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Filling the boundary: NUTs and Bolts and AdS black holes

Ecole Normale Superieure

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Based on work in collaboration with B. Willett and K. Hristov, S. Katmadas

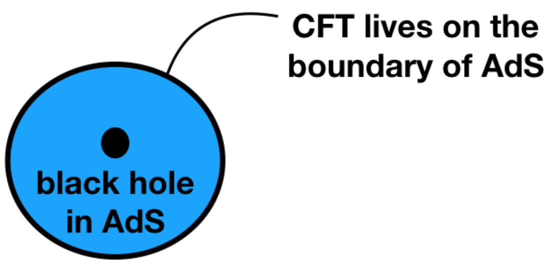
Holography and AdS/CFT correspondence

AdS/CFT correspondence [Maldacena, '97]: gravitational physics in d−dimensional Anti-de Sitter spacetime ("bulk") dual to conformal field theories in $d-1$ dimensions ("boundary")

- **Weak-strong** coupling duality. Strong coupled QFT dual to classical gravity in the bulk
- When bulk and boundary are supersymmetric perform detailed computations e.g. statistical interpretation of AdS black hole entropy

AdS black holes: entropy related to the **counting of states** in the dual CFT, living on the boundary.

Exact quantities in field theory (i.e. partition function) computed via *supersymmetric localization*.



Static AdS black holes

Recent success: microstate counting for susy AdS_4 black holes [Cacciatori, Klemm '09]

- black holes are flows from AdS_4 to $AdS_2 \times \Sigma_g$ near horizon geometry
- magnetic gauge field cancels spin connection in the susy equations (topological twist)

Boundary is $S^1 \times \Sigma_g$: ABJM partition function on $S^1 \times \Sigma_g$ with magnetic fluxes s_i on Σ_g computed via susy localization, in the large N limit [Benini, Hristov, Zaffaroni '15], [Benini, Zaffaroni '16]

$$\log Z_{S^1 \times S^2} \approx -\frac{2\pi N^{3/2}}{3} \sqrt{2 m_1 m_2 m_3 m_4} \sum_{i=1}^{4} \frac{s_i}{m_i} \qquad \sum m_i = 2\pi$$

reproduces Bekenstein–Hawking entropy upon extremization on m_i.

Holographic matching

This talk: models of 4d $\mathcal{N}=2$ gauged supergravity coming from M-theory on SE_7 (e.g S^7).

Dual 3d CFT in the class of ABJM [Aharony, Bergman, Jafferis, Maldacena '08]

Aim: study supersymmetric partition functions of $N=2$ 3D SCFTs on M_d with holographic dual. We will learn from both sides of the duality

Asymptotically locally AdS solutions with asymptotics

$$ds^2_{d+1} = \frac{dr^2}{r^2} + (r^2 ds^2_{M_d} + O(r)) \qquad A_\mu = A_{M_d} + O(1/r)$$

correspond to CFTs on a d-manifold M_d and a non trivial background field for the R-symmetry

Holographic matching

1. Field theory on **product** manifolds $S^1 \times \Sigma_g$
   - Static (magnetic) black holes

2. $U(1)$ **p-bundle** over 2d Riemann surfaces Σ_g

$$ds^2 = (d\psi + a)^2 + d\Omega_\kappa^2 \qquad -\frac{1}{4\pi}\int_{\Sigma_g} da = p$$

   - "Bolt" solutions with NUT charge [Toldo, Willett '17]

3. Refinement by **angular momentum**

$$ds^2 = d\theta^2 + f(\theta)(d\phi - \zeta dt)^2 + dt^2$$

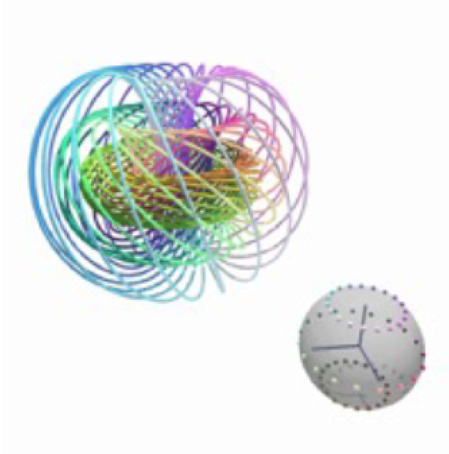
   - Rotating BPS black holes [Hristov, Katmadas, Toldo '18–'19]

S^1 p-bundle over 2d Riemann surfaces Σ_g ($\mathcal{M}_{g,p}$)

Partition function of $\mathcal{N} = 2$ SCFTs on infinite family of manifolds $\mathcal{M}_{g,p}$ [Closset, Kim, Willett '17] was recently computed via localization

$$ds^2 = (d\psi + a)^2 + d\Omega_\kappa^2 \qquad -\frac{1}{4\pi}\int_{\Sigma_g} da = p$$

- $\mathcal{M}_{g,0} \simeq S^1 \times \Sigma_g$ making contact with black holes
- $\mathcal{M}_{0,1} \simeq S^3$ partition function: F-theorem, entanglement entropy
- $\mathcal{M}_{0,p} \simeq S^3/\mathbb{Z}_p$ Lens space partition function



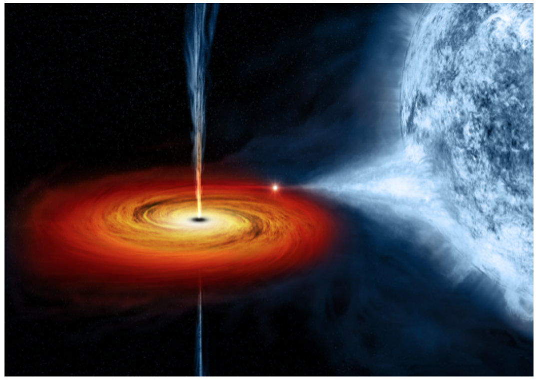
Aim: to provide regular supersymmetric fillings and match their free energy. $\mathcal{M}_{g,p}$ boundary due to presence of NUT charge

Rotating AdS black holes

In AdS_4 extremal black holes with angular momentum can preserve susy!

Not possible in 4D Minkowski

Extremal rotating AdS_4 black holes have Near Horizon geometry in the same class as the Near-Horizon Extremal Kerr, present in our universe.



Amount to solving first order PDE coming from Killing spinor equations (harder than static)

Holography:

Lot of studies for 5d black holes [Kunduri,Lucietti], [Gutowski, Reall '07]

4d black holes: ABJM indices $S^1 \times S^2$ with angular momentum refinement less studied

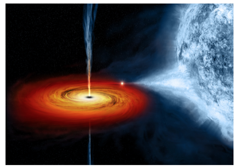
Outline

NUTS AND BOLTS

- Supersymmetric solutions for NUTs and Bolts
- Free energy and matching with dual CFT

ROTATING ADS BLACK HOLES

- Solutions and entropy function



Minimal $\mathcal{N} = 2$ gauged supergravity

Euclidean minimal 4d gauged supergravity contains only the gravity multiplet (no vector multiplets or hypermultiplets). Bosonic action is Einstein-Maxwell-Λ:

$$S = \int d^4x \sqrt{g} \left[R - F_{\mu\nu}F^{\mu\nu} + \frac{6}{l^2} \right]$$

For a BPS solution the gravitino susy variation is zero

$$\delta_\epsilon \psi = \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma^{ab} + \frac{1}{2l}\gamma_\mu + iA_\mu + \frac{i}{4}F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu \right) \epsilon = 0$$

where $\gamma_\mu \in \mathrm{Cliff}(4, 0)$ and $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. Any solution M_4 of this theory uplifts locally to a solution of $M_4 \times Y^7$ of 11d sugra

$\mathcal{M}_{g,p}$ boundary: related to the presence of NUT charge s. General* solution to the equations of motion [Chamblin, Emparan, Johnson,Myers '98] depends on M, Q, P, κ, s

$$ds^2 = \frac{\lambda(r)}{r^2 - s^2}(d\tau + 2s\, f(\theta) d\phi)^2 + \frac{dr^2(r^2 - s^2)}{\lambda(r)} + (r^2 - s^2) d\Omega_\kappa^2$$

$$\lambda(r) = (r^2 - s^2)^2 + (\kappa - 4s^2)(r^2 + s^2) - 2Mr + P^2 - Q^2$$

and

$$d\Omega_\kappa^2 = d\theta^2 + f'(\theta)^2 d\phi^2 \qquad f(\theta) = \begin{cases} \cos\theta & \text{for} \quad \kappa = 1 \\ -\theta & \text{for} \quad \kappa = 0 \\ -\cosh\theta & \text{for} \quad \kappa = -1 \end{cases}$$

The gauge field is

$$A_\tau = \frac{-2sQr + P(r^2 + s^2)}{r^2 - s^2} \qquad\qquad A_\phi = f(\theta)\frac{P(r^2 + s^2) - 2s\, Q\, r}{r^2 - s^2}$$

Compact Σ_g by appropriately taking the quotient of $\mathbf{R}^2$ and $\mathbf{H}^2$

NUTs and Bolts

Boundary ($r \to \infty$) is a circle bundle over a constant curvature Σ_g

$$ds^2 = \frac{dr^2}{r^2} + r^2 \left(4s^2(d\psi + f(\theta)d\phi)^2 + d\Omega_\kappa^2\right)$$

For the choice $p = 1, g = 0$, imposing periodicity $\Delta\psi = 4\pi$ the boundary is a biaxially squashed S^3.

Impose regularity (non-singular solutions). Two classes depending on the dimension of the fixed point for the Killing vector ∂_ψ.

- "NUT" (AdS-Taub-NUT): isolated fixed point
- "BOLT" (AdS-Taub-Bolt): 2d fixed point

NUTs and Bolts

"NUT" ($\mathbf{R}^4$ topology)

- $\lambda(r)$ has double root at $r = s$, identified with the origin of $\mathbf{R}^4$
- Rregularity condition is $\Delta\psi = 4\pi$ ("mildly singular" $\Delta\psi = 4\pi/p$). Boundary is squashed S^3

"Bolt" (topology $\mathcal{O}(-p) \to S^2$)

- $\lambda(r)$ has simple root at $r_b > s$
- regularity condition at the Bolt $r = r_b$ requires: $\left|\frac{r_b^2 - s^2}{s\lambda'(r_b)}\right| = \frac{2}{p}$. This gives $\Delta\psi = 4\pi/p$ so that boundary is squashed Lens space $S^3/\mathbb{Z}_p$

Spherical Bolts can be generalized into Bolts with higher genus Riemann surface Σ_g base.

Supersymmetry and regularity

Impose supersymmetry (1/4 BPS) : $P = \frac{1}{2}(4s^2 - \kappa) \qquad M = 2sQ$

[Alonso-Alberca et al '00], [Martelli et al.,'12]

+ Regularity:

- NUT solutions with double root at $r = s$, giving $Q = P = \frac{1}{2}(4s^2 - 1)$.
- For Bolts we need to impose the regularity condition

$$\left.\frac{r^2 - s^2}{2s\lambda'(r)}\right|_{r_b} = \frac{2}{\mathbf{p}}$$

which constrains the value of Q:

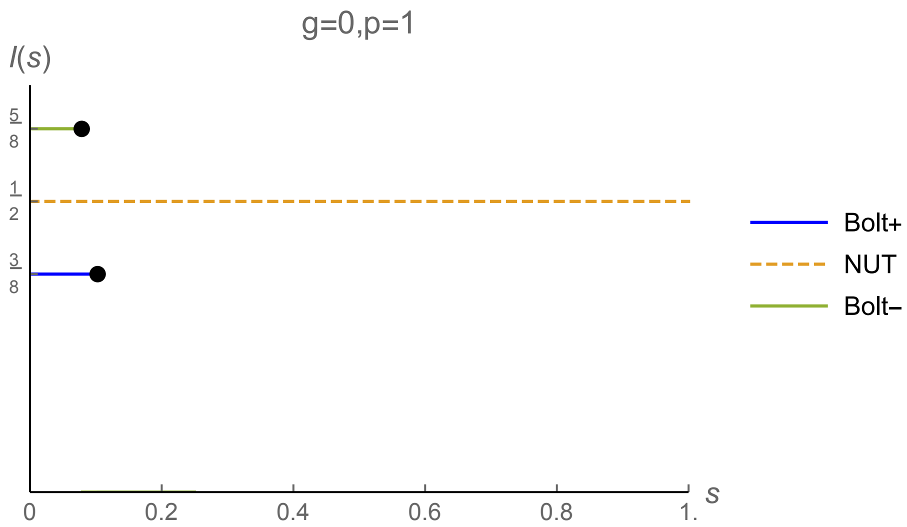
$$Q_+^\pm = \frac{p^2 \mp (16s^2 - p)\sqrt{f_+}}{128s^2} \qquad Q_-^\pm = -\frac{p^2 \mp (16s^2 + p)\sqrt{f_-}}{128s^2}$$

$$f_\pm = (16s^2 \pm p)^2 - 128\kappa s^2$$

Up to four different branches, denoted with $\text{Bolt}_\pm$.

In general a solution exists for a range of parameter s. Different branches of solutions joining each others at special points.

- NUT ($g = 0$) exists for every value of squashing s. For $s = 1/2$ boundary is round S^3
- Bolt solutions exist for finite range of s for $p = 1, 2$. For $p \geqslant 3$ Bolts are present for every s



Flux and uplift in 11d

Flux through the Bolt computed as

$$\mathcal{F}_{\mathrm{Bolt}\pm} = \int_{\Sigma_g} \frac{F}{2\pi} = (g-1) \pm \frac{p}{2}$$

For the uplift to 11d on SE^7 [Gauntlett, Varela '07] to be well defined

$$ds_{11}^2 = R^2 \left(\frac{1}{4} ds_4^2 + (d\chi + \sigma + \frac{1}{2} A)^2 + ds_6^2 \right) \qquad G = R^3 (\frac{3}{8} \mathrm{vol}_4 - \star dA \wedge d\eta)$$

The flux through the Bolt 2-cycle needs to be quantized

$$\chi \sim \chi + 2\pi \frac{I}{4} \qquad \frac{4}{2\pi I} \int_{\Sigma_g} \frac{F}{2} \in \mathbb{Z}$$

I = "Fano number" of Y^7

For ABJM: $\pm p + 2(g-1) = 0 \mod I(Y^7)$

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For ABJM: $\pm p + 2(g-1) = 0 \mod 4$

Holographic free energy of NUTs and Bolts

Evaluating bulk supergravity action on a NUT/Bolt leads to divergencies.

Regularize with the introduction of cutoff r_{inf} and add the boundary term from holographic renormalization [Skenderis, '02]

$$I = I_{bulk} + I_{ct}$$

$$I_{bulk} = -\frac{1}{16\pi G_4}\int d^4x\sqrt{g}(R + 6 - F^2) \qquad I_{ct} = \frac{1}{8\pi G_4}\int_{\partial M} d^3x\sqrt{\gamma}\,(2 + \frac{1}{2}R(\gamma) - K)$$

Manifold closes off at $r_0 = s$ for NUT and $r_0 = r_b$ for Bolt.

Evaluating bulk terms we obtain

$$I_{grav}^{bulk} = \frac{1}{8\pi G_4}\frac{16\pi^2}{p}[2sr_{inf}^3 - 6s^3r_{inf} - 2sr_0 + 6s^3r_0]$$

Terms from gauge fields:

$$I_{F,NUT} = \frac{2\pi}{G_4}\frac{(1-4s^2)^2}{4}$$

$$I^{bulk}_{F,Bolt} = \frac{\pi s r_b \left(r_b^2\left(\left(\kappa-4s^2\right)^2+4Q^2\right)+8\left(4s^2-\kappa\right)sQr_b\right)}{2G_4\left(s^2-r_b^2\right)^2 p} + \frac{\pi s r_b\left(s^2\left(\left(\kappa-4s^2\right)^2+4Q^2\right)\right)}{2G_4\left(s^2-r_b^2\right)^2 p}$$

Boundary counterterms:

$$I_{ct} = \frac{1}{8\pi G_4}\frac{16\pi^2}{p}[4Qs^2 - 2sr_{inf}^3 + 6s^3 r_{inf} + O(r_{inf}^{-1})]$$

...Putting all together:

$$I_{NUT} = \frac{\pi}{2G_4}, \qquad I_{Bolt_\pm} = \frac{\pi\left(4(1-g)\mp p\right)}{8G_4}$$

g=0 case reduces to [Martelli et al. '12]. No dependence on s !

Recap (ABJM)

- NUT solutions with S^3 $(g = 0, p = 1)$ boundary

$$I_{\text{NUT}} = \frac{\pi}{2G_4}$$

- Bolt solutions with $\mathcal{M}_{g,p}$ boundary

$$I_{\text{Bolt}_\pm} = \frac{\pi\,(4(1-g) \mp p)}{8G_4}$$

with quantization condition

$$\pm p + 2(g-1) = 0 \mod 4$$

Free energy result for S^3

$\mathcal{N} = 2$ CS theories on the squashed S^3 studied in [Hama, Hosomichi, Lee '11]: one family found to be independent of the squashing s.

Computation of the ABJM partition function on S^3 gives

$$F_{S^3} = -\mathrm{Log}(Z_{S^3}) = \frac{\sqrt{2}\pi N^{3/2}}{3} = I_{\mathrm{NUT}}$$

Matching already noticed in [Martelli, Passias, Sparks '12].

- What about the (infinite tower of) Bolts with $\mathcal{M}_{g,p}$ boundary?

$\mathcal{M}_{g,p}$ partition function

Recently the partition function of superconformal $\mathcal{N}=2$ theories on $\mathcal{M}_{g,p}$ computed in [Closset, Kim, Willett '17]. It has the form of sum over vacua

$$Z_{\mathcal{M}_{g,p}} = \sum_{u_a \in S_{BE}} \mathcal{F}_I(u_a, m_i)^p \, \mathcal{H}_I^{g-1}(u_a, m_i) \, \Pi_I^i(u_a, m_i)^{s_i}$$

where

- $\mathcal{F}_I(u_a, m_i) = \exp\left(2\pi i \left(\mathcal{W}^I - \sum_i m_i \frac{\partial \mathcal{W}^I}{\partial m_i}\right)\right)$ "fibering operator"
- $\mathcal{H}_I = \exp\left(2i\pi\Omega^I\right) \det_{ab} \frac{\partial^2 \mathcal{W}}{\partial u_a \partial u_b}$ "handle gluing operator"
- $\Pi_I^i = \exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial m_i}\right)$ "flux operator"

Large N limit of the partition function is

$$\log Z_{\mathcal{M}_{g,p}}(m_i, n_i) = pW_{ext} - \sum_i (pm_i - s_i - (g-1)r_i)\partial_i W_{ext}$$

$\mathcal{M}_{g,p}$ partition function

Insert extremal superpotential $\mathcal{W}_{\text{ext}}$

$$\log Z_{\mathcal{M}_{g,p}} = \frac{2\pi N^{3/2}}{3}\sqrt{2[m_1][m_2][m_3][m_4]}\left(2p - \sum_i \frac{-pn_i + s_i + (g-1)r_i}{[m_i]}\right)$$

with $\quad \sum_{j=0}^{3} m_j = \sum_j s_j = 0\,, \qquad \sum_j [m_j] = 1\,, \qquad \sum_j (r_j - 1) + 2 = 0$

Our supergravity setup: constant scalars and equal fluxes

$$[m_i] \;\; \text{and} \;\; -pn_i + s_i + (g-1)r_i \;\; \text{independent of} \;\; i$$

gives $p + 2(g-1) = 0 \mod 4$ and

$$-\log Z^{\text{ABJM}}_{\mathcal{M}_{g,p}} = \frac{\pi N^{3/2}}{6\sqrt{2}}\left(4(1-g) - p\right) = I_{\text{Bolt}+}$$

which matches the supergravity result!

Recap: results for ABJM

- For $k = 1$ ABJM theory Bolt free energy matches [Toldo, Willett '17] the large N localization result for $\mathcal{M}_{gp}$ with

$$\pm p + 2(g-1) = 0 \mod 4$$

  For $p = 0$ reduces to [Azzurli, Bobev, Crichigno, Min, Zaffaroni '17] which gives the entropy of minimal supergravity black holes with Σ_g horizons.

- S^3 boundary treated separately, computed (also $S^3/\mathbb{Z}_p$ (no flux) in [Alday,Fluder,Sparks]):

$$I_{\rm NUT} = I_{\rm EAdS4} = -Z_{\rm ABJM}(S^3)$$

Everything good so far... How about the $p = 1$ Bolt?

$p = 1$ Bolt

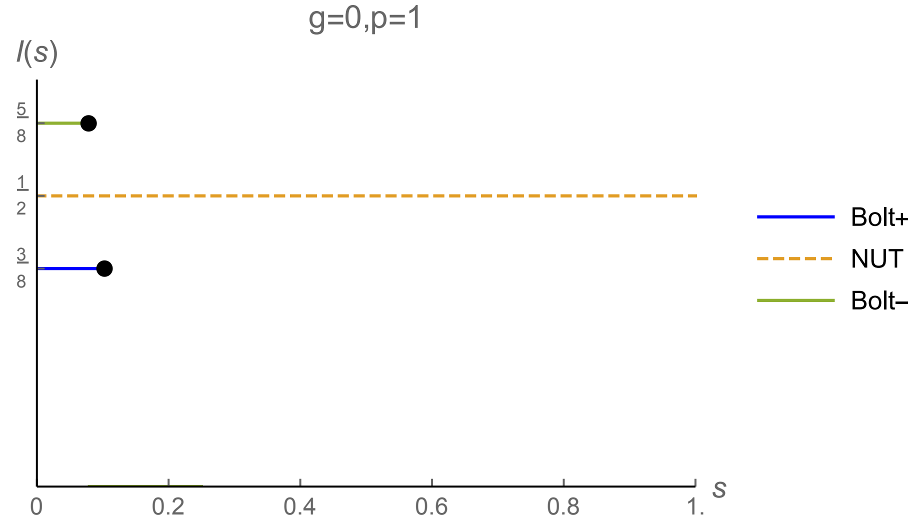
There are cases in which the S^3 boundary ($g = 0, p = 1$) admits Bolt fillings for certain parameter ranges $s \in [s_-, s_+]$. Uplift condition is

$$\pm p + 2(g-1) = 0 \mod I(Y^7)$$

Bolt free energy is

$$I_{\mathrm{Bolt}+} = \pi \frac{4-p}{8G_4} = \frac{3\pi}{8G_4}$$

$$I_{\mathrm{Bolt}-} = \pi \frac{4+p}{8G_4} = \frac{5\pi}{8G_4}$$



Curious case of $p = 1$ Bolt

- M-theory on $M^{3,2}$: both Bolt$\pm$ with $p = 1$ uplift. Dual is chiral quiver, computation not under control.

- M-theory on $Y^7 = V^{5,2}$

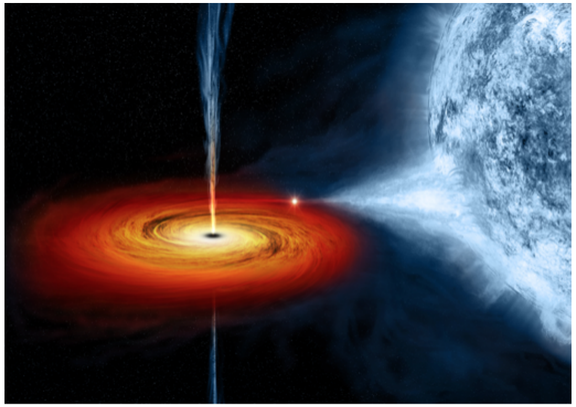
$$\pm p + 2(g-1) = 0 \quad \text{mod } 3$$

hence $p = 1$ Bolt- uplifts! Its free energy is $I_{\text{Bolt}-} = \frac{5}{4} I_{\text{NUT}}$.

Free energy for $V^{5,2}$ dual theory on S^3 computed by [Martelli, Sparks '08] coincides with I_{NUT}

However... Our result for $Z(\mathcal{M}_{g,p})$ reproduces *subleading* saddle point $I_{\text{Bolt}-} = \frac{5}{4} I_{\text{NUT}}$

ROTATING ADS$_4$ BLACK HOLES



based on collaboration with K. Hristov and S. Katmadas

Rotating supersymmetric black holes

First studies in minimal 4d gauged supergravity. **Two classes** of BPS solutions:

- electric KN AdS black hole: no static limit [Kostelecky,Perry '92]. Charges satisfy

$$M = \frac{J}{l_{AdS}} + Q$$

- magnetic KN AdS black hole with $P = \frac{l_{AdS}}{2}$ exist only with noncompact horizon: has static limit [Caldarelli,Klemm'98]. For compact horizon we need scalars with radial profile

U(1) FI gauged supergravity + vector multiplets: only isolated examples [Cvetic et al,'04],[Klemm '11]. Lack of systematic! Solutions with both compact horizon and static limit possible.

Matter-coupled rotating black holes

Solve BPS first order PDE of [Meessen,Ortin '12] with appropriate ansatz on metric, symplectic sections (scalar fields) and vectors

- obtain general STU 1/4 BPS rotating black hole solutions of FI-gauged coupled to vector-multiplets [Hristov, Katmadas, Toldo, '18-19]. Admit embedding in M-theory

Start from metric with timelike Killing vector

$$ds^2 = -e^{2U}(dt + \omega)^2 + e^{-2U} ds_3^2$$

and base space

$$ds_3^2 = d\rho^2 + e^{2\phi}(dx^2 + dy^2)$$

Assume further isometry ∂_y such that $\omega = f(\rho, x)dy$ and $\phi = \phi(\rho, x)$. Two classes with different choices for $\phi(\rho, x)$.

Two classes of solutions

- Magnetic AdS solutions

$$e^{2\phi} = \Phi(x) e^{2\psi(\rho)}$$

ρ radial variable, x, y are coordinates on the sphere.

Realize the topological twist in sugra, $g_I P^I = 1$

e.g. for T^3 model

$$S_{BH} = \pi \sqrt{\frac{\sqrt{(1+12p^1)(1+4p^1)^3 - 4J^2 l_{AdS}^{-4}} - (24(p^1)^2 + 12g_1 p^1 + 1)}{2l_{AdS}^{-4}}}$$

to be reproduced by the twisted $S^1 \times S^2$ partition function with angular momentum refinement

Two classes of solutions

- Electric Kerr–Newman solutions

$$e^{2\phi} = Q(q)P(p) \qquad \rho = qp \qquad x = \alpha(q) + \beta(p)$$

such that

$$ds_3^2 = (q^2 P(p) + p^2 Q(q)) \left(\frac{dp^2}{P(p)} + \frac{dq^2}{Q(q)} \right) + Q(q)P(p) dy^2$$

q radial variable, p, y are coordinates on the sphere. No twist, $g_I P^I = 0$

Entropy function $\mathcal{S}$ gives S_{BH} upon extremization wrt fugacities [Choi et al. '18]

$$\mathcal{S}(\omega, X^I) = i\frac{4\sqrt{2}}{3}\frac{\sqrt{m_1 m_2 m_3 m_4}}{\omega} + \sum_I m^I q_I + \omega J \qquad \sum_I m^I + \omega + 2\pi i = 0$$

To be reproduced by the superconformal index.

Dual field theory results

Boundary geometry:

$$ds^2 = r^2\Delta(\theta)\left[-\frac{dt^2}{l_{AdS}^2} + \frac{d\theta^2}{\Delta(\theta)^2} + \frac{\sin\theta^2}{\Delta(\theta)}\left(d\phi + \frac{j}{l_{AdS}^3}dt\right)^2\right]$$

Electric case:

- Superconformal index $\sim O(1)$ [S. Kim '09]. Should be instead $\sim N^{3/2}$
- Taking complex fugacities (already considered in 5d [Cabo-Bizet et al '18, Benini '18, S. Kim et al '18, Honda '18, Ardehali '18]) one obtains $N^{3/2}$ scaling in the Cardy limit, and match large BH entropy [S. Kim et al, '19]. **Exciting**! still some subtleties though

Magnetic (twisted case):

- Large N twisted index with angular momentum refinement [Benini, Zaffaroni '16] difficult to compute, though some progress in [Closset, Kim, Willett '18]

Conclusions and perspectives

Conclusions and perspectives

Holography provides new tools for microstate counting

- Rotating AdS black holes have NH geometry in the same class as the Near-Horizon Extremal Kerr, present in our universe. Exciting! but lots of work to do
- Localization: all corrections to macroscopic black hole entropy. Wealth of information about quantum gravity! Supergravity captures leading N contribution, precision tests i.e. logarithmic corrections [Liu et al '17] to be done

Rotating supersymmetric black holes also present in higher dimensions: 5d, 6d etc. Microstate description? Perhaps general lessons to learn

From here where?

Euclidean instantons of gauged supergravity give non-trivial predictions for the large N limit of matrix models. Geometry probes **new phases of dual QFT**.

Computation of supersymmetric observables: insights on strong coupling behaviour of dual field theories and check dualities

- Higher dimensional NUTs and Bolts and partition functions on $\mathcal{M}_{g,p} \times S^1$, $\mathcal{M}_{g,p} \times \Sigma_g$
- Matter-coupled supergravity and match of full CFT formula [Passias, CT, Willett, in progress]

the end. Thank you!

Backup

First Method due to [Jafferis, Klebanov, Pufu, Safdi]. Free energy functional is

$$Z(S^3) = \frac{1}{\mathcal{W}} \int d^N u \, \mathcal{G}(u_a, m_i)$$

with

$$\mathcal{G}(u_a, m_i) = \mathrm{Exp}\left(2\pi i \left(\mathcal{W}(u_a, m_i) - u_a \partial_a \mathcal{W}(u_a, m_i) - m_i \partial_i \mathcal{W}(u_a, m_i) - \Omega(u_a, m_i)\right)\right) \quad (1)$$

Look for saddle points in the u_a plane, with ansatz

$$u_a = v_a + i N^{1/2} t_a$$

Large N limit, eigenvalue become dense, parameterize with

$$\rho(t) = \frac{1}{N} \sum_{a=1}^{N} \delta(t - t_a) \qquad \text{with} \qquad \int dt \rho(t) = 1$$

The integral over t

$$\log \mathcal{G}[\rho, m_i] = \int dt\, f(\rho(t), m_i, t)$$

and to leading order in N the partition function is given by the extremal value

$$\log Z(S^3) = \log \mathcal{G}[\rho*, m_i]$$

$\rho*$ is the eigenvalue density extremizing the functional.

Second method [Benini,Zaffaroni] applied to $\mathcal{M}_{g,p}$

The initial integrand can be rewritten as a sum over Bethe vacua

$$Z_{S^3} = \sum_{\tilde{u}_a \in \mathcal{S}_{BE}} \mathcal{G}(\tilde{u}_a, m_i) H(\tilde{u}_a, m_i)^{-1}$$

with $H = \det_{ab} \frac{\partial^2 \mathcal{W}}{\partial u_a \partial u_b}$. Bethe vacua are

$$\exp(\partial_a \mathcal{W}(u_a, m_i)) = 1$$

Take same ansatz (1) and compute the extremum of $\mathcal{W}$,

$$\mathcal{W}[\rho, m_i] = \int dt\, g(\rho(t), m_i, t) \qquad \mathcal{W}_{extr}(m_i) = \mathcal{W}[\rho_{**}, m_i]$$

and plug in to get $Z(S^3)$.