On genus one mirror symmetry

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Calabi–Yau manifolds

Let X be a connected compact Kähler manifold of dimension n. We say that X is a *Calabi–Yau (CY) manifold* if:

- X has a nowhere vanishing holomorphic differential form, *i.e.* the canonical bundle is trivial:

$$K_X = \wedge^n \Omega_X \simeq \mathcal{O}_X.$$

- for $0 < p < n$, there are no non-trivial global holomorphic p-forms:

$$H^0(X, \Omega_X^p) = 0, \text{ or equivalently } H^p(X, \mathcal{O}_X) = 0.$$

A Calabi–Yau manifold is automatically algebraic and projective.

In any given Kähler cohomology class $\nu \in H^{1,1}_{\mathbb{R}}(X)$, there exists a unique Ricci flat Kähler metric.

The infinitesimal deformations of the complex structure of X are parametrized by $H^1(X, T_X)$. This has dimension $h^{n-1,1}$:

$$H^1(X, T_X) \simeq H^1(X, T_X \otimes K_X) \simeq H^1(X, \Omega_X^{n-1}).$$

The deformations of X are unobstructed. There is an open neighborhood of 0, $\mathrm{Def}(X) \hookrightarrow H^1(X, T_X)$ and a local universal deformation of X over $\mathrm{Def}(X)$:

$$\mathfrak{X} \longrightarrow \mathrm{Def}(X), \quad \mathfrak{X}_0 \simeq X.$$

If we fix a polarization L, there is a quasi-projective coarse moduli space $\mathcal{M}$ of (X, L). In a neighborhood of the point corresponding to (X, L), $\mathcal{M}$ is a quotient of $\mathrm{Def}(X)$ by a finite group.

Let $X \to S$ be a morphism of complex manifolds, with Calabi–Yau fibers.

For every $s \in S$, after possibly shrinking S around s, there is a Cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathrm{Def}(X_s). \end{array}$$

We say that the family $X \to S$ is *maximal* if the morphism $S \to \mathrm{Def}(X_s)$ is a local biholomorphism.

Mirror families

Let X be a CY manifold of dimension n.

Mirror symmetry predicts the existence of a *mirror family* of CY n-folds $\varphi\colon \mathcal{X}^\vee \to \mathbf{D}^\times$:

- $\mathbf{D}^\times = (\mathbb{D}^\times)^d$ is a punctured multi-disc, $d = h^{n-1,1}(\mathcal{X}_q^\vee)$.
- $\varphi\colon \mathcal{X}^\vee \to \mathbf{D}^\times$ is maximal.
- the monodromy T on $R^n\varphi_*\underline{\mathbb{C}}$ is maximal unipotent (MUM): if $d = 1$, this means

$$(T-1)^n \neq 0 \quad \text{and} \quad (T-1)^{n+1} = 0.$$

- mirror Hodge numbers: $h^{p,q}(X) = h^{n-p,q}(\mathcal{X}_q^\vee)$.

Informally:

$$\varphi\colon \mathcal{X}^\vee \to \mathbf{D}^\times \leftrightsquigarrow \quad \text{cusp in a moduli space of CY varieties.}$$

There should also exist a bihomolorphic map, called *mirror map*,

$$\tau\colon \mathbf{D}^\times \longrightarrow \mathcal{H}_X := H^{1,1}_{\mathbb{R}}(X)/H^{1,1}_{\mathbb{Z}}(X) + i\mathcal{K}_X,$$

where $\mathcal{K}_X$ is the Kähler cone of X, which relates the variation of complex structures of $\mathcal{X}^\vee \to \mathbf{D}^\times$ and the Kähler "moduli" of X.

When $d = 1$, τ can be identified with a multi-valued function of the form

$$\tau(q) = \frac{1}{2\pi i}\frac{\int_{\gamma_1}\eta}{\int_{\gamma_0}\eta} \in \mathbb{H},$$

where η is a local holomorphic basis of $\varphi_* K_{\mathcal{X}^\vee/\mathbf{D}^\times}$ and γ_0, γ_1 are well-chosen homology n-cycles (Morrison):

- $T\gamma_0 = \gamma_0$.
- $T\gamma_1 = \gamma_1 + \gamma_0$.

Mirror symmetry (rough version)

Variation of Hodge structures on $R^n\varphi_*\underline{\mathbb{C}}$ close to 0. $\leftrightsquigarrow$ Curve counting invariants in X.

For instance, the mirror symmetry conjecture at genus 0 predicts that the *Yukawa coupling* of the mirror family has a holomorphic series expansion in $\tau \in \mathcal{H}_X$, whose coefficients are given by genus zero Gromov–Witten invariants of X.

In this talk we discuss a mirror symmetry conjecture at genus one, where the role of the Yukawa coupling is played by a functorial lift of the Grothendieck–Riemann–Roch theorem.

Functorial BCOV conjecture

The Grothendieck–Riemann–Roch theorem

Let X be a compact complex manifold.

Let E be a holomorphic vector bundle on X.

Recall the Hirzebruch–Riemann–Roch theorem (HRR):

$$\sum_q (-1)^q \dim H^q(X,E) = \int_X \mathrm{ch}(E)\,\mathrm{td}(T_X).$$

The cohomology groups $H^q(X,E)$ can be realized as Dolbeault cohomology:

$$H^q_{\overline{\partial}}(X,E) = \frac{\ker\left(\overline{\partial}\colon A^{0,q}(E) \to A^{0,q+1}(E)\right)}{\mathrm{Im}\left(\overline{\partial}\colon A^{0,q-1}(E) \to A^{0,q}(E)\right)}.$$

The Grothendieck–Riemann–Roch theorem (GRR) is a variant of HRR in an algebraic and relative setting.

Let $f : X \to S$ be a projective submersion of complex connected algebraic manifolds.

Let E be a vector bundle on X.

Theorem (Grothendieck–Riemann–Roch)

The following equality holds in $\mathrm{CH}^\bullet(S)_{\mathbb{Q}}$*:*

$$\mathrm{ch}(Rf_*E) = f_*\left(\mathrm{ch}(E)\,\mathrm{td}(T_{X/S})\right).$$

In particular, for the determinant of cohomology:

$$c_1(\det Rf_*E) = f_*\left(\mathrm{ch}(E)\,\mathrm{td}(T_{X/S})\right)^{(1)} \quad \textit{in} \quad \mathrm{CH}^1(S)_{\mathbb{Q}} \simeq \mathrm{Pic}(S)_{\mathbb{Q}}.$$

In the statement of GRR, Rf_*E stands for the direct image of E in the derived sense.

When $s \mapsto \dim H^q(X_s, E_{|X_s})$ is constant for all q, the cohomology spaces $H^q(X_s, E_{|X_s})$ organize into holomorphic vector bundles $R^q f_* E$. Then

$$\mathrm{ch}(Rf_*E) = \sum_q (-1)^q \,\mathrm{ch}(R^q f_* E)$$

and

$$\det Rf_*E = \bigotimes_q (\wedge^{\max} R^q f_* E)^{(-1)^q}.$$

For instance, this is the case of Hodge bundles, *i.e.* when we take $E = \Omega^p_{X/S}$.

Assume now that the fibers X_s are Calabi–Yau (CY).

Define the virtual vector bundle

$$D\Omega^{\bullet}_{X/S} = \bigoplus_p (-1)^p p\Omega^p_{X/S}.$$

Hence

$$\det Rf_* D\Omega^{\bullet}_{X/S} = \bigotimes (\det R^q f_* \Omega^p_{X/S})^{(-1)^{p+q}p}$$

is a weird combination of determinants of Hodge bundles.

For this datum, GRR simplifies to:

$$c_1(\det Rf_* D\Omega^{\bullet}_{X/S}) = \frac{\chi}{12} c_1(f_* K_{X/S}) \quad \text{in} \quad \mathrm{CH}^1(S)_{\mathbb{Q}} \simeq \mathrm{Pic}(S)_{\mathbb{Q}},$$

with $\chi = \chi(X_s)$ the topological Euler characteristic of the fibers.

Definition (BCOV line bundle)

The BCOV line bundle on S is defined by

$$\begin{aligned}\lambda_{BCOV}(f) &= \det Rf_* D\Omega^{\bullet}_{X/S} \\ &= \bigotimes_p \left(\det R^q f_* \Omega^p_{X/S}\right)^{(-1)^{p+q}p}.\end{aligned}$$

It commutes with arbitrary base change.

Corollary (of GRR)

There exists an isomorphism of $\mathbb{Q}$-line bundles on S

$$\lambda_{BCOV}(f)^{\otimes 12} \xrightarrow{\sim} (f_* K_{X/S})^{\otimes \chi}.$$

But: there are as many as $H^0(S,\mathcal{O}_S^\times)$ such isomorphisms.

Theorem (Eriksson, Franke)

There exists a canonical isomorphism of $\mathbb{Q}$-line bundles

$$\mathrm{GRR}\colon \lambda_{BCOV}(f)^{\otimes 12} \xrightarrow{\sim} (f_* K_{X/S})^{\otimes\chi},$$

commuting with arbitrary base change. If $f: X \to S$ is defined over $\mathbb{Q}$, GRR *is defined over $\mathbb{Q}$ as well.*

The arithmetic Riemann–Roch theorem of Gillet–Soulé provides a weak variant, enough for most purposes:

- natural isomorphism up to a constant of norm one.
- isometry for auxiliary hermitian structures (Quillen metric).
- over $\mathbb{Q}$, the constant is necessarily ± 1.
- compatible with Eriksson–Franke.

Mirror symmetry at genus one

Let X be a Calabi–Yau manifold, and $\varphi\colon \mathcal{X}^\vee \to \mathbf{D}^\times$ its conjectural mirror family.

Definition

Define the formal generating series of genus one Gromov–Witten invariants of X by

$$F_1(\tau) = -\frac{1}{24}\int_X c_{n-1}(X) \cap 2\pi i\tau + \sum_\beta \mathrm{GW}_1(X,\beta) e^{2\pi i\langle\beta,\tau\rangle},$$

where

- $\tau \in \mathcal{H}_X$.
- $\beta \in H_2(X,\mathbb{Z})$ runs over curve classes.
- $\mathrm{GW}_1(X,\beta) =$ genus one Gromov–Witten invariant of class β.

Conjecture (Optimistic functorial BCOV)

Let X and $\varphi\colon \mathcal{X}^\vee \to \mathbf{D}^\times$ be mirrors as above.

Assume that φ can be extended over an algebraic base.

Let $\tau\colon \mathbf{D}^\times \to \mathcal{H}_X$ be the mirror map.

Then:

- *there exist canonical trivializations of $\lambda_{BCOV}(\varphi)$ and $\varphi_* K_{\mathcal{X}^\vee/\mathbf{D}^\times}$.*
- *in these trivializations,* GRR *can be identified to a holomorphic function in $q \in \mathbf{D}^\times$ of the form*

$$\mathrm{GRR}(q) = \exp\Big((-1)^n F_1(\tau(q))\Big)^{24}.$$

The conjecture is optimistic in that:

- we might need to impose further conditions on the Hodge numbers of X (*e.g.* $h^{p,q} = 0$ outside the central cross).
- the usual MUM condition might not be sufficient.
- it could be that the predicted formula for GRR only holds up to a constant.

An alternative would be a conjecture for $d \log \mathrm{GRR}$. This still captures the genus one enumerative invariants.

The BCOV invariant

Construction of τ_{BCOV}

Let $f\colon X \to S$ be a family of Calabi–Yau manifolds as before.

Recall the canonical GRR isomorphism of $\mathbb{Q}$-line bundles

$$\mathrm{GRR}\colon \lambda_{BCOV}(f)^{\otimes 12} \xrightarrow{\sim} (f_* K_{X/S})^{\otimes \chi}.$$

To attack the functorial BCOV conjecture we need methods of explicitly computing GRR.

Idea:

- introduce natural hermitian metrics on $\lambda_{BCOV}(f)$ and $f_* K_{X/S}$ (Hodge theory).
- compute the norm of GRR with respect to these metrics.
- how? Arithmetic Riemann–Roch of Gillet–Soulé.

Hermitian metrics via Hodge theory

We endow the line bundle $f_*K_{X/S}$ with the L^2 (or Hodge) metric:

$$h_{L^2,s}(\alpha,\beta) = \frac{i^{n^2}}{(2\pi)^n}\int_{X_s}\alpha\wedge\overline{\beta},$$

for α,β are local sections of $f_*K_{X/S}$.

The normalization $(2\pi)^n$ is standard in Arakelov geometry.

The line bundle $\lambda_{BCOV}(f)$ has a *canonical* metric:

- by the Hodge decomposition, there is a $\mathcal{C}^\infty$ isomorphism

$$\lambda_{BCOV}(f)^{\otimes 2} \otimes \overline{\lambda_{BCOV}(f)}^{\otimes 2} \xrightarrow{\sim} \bigotimes_k (\det R^k f_* \underline{\mathbb{C}})^{(-1)^k 2k}.$$

- using the lattice of integral cohomology:

$$\begin{aligned}\bigotimes_k (\det R^k f_* \underline{\mathbb{C}})^{(-1)^k 2k} &= \bigotimes_k (\det R^k f_* \underline{\mathbb{Z}})_{\mathrm{nt}}^{(-1)^k 2k} \otimes \mathbb{C} \\ &\simeq \underline{\mathbb{C}}.\end{aligned}$$

The isomorphism is canonical, since $(\det R^k f_* \underline{\mathbb{Z}})_{\mathrm{nt}}^{\otimes 2} \simeq \underline{\mathbb{Z}}$ canonically.

- All in all, we have a $\mathcal{C}^\infty$ isomorphism

$$\lambda_{BCOV}(f)^{\otimes 2} \otimes \overline{\lambda_{BCOV}(f)}^{\otimes 2} \xrightarrow{\sim} \underline{\mathbb{C}},$$

which actually defines a smooth hermitian metric on $\lambda_{BCOV}(f)^{\otimes 2}$, and hence on $\lambda_{BCOV}(f)$.

Definition (L^2-BCOV metric)

The above canonical hermitian metric on $\lambda_{BCOV}(f)$ is called the L^2-BCOV metric, and denoted $h_{L^2,BCOV}$.

Now the BCOV invariant of the family $f\colon X \to S$ is defined as the norm of GRR with respect to h_{L^2} and $h_{L^2,BCOV}$:

Definition (BCOV invariant)

We define the BCOV invariant of the family $f\colon X \to S$ as the $\mathcal{C}^\infty(S)$ function

$$s \mapsto \tau_{BCOV}(X_s) := \frac{\| \operatorname{GRR}(\theta)\|^2_{L^2,s}}{\|\theta\|^2_{L^2,BCOV,s}},$$

where θ is any local trivialization of $\lambda_{BCOV}(f)$.

Relation to holomorphic analytic torsion

Assume for simplicity that X has a Kähler form ω, whose restriction to fibers is Ricci flat.

With respect to ω, we form the $\overline{\partial}$-Laplacian $\Delta^{p,q}_{\overline{\partial},s}$ on $A^{p,q}(X_s)$.

Theorem (Arithmetic Riemann–Roch + ε)

There exists a constant C such that

$$\tau_{BCOV}(X_s) = C \prod_{p,q} (\det \Delta^{p,q}_{\overline{\partial},s})^{(-1)^{p+q} pq},$$

where $\det \Delta^{p,q}_{\overline{\partial},s}$ *is the ζ-regularized determinant of* $\Delta^{p,q}_{\overline{\partial},s}$.

- The arithmetic Riemann–Roch relationship

$$\| \mathrm{GRR} \|^2 = C \prod_{p,q} (\det \Delta^{p,q}_{\bar{\partial},s})^{(-1)^{p+q} pq}$$

is a $\mathcal{C}^\infty$, spectral evaluation of GRR.

- The original BCOV conjecture was formulated in terms of the function

$$\mathcal{F}_1(s) := \frac{1}{2} \log \prod_{p,q} (\det \Delta^{p,q}_{\bar{\partial},s})^{(-1)^{p+q} pq}.$$

- Determining the singularities of τ_{BCOV} for degenerations of CY's is central in our approach to the conjecture. This relies on the spectral interpretation.

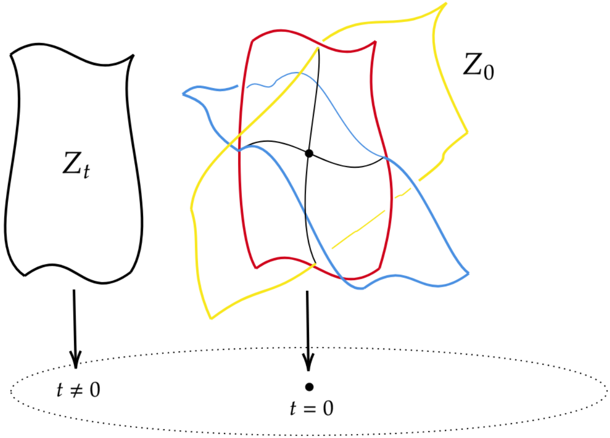
The singularities of τ_{BCOV}

Let $f\colon Z \to \mathbb{D}$ be a *germ of a degeneration of CY manifolds*:

- f is a projective morphism of complex manifolds, and it can be extended to a morphism of algebraic varieties.
- f is a submersion over the punctured disc $\mathbb{D}^\times$, with CY fibers.

Problem: describe the behaviour of $t \mapsto \log \tau_{BCOV}(Z_t)$ as $t \to 0$.

Here is an attempt of picture of a normal crossings degeneration:



Theorem (CDG 2018)

The following limit exists:

$$\kappa_f := \lim_{t \to 0} \frac{\log \tau_{BCOV}(Z_t)}{\log |t|^2} \in \mathbb{Q}.$$

We expect that the coefficient κ_f encodes interesting topological invariants.

We can compute κ_f when we have a good control on the singularities of the degeneration.

Theorem (CDG 2018–2019)

- *If f acquires at most ordinary quadratic singularities, i.e. modelled on $z_0^2 + \ldots + z_n^2 = t$, then:*

$$\kappa_f = \begin{cases} \frac{n+1}{24}\,\#\,\mathrm{sing}(Z_0) & \textit{if} \quad n \textit{ is even,} \\ -\frac{n-2}{24}\,\#\,\mathrm{sing}(Z_0) & \textit{if} \quad n \textit{ is odd.} \end{cases}$$

- *If $K_Z = \mathcal{O}_Z$ and f is semi-stable, with central fiber $Z_0 = \sum_k D_k$ a reduced NCD, then*

$$\kappa_f = \sum_{k \geq 1} (-1)^k \frac{k(k-1)}{24} \chi(D(k)),$$

where $D(k) = \sqcup_{|J|=k} D_J$ and $D_J = D_{j_1} \cap \ldots \cap D_{j_k}$.

The BCOV conjecture for hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$

In the remaining of the talk, we elaborate on the following:

Theorem (CDG 2019)

The functorial BCOV conjecture holds, up to a constant, for CY hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$ and their mirror family.

The 3-dimensional case was obtained by Fang–Lu–Yoshikawa.

First instance of higher dimensional mirror symmetry of BCOV type at genus one.

It builds on our work on the singularities of τ_{BCOV} for degenerations of CY's, and some refinements of Schmid's asymptotics of Hodge metrics.

The mirror family

Let X_{n+1} be a general degree $n+1$ hypersurface in $\mathbb{P}^n_{\mathbb{C}}$.

A concrete mirror family $f\colon Z \to \mathbb{D}^\times$ (over a small punctured disc) is constructed in three steps:

- for $q \in \mathbb{D}^\times$, define a CY hypersurface $X_q \subseteq \mathbb{P}^n_{\mathbb{C}}$ by
$$q\sum_{j=0}^{n} x_j^{n+1} - (n+1)\prod_{j=0}^{n} x_j = 0.$$
- kill the symmetries: $Y_q = X_q/G$, where
$$G = \{(\xi_0,\ldots,\xi_n) \in \mu_{n+1}^{\times(n+1)} \mid \prod_j \xi_j = 1\}/\text{diagonal}.$$
- Y_q is a mildly singular CY variety, but it admits a natural CY desingularisation $Z_q \to Y_q$. The construction can be done in families, and yields $f\colon Z \to \mathbb{D}^\times$.

The mirror family can be extended to a morphism of projective manifolds $\widetilde{Z} \to \mathbb{P}^1_{\mathbb{C}}$, with some singular fibers:

- the fibers at $q \in \mu_{n+1}$ have ordinary quadratic singularities (ODP points); each such fiber has a unique singular point.
- the fiber at $q = 0$ (MUM point), whose geometry we don't control.

For the BCOV conjecture, we need to canonically trivialize the Hodge bundles $R^q f_* \Omega^p_{Z/\mathbb{D}^\times}$.

Relevant Hodge bundles: $(R^q f_* \Omega^p_{Z/\mathbb{D}^\times})_{\mathrm{prim}}$ with $p + q = n - 1$.

The relevant Hodge bundles have rank one.

Let $F_\infty^\bullet$ be the limiting Hodge filtration and $W_\bullet$ the monodromy weight filtration, on $H^{n-1}_{\lim} =$ Schmid's LMS on $(R^{n-1}f_*\underline{\mathbb{C}})_{\mathrm{prim}}$.

For all $k = 0, \ldots, n-1$,

$$\mathrm{Gr}^W_{2k} H^{n-1}_{\lim} = \mathrm{Gr}^k_{F_\infty} \mathrm{Gr}^W_{2k} H^{n-1}_{\lim} \simeq \mathrm{Gr}^k_{F_\infty} H^{n-1}_{\lim}$$

is 1-dimensional. All the other possible graded pieces are trivial.

The flag $W_\bullet$ hence looks like

$$W_0 = \underbrace{W_1 \subseteq W_2}_{\dim 1} = W_3 \subseteq \ldots \subseteq W_{2n-4} = \underbrace{W_{2n-3} \subseteq W_{2n-2}}_{\dim 1} = H^{n-1}_{\lim}.$$

The monodromy weight filtration on $(H_{n-1})_{\lim} \simeq H^{n-1}_{\lim}$ has the same structure.

Theorem

Fix a basis $\gamma_\bullet = \{\gamma_k\}_k$ *of* $(H_{n-1})_{\mathrm{lim}}$ *adapted to the weight filtration:* $\gamma_k \in W'_{2k} \setminus W'_{2k-2}$.

Then there exists a unique holomorphic trivialization $\vartheta_\bullet$ *of* $(R^{n-1} f_* \Omega^\bullet_{Z/\mathbb{D}^\times})_{\mathrm{prim}}$, *adapted to the Hodge filtration, with*

$$\int_{\gamma_j} \vartheta_k = \begin{cases} 0 & \text{if } j < k \\ 1 & \text{if } j = k. \end{cases}$$

Here, the convention is $\vartheta_k \in \mathcal{F}^{n-1-k}(R^{n-1} f_* \Omega^\bullet_{Z/\mathbb{D}^\times})_{\mathrm{prim}}$.

Notice that, up to constants, $\vartheta_\bullet$ only depends on the weight filtration $W_\bullet$ on H^{n-1}_{lim}.

Definition

We denote by η_k the trivializing section of $(R^k f_* \Omega^{n-1-k}_{Z/\mathbb{D}^\times})_{\mathrm{prim}}$ obtained by projecting ϑ_k.

By the theorem and the observation above, the η_k depend only on the limiting mixed Hodge structure, up to constants.

One can actually show that the η_k extend to trivializations of the Deligne extensions of the Hodge bundles.

Let $\|\cdot\|_{L^2}$ be the L^2 norm on $(R^k f_* \Omega^{n-1-k}_{Z/\mathbb{D}^\times})_{\mathrm{prim}}$.

Let $q \mapsto \tau(q)$ be the mirror map and $F_1(\tau)$ the generating series of genus one Gromov–Witten invariants of X_{n+1}.

Theorem (CDG 2019)

The BCOV invariant of the mirror family $f : Z \to \mathbb{D}^\times$ of X_{n+1} has the form

$$\tau_{BCOV}(Z_q) = C \left| \exp\left((-1)^{n-1} F_1(\tau(q)) \right) \right|^4 \|\Theta\|^2,$$

where

$$\|\Theta\| := \left(\frac{\|\eta_0\|_{L^2}^{\chi(X_{n+1})/12}}{\prod_{k=0}^{n-1} \|\eta_k\|_{L^2}^{(n-1-k)}} \right)^{(-1)^{n-1}}$$

and C is some constant.

Some key points of the proof:

- extension of the mirror family to $\widetilde{Z} \to \mathbb{P}^1_{\mathbb{C}}$, with controlled singularities except for $q = 0$ (MUM point).
- one-dimensional parameter space and complete knowledge of its meromorphic functions.
- relation of τ_{BCOV} to Grothendieck–Riemann–Roch (arithmetic Riemann–Roch in a non-arithmetic setting...).
- behaviour of τ_{BCOV} for degenerations with ordinary quadratic singularities (ODP points).
- good understanding of the sections of the relevant Hodge bundles (explicit constructions, known divisors).
- Zinger's theorem: computation of F_1 in terms of hypergeometric functions.

From Schmid's asymptotics for L^2 metrics, one derives:

Corollary

As $\tau \to i\infty$, there is an asymptotic expansion

$$\frac{1}{2}\partial_\tau \log \tau_{BCOV}(Z_{q(\tau)}) = \underbrace{(-1)^{n-1}\partial_\tau F_1(\tau)}_{\textit{holomorphic part}} + \underbrace{\frac{\rho_\infty}{\operatorname{Im}\tau}\left(1 + O\left(\frac{1}{\operatorname{Im}\tau}\right)\right)}_{\textit{real analytic in } (\operatorname{Im}\tau)^{-1}},$$

where ρ_∞ is explicit and depends only on $H^{n-1}_{\lim}$.

Taking $\partial \log$ removes all the indeterminacies, and produces a canonical expression.