Geometric Extremization for AdS/CFT and Black Hole Entropy

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JPG, Dario Martelli, James Sparks x 3

SCFTs with abelian R-symmetry

$$\{Q, Q\} \sim P \qquad \{S, S\} \sim K$$

$$\{Q, S\} \sim M + D + R$$

The R-symmetry encodes important exact results for physical observables. E.g.

- For chiral primary operators $\Delta(\mathcal{O}) = nR(\mathcal{O})$
- Central charges/free energies can be obtained from R

Furthermore, the R-symmetry can be obtained by variational techniques.

$\mathcal{N} = 1, d = 4$ a-maximization [Intriligator,Wecht 03]

$$a(R_T) = \frac{9}{32} Tr R_T^3 - \frac{3}{32} Tr R_T \quad \text{and} \quad a = a(R_*)$$

$\mathcal{N} = 2, d = 3$ F-extremization [Jafferis 10]

$$F_{S^3}(R_T) \quad \text{and} \quad F_{S^3} = F_{S^3}(R_*)$$

$\mathcal{N} = (0,2), d = 2$ c-extremization [Benini,Bobev 12]

$$c_R(R_T) = 3 Tr \gamma_3 R_T^2 \quad \quad c_R = c_R(R_*)$$

$\mathcal{N} = 2, d = 1$ Is there a general extremization principle for susy QM??

$\mathcal{N} = 2, d = 1$

$\mathscr{I}$-extremization conjecture: [Benini,Hristov,Zaffaroni 15]

Compactify $\mathcal{N} = 2, d = 3$ SCFT on Riemann surface Σ_g with a topological twist - flows to SQM in IR

"Topologically twisted index" on $S^1 \times \Sigma_g$ [Benini,Zaffaroni 15] can be obtained via an extremization principle in large N limit

Also: if the N=2 SCFT has an AdS_4 dual, the on-shell result for the index is conjectured to give entropy of susy magnetically charged AdS_4 black holes with $AdS_2 \times \Sigma_g$ near horizon

Provides a microscopic state counting derivation of the Bekenstein-Hawking entropy for asymptotically AdS black holes!

Holographic dual for these extremization principles?

Well established in the context of Sasaki-Einstein solutions:

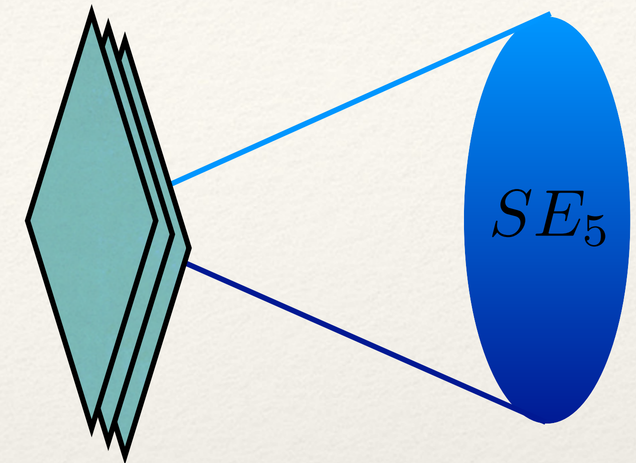
Type IIB

$$ds^2_{10} = L^2[ds^2(\mathrm{AdS}_5) + ds^2(SE_5)]$$

$$F_5 = -L^4\left[vol_{\mathrm{AdS}_5} + vol_{SE_5}\right]$$

Dual to N=1SCFT in d=4

D3



CY_3

$dr^2 + r^2 ds^2(SE_5)$

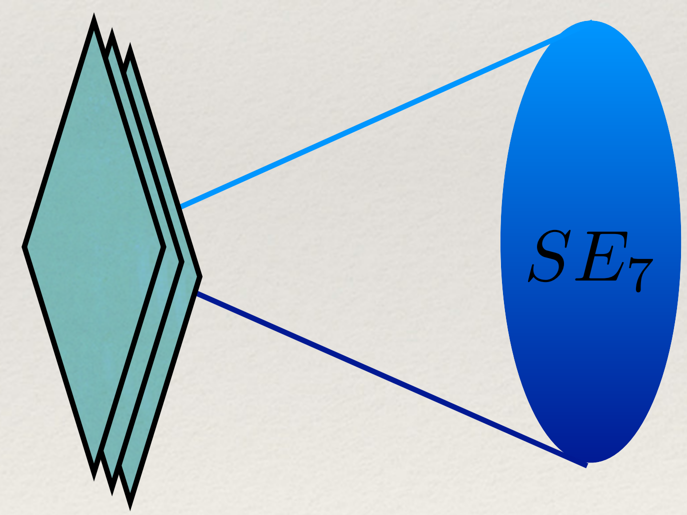
D=11

$$ds^2_{11} = L^2[ds^2(AdS_4) + ds^2(SE7)]$$

$$G = L^3 vol_{\mathrm{AdS}_4}$$

Dual to N=2 SCFT in d=3

M2



CY_4

Fact: SE have canonical Killing vector ξ

$\mathcal{N}=1, d=4$ SCFT dual to $AdS_5 \times SE_5$: $a \propto \frac{1}{Vol(SE_5)}$

$\mathcal{N}=2, d=3$ SCFT dual to $AdS_4 \times SE_7$: $F_{S^3} \propto \frac{1}{\sqrt{Vol(SE_7)}}$

R can be obtained using volume minimization: [Martelli,Sparks,Yau 05]

Go off-shell: Consider Sasaki metrics (cone is Kahler)

Extremize $Vol(Sas)(\xi)$

Very useful for identifying dual SCFTs!

Here: geometric duals for

c-extremization for AdS_3 solutions dual to $\mathcal{N}=(0,2), d=2$

New principle for AdS_2 solutions dual to $\mathcal{N}=2, d=1$

Includes a dual of $\mathscr{I}$- extremization as a special case and hence microstates of infinite classes of AdS4 black holes

Type IIB

$$ds^2_{10} = L^2 e^{-B/2}[ds^2(\mathrm{AdS}_3) + ds^2(Y_7)]$$

$$F_5 = -L^4\,[vol_{\mathrm{AdS}_3} \wedge F + *_7 F]$$

[Kim 05]

Dual d=2 SCFT has (0,2) supersymmetry

D=11

$$ds^2_{11} = L^2 e^{-2B/3}\left[ds^2(\mathrm{AdS}_2) + ds^2(Y_9)\right]$$

$$G_4 = L^3 vol_{\mathrm{AdS}_2} \wedge F$$

[Kim,Park 06]

Dual SCQM has 2 supersymmetries with R-symmetry

Also can arise as near horizon limits of magnetically charged supersymmetric black holes in $AdS_4 \times SE_7$

[Gauntlett,Kim 07]

Both special cases of GK geometry $(Y_{2n+1}, g_{\mu\nu}, B, F)$

Infinite classes of explicit $AdS_3 \times Y_7$ and $AdS_2 \times Y_9$

solutions have been known for a while

[Gauntlett,MacConamhna,Mateos,Waldram 06]

[Gauntlett,Kim,Waldram 06]

[Donos,Gauntlett,Kim 08]

Until recently dual field theories essentially unknown!

Some results for Y_7 [Benini,Bobev,Crichigno 15]

Plan

- Introduce GK geometry
- Go off-shell and derive new geometric extremization principles
- Utilise toric geometry to further analyse special classes

Identify all dual field theories and black hole microstates (almost!)

GK Geometry $(Y_{2n+1}, g_{\mu\nu}, B, F)$

[Gauntlett,Kim 07]

Action: $F = dA$

$$S = \int_{Y_{2n+1}} e^{(1-n)B} \left[R_{2n+1} - \frac{2n}{(n-2)^2} + \frac{n(2n-3)}{2} (dB)^2 + \frac{1}{4} e^{2B} F^2 \right] vol_{2n+1}$$

Equations of motion: $\delta S = 0$

Supersymmetry - existence of certain Killing spinors

GK Geometry (Y_{2n+1}, B, F) $n \geq 3$

Supersymmetry implies:

- Killing vector ξ (R-symmetry) $||\xi||^2 = 1$
- Define one-form η dual to Killing vector: $\xi^a \eta_a = 1$
- Local coordinates $\xi = \frac{1}{c}\partial_z$ and $\eta = c(dz + P)$ $c = \frac{1}{2}(n-2)$
- In general

$$ds^2_{2n+1} = \eta^2 + e^B ds^2_{2n}$$

⟵ Kahler J, ρ

$$\rho_{ij} = J_i{}^k R_{kj}$$

$$d\eta = c\rho \qquad e^B = \frac{c^2}{2} R > 0 \qquad F = -\frac{1}{c} J + d\left(e^{-B}\eta\right)$$

- Supersymmetric solution if

$$\delta S = 0 \quad \Leftrightarrow \quad \Box R = \frac{1}{2} R^2 - R_{ij} R^{ij}$$

Off shell GK Geometry

Continue off-shell: just demand a "supersymmetric geometry"

(doesn't extremize action S)

$$\xi^a \eta_a = 1 \qquad \eta = c(dz + P) \qquad d\eta = c\rho$$

$$ds^2_{2n+1} = \eta^2 + e^B ds^2_{2n}$$ ← Kahler

$$d\eta = c\rho \qquad e^B = \frac{c^2}{2} R > 0 \qquad F = -\frac{1}{c} J + d\left(e^{-B}\eta\right)$$

$$\delta S = 0 \quad \Rightarrow \quad \Box R = \frac{1}{2} R^2 - R_{ij} R^{ij}$$

Off-shell GK Geometry

- Consider cone metric on $C(Y_{2n+1}) \equiv \mathbb{R}_{>0} \times Y_{2n+1}$

$$ds^2_{2n+2} = dr^2 + r^2 ds^2(Y_{2n+1})$$

- Cone has an integrable complex structure
- R symmetry vector ξ is holomorphic
- No-where vanishing (n+1,0) form Ψ with

$$d\Psi = 0 \qquad \mathcal{L}_\xi \Psi = \frac{i}{c}\Psi \qquad c = \tfrac{1}{2}(n-2)$$

Off-shell GK geometry and extremization - part I

Fix Y_{2n+1} and the complex cone $C(Y_{2n+1})$

Choose holomorphic $\xi \neq 0$ with $\mathcal{L}_\xi \Psi = \frac{i}{c}\Psi$

as well as a Kahler metric ds^2_{2n}

Gives a supersymmetric geometry on Y_{2n+1}

$$ds^2_{2n+1} = \eta^2 + e^B ds^2_{2n}$$

$$d\eta = c\rho \qquad e^B = \frac{c^2}{2} R > 0 \qquad F = -\frac{1}{c} J + d\left(e^{-B}\eta\right)$$

- To get a supersymmetric solution: need to vary action S

$$S_{\text{SUSY}} = \int_{Y_{2n+1}} \eta \wedge \rho \wedge \frac{J^{n-1}}{(n-1)!}$$

$$= S_{\text{SUSY}}(\xi, [J])$$

Nice! $[J] \in H^2_B(\mathcal{F}_\xi)$

Off-shell GK geometry and extremization - part II

Need to consider flux quantization

$$\frac{1}{(2\pi\ell_s)^4 g_s}\int_{\Sigma_A} F_5 = N_A \qquad \text{or} \qquad \frac{1}{(2\pi\ell_p)^6}\int_{\Sigma_A} *_{11}G = N_A$$

Requires $dF_5 = 0$ or $d *_{11} G_4 = 0$

We need to impose the integral of $\Box R = \frac{1}{2}R^2 - R_{ij}R^{ij}$

$$\int_{Y_{2n+1}} \eta\wedge\rho^2\wedge\frac{J^{n-2}}{(n-2)!} = 0$$

"topological constraint"

and then

$$\int_{\Sigma_A} \eta\wedge\rho\wedge\frac{J^{n-2}}{(n-2)!} = \begin{cases} \dfrac{2(2\pi\ell_s)^4 g_s}{L^4} N_A\,, & n = 3 \\ \dfrac{(2\pi\ell_p)^6}{L^6} N_A\,, & n = 4 \end{cases}$$

Geometric extremal problem- Summary I

[Couzens,Gauntlett, Martelli,Sparks 18]

- Complex cone $C(Y_{2n+1})$ with (n+1,0) form Ψ
- Choose holomorphic $\xi \neq 0$ and $\mathcal{L}_\xi \Psi = \frac{i}{c}\Psi$
- Consider a supersymmetric geometry on Y_{2n+1}
- Choose basic class $[J]$ for transverse Kahler metric

- Impose constraint: $\int_{Y_{2n+1}} \eta \wedge \rho^2 \wedge \frac{J^{n-2}}{(n-2)!} = 0$

- Impose flux quantization: $\int_{\Sigma_A} \eta \wedge \rho \wedge \frac{J^{n-2}}{(n-2)!} = \begin{cases} \frac{2(2\pi\ell_s)^4 g_s}{L^4} N_A \,, & n = 3 \\ \frac{(2\pi\ell_p)^6}{L^6} N_A \,, & n = 4 \end{cases}$

- Extremise action: $S_{\mathrm{SUSY}}(\xi, [J]) = \int_{Y_{2n+1}} \eta \wedge \rho \wedge \frac{J^{n-1}}{(n-1)!}$

Geometric extremal problem- Summary II

For $AdS_3 \times Y_7$ define

$$\mathscr{Z} = \frac{3L^8}{(2\pi)^6 g_s^2 \ell_s^8} S_{\text{susy}} \qquad \text{and} \qquad \mathscr{Z}|_{\text{on-shell}} = c_{\text{sugra}} = \frac{3L}{2G_3}$$

For $AdS_2 \times Y_9$ define "entropy function"

$$\mathscr{S} = \frac{4\pi L^9}{(2\pi)^8 \ell_p^9} S_{\text{susy}} \qquad \text{and} \qquad \mathscr{S}|_{\text{on-shell}} = \frac{1}{4G_2}$$

Generically expect $\mathscr{S}|_{\text{on-shell}} = \ln Z, \quad \mathscr{I}$

For black hole horizons $\mathscr{S}|_{\text{on-shell}} = S_{BH}$

Special Cases and Toric Geometry

Type IIB $AdS_3 \times Y_7$ with $Y_5 \hookrightarrow Y_7 \rightarrow \Sigma_g$

Physical picture:

- Start with $AdS_5 \times Y_5$ and SE metric on Y_5

  Dual to d=4 N=1 SCFT

  Isometries of Y_5 give rise to global (mesonic) symmetries

  3-cycles of Y_5 give rise to global (baryonic) symmetries

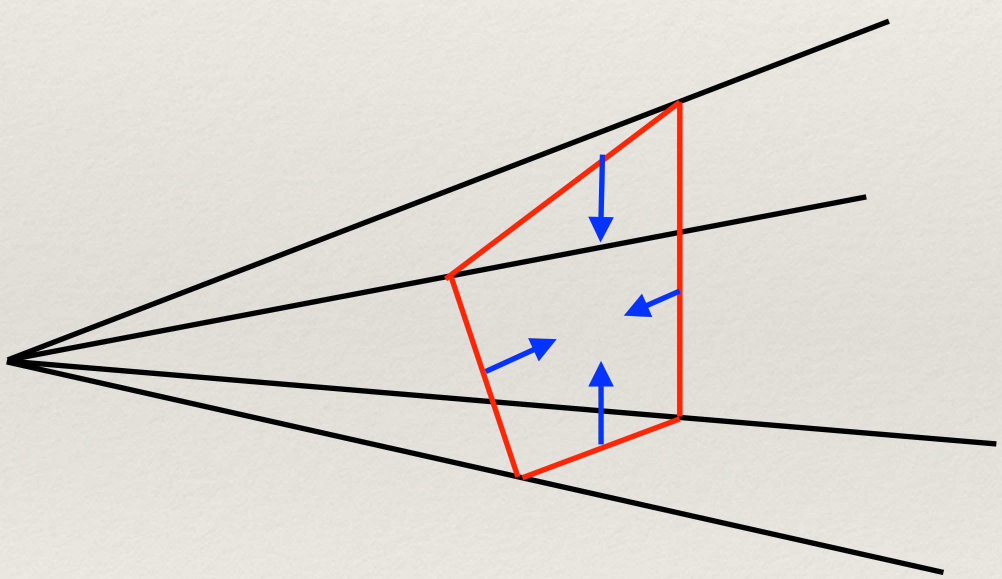
- Compactify d=4 SCFT on Σ_g and switch on magnetic fluxes for the global symmetries (including topological twist)

- IF we flow to d=2 SCFT in IR then expect it is dual to $AdS_3 \times Y_7$ with Y_7 fibred as above

Focus on $AdS_5 \times Y_5$ with the complex cone $C(Y_5)$ admitting a toric Kahler cone metric: [Martelli,Sparks,Yau 05]

- Three holomorphic Killing vectors ∂_{φ_i} generate $U(1)^3$

- There is an associated polyhedral cone with d facets specified by inward pointing normal vectors $\vec{v}_a \in \mathbb{Z}^3$



$\vec{v}_a$ specifies which $U(1)$ collapses along that facet

- The extremization problem for the $AdS_3 \times Y_7$ solutions with $Y_5 \hookrightarrow Y_7 \rightarrow \Sigma_g$ becomes algebraic in the $\vec{v}_a$!

Master volume

[Gauntlett,Martelli,Sparks 18]

$$\mathcal{V}(\vec{b};\{\lambda_a\}) = \frac{(2\pi)^3}{2}\sum_{a=1}^{d}\lambda_a \frac{\lambda_{a-1}(\vec{v}_a,\vec{v}_{a+1},\vec{b}) - \lambda_a(\vec{v}_{a-1},\vec{v}_{a+1},\vec{b}) + \lambda_{a+1}(\vec{v}_{a-1},\vec{v}_a,\vec{b})}{(\vec{v}_{a-1},\vec{v}_a,\vec{b})(\vec{v}_a,\vec{v}_{a+1},\vec{b})}$$

Extremization problem

$A \sim$ Kahler class for Σ_g

$$S_{\mathrm{SUSY}}(\vec{b};\{\lambda_a\};A) = -A\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a} - 4\pi\sum_{i=1}^{3}n_i\frac{\partial\mathcal{V}}{\partial b_i}$$

$$0 = A\sum_{a,b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b} - 2\pi n_1\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a} + 4\pi\sum_{a=1}^{d}\sum_{i=1}^{3}n_i\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial b_i}$$

$$\frac{2(2\pi\ell_s)^4 g_s}{L^4}N = -\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a}$$

Flux on Y_5

$$\frac{2(2\pi\ell_s)^4 g_s}{L^4}M_a = \frac{1}{2\pi}A\sum_{b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b} + 2\sum_{i=1}^{3}n_i\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial b_i}$$

Flux on $\Sigma_g \times (\Sigma_a \subset Y_5)$

Results

- For arbitrary toric Y_5

  Can calculate c_{sugra} for the $AdS_3 \times Y_7$ solutions as a function of the geometric twists and fluxes using geometric extremization

- Can compare with known dual quiver gauge theories using field theory c-extremization procedure

Find exact agreement (even off-shell)!

[JPG,Martelli,Sparks 19]

[Hosseini,Zaffaroni 19]

Comments

This provides an identification of an infinite classes of d=4 quiver field theories compactified on Σ_g with these $AdS_3 \times Y_7$ solutions!

Caveat: provided that they both exist…

- Geometry: there can be obstructions to the existence of Y_7
- Field theory: the field theory may not flow in the IR to a SCFT of the type we are considering

Conjecture: sufficient to check $c > 0$ and $R_a > 0$

Comments

- There is an analogous story for $AdS_2 \times Y_9$ solutions with with $Y_7 \hookrightarrow Y_9 \to \Sigma_g$ and Y_7 toric [JPG,Martelli,Sparks 19] [Hosseini,Zaffaroni 19]

- Using toric data can calculate an off shell entropy function as a function of geometric twists and fluxes

- This can be identified with the entropy of a magnetically charged black hole in $AdS_4 \times Y_7$ (provided that they exist)

- Field theory: off-shell calculation of topological index $\mathscr{I}$ for certain quiver gauge theories compactified on Σ_g calculated in [Hosseini,Zaffaroni 16

When there are no baryonic twists:

Find exact agreement, even off-shell!

[JPG,Martelli,Sparks 19]
[Hosseini,Zaffaroni 19]
[Kim,Kim19]

Summary and outlook

- Geometric dual of c-extremization for type IIB $AdS_3 \times Y_7$
- Geometric extremization for SCQM dual to D=11 $AdS_2 \times Y_9$
    - What is the field theory story? does it exist for finite N?
    - If arise as black hole horizons, get entropy via extremization
- Interesting sub-class of examples $Y_5 \hookrightarrow Y_7 \rightarrow \Sigma_g$

  $Y_7 \hookrightarrow Y_9 \rightarrow \Sigma_g$

    - Toric case: striking agreement with field theory and new microstate counting of entropy of mag. charged AdS4 black holes
    - Obstructions? Geometry/black hole solutions/field theory
    - Novel features arise in toric geometry - develop
    - Non-toric class?
    - Dyonic black holes? Rotating black holes?