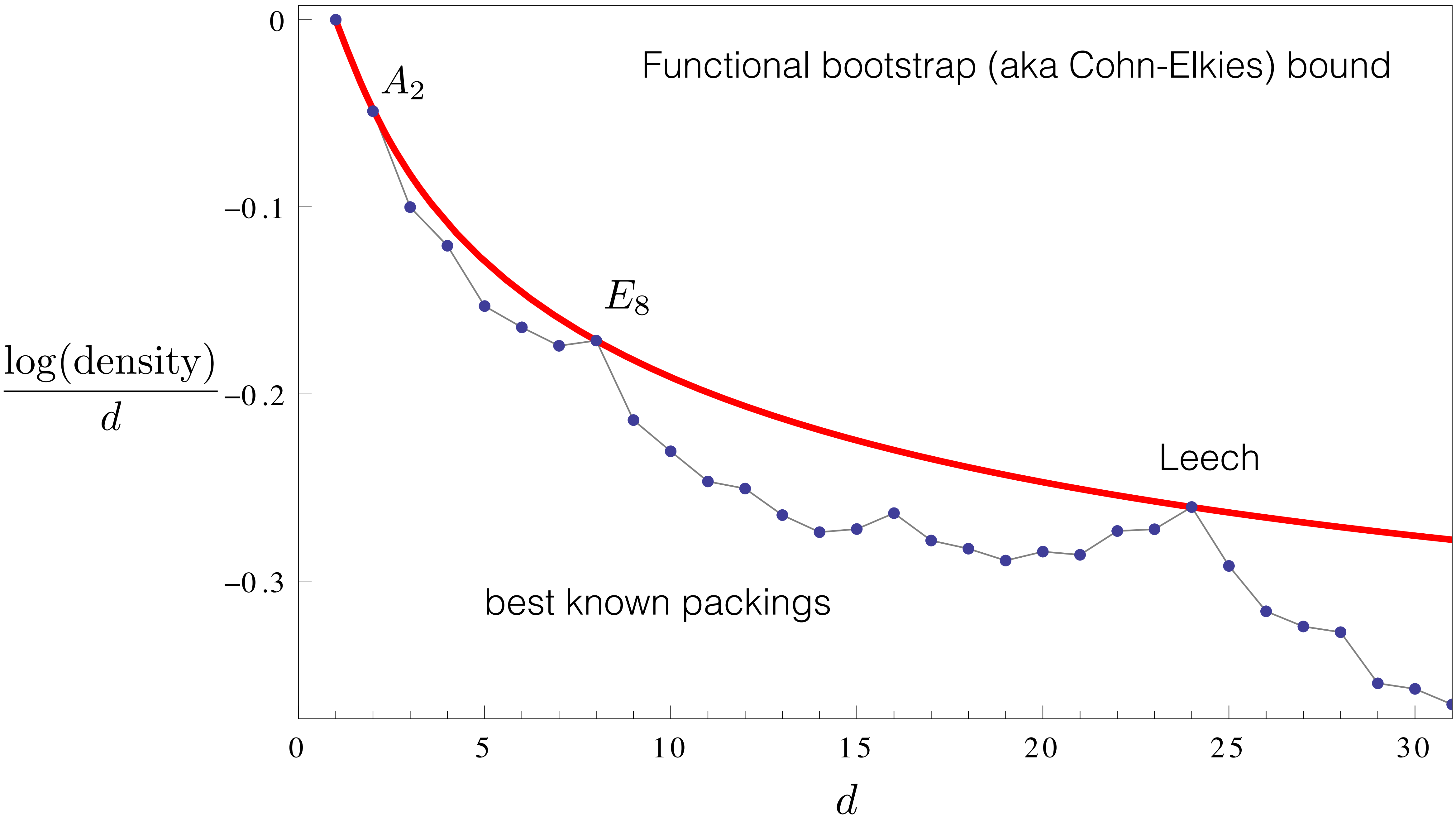

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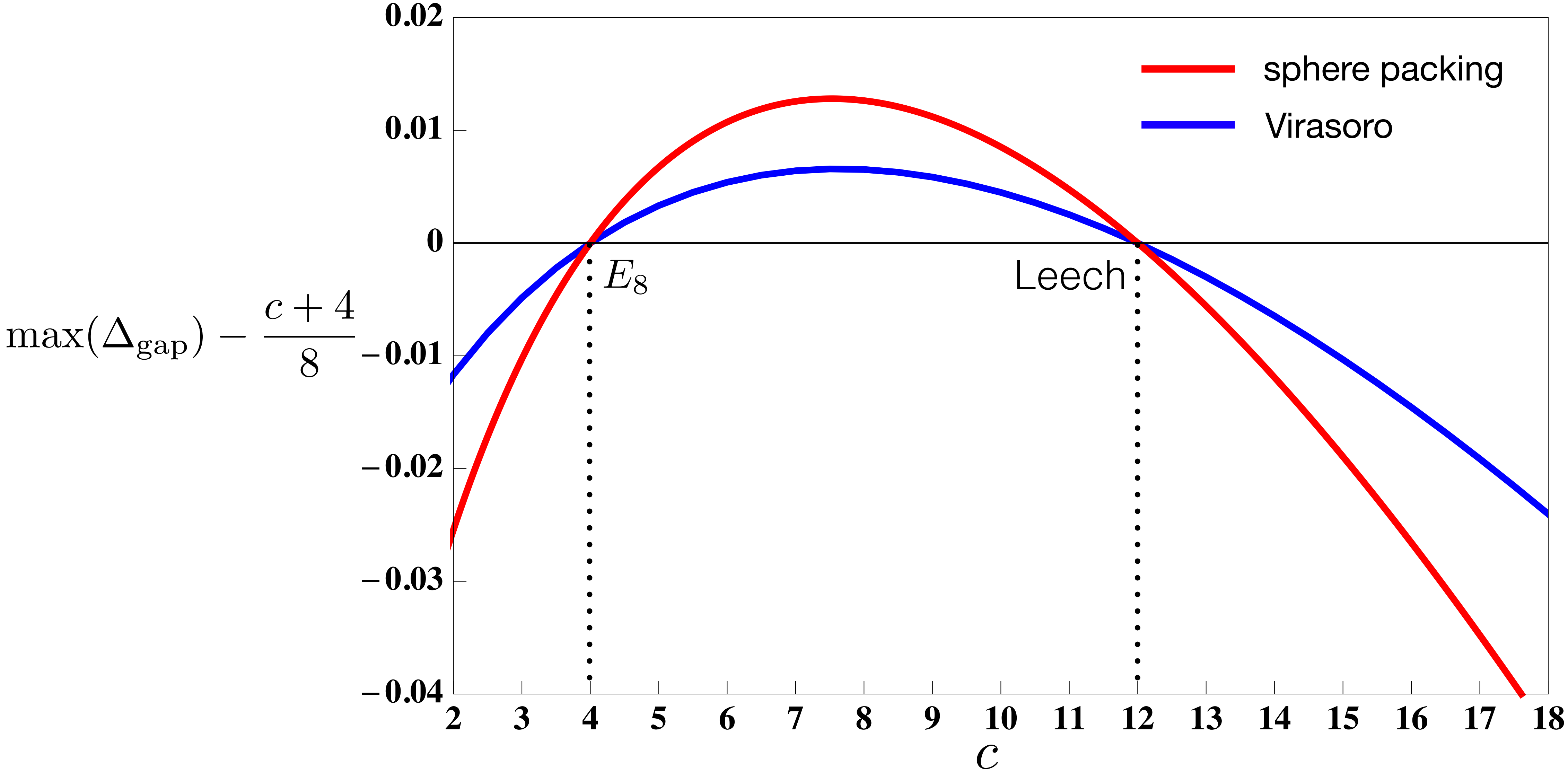

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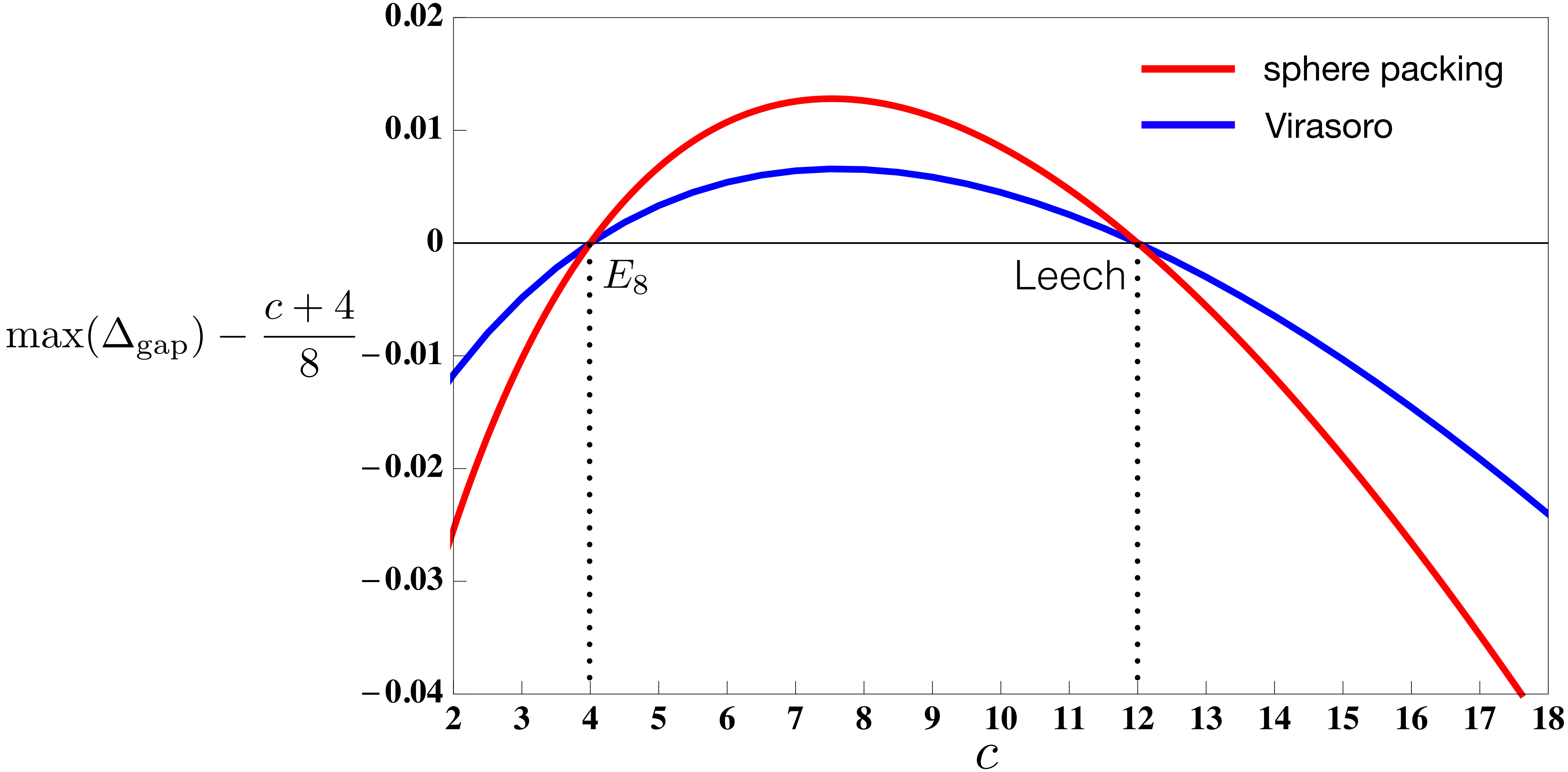

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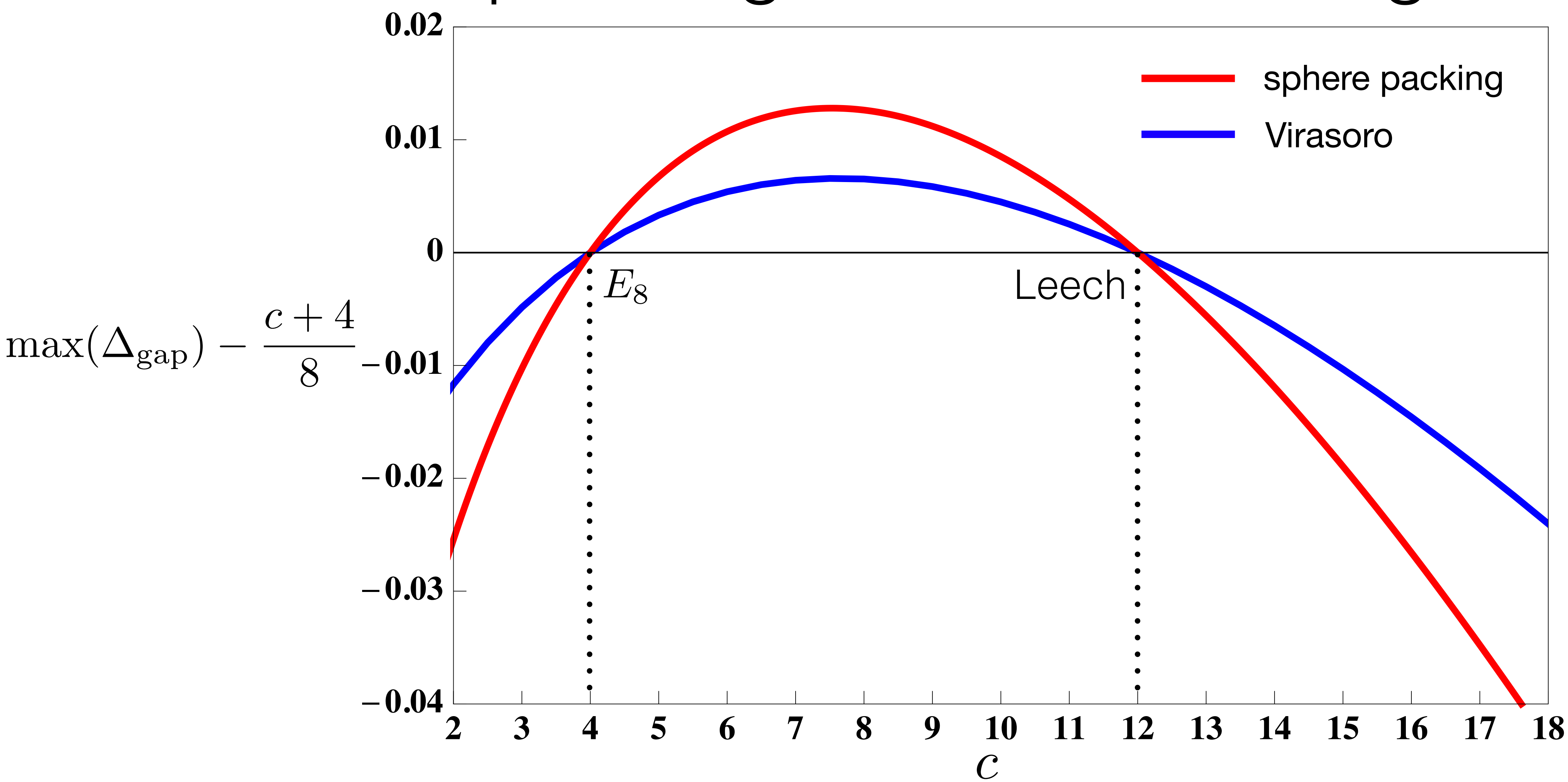

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What I have described is a condensed version of Viazovska's solution.

Dictionary

	3D quantum gravity	sphere packing
parameter	central charge c	dimension of space $d = 2c$
symmetry	Virasoro^2	$U(1)^c \times U(1)^c$
partition function	$Z(\tau) = \sum_{\text{primaries}} \frac{e^{2\pi i \tau (\Delta - \frac{c-1}{12})}}{\eta(\tau)^2}$	$Z(\tau) = \sum_{\substack{\text{pairs of} \\ \text{spheres}}} \frac{e^{\pi i \tau \lvert x_i - x_j \rvert^2}}{\eta(\tau)^d}$
scaling dimension	Δ	distance in $\mathbb{R}^d$ $\quad r = \sqrt{2\Delta}$
optimal bounds	$c = 4$: $\quad \Delta_{\text{gap}} \leq 1$ $c = 12$: $\quad \Delta_{\text{gap}} \leq 2$	E_8 lattice optimal in $d = 8$ Leech lattice optimal in $d = 24$

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In this context, the analytic functionals were discovered independently (and earlier!) by Viazovska under the name magic functions, building on the work of Cohn, Elkies and others. This lead to the solution of the sphere-packing problem in 8 and 24 dimensions.

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Can pure gravity be ruled out, perhaps with some extra assumptions?

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Large scaling dimensions (UV)	$\sim$	Large distances in the packing (IR)

Thank you!