Noncommutative topological approach to topological phases

Johannes Kellendonk

Institut Camille Jordan, Université Lyon 1

Paris, 04/02/2021

Plan of the talk

(Partial) Overview on the theory of topological phases in the C^*-algebraic approach to solid state physics proposed by J. Bellissard in 1984.

1. Topological phases in the 1-particle approximation
2. K-theoretic formulation of topological phases
3. Symmetry protection
4. Boundary invariants and KK-classes
5. Bulk boundary correspondance
6. Numerical invariants (outlook)

A theory of topological phases for interacting fermions in a solid is an active area of research, but not discussed here (second quantization, study of the topology of the ground state separated from the rest by a gap, captured by (higher) category theory).

What is a topological insulator?

Topological phases in the quasi-particle picture

Deep inside the material (bulk): configuration space $\mathbb{R}^d$ or $\mathbb{Z}^d$.

Electron-electron interaction gives rise to an effective potential V for a single quasi-particle described by a stationary Schrödinger equation in $\mathcal{H} = L^2(\mathbb{R}^d, \mathbb{C}^N)$ or $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ ($\mathbb{C}^N$ internal (spin) degrees of freedom).

$$H\Psi = E\Psi, \quad H = -\Delta + V$$

Examples: Landau Hamiltonian for a free electron in a magnetic field, Hofstadter Hamiltonian (its tight binding analog), but also models without magnetic field (Kane-Mele).

Main assumption: the system is (bulk) insulating: the Fermi energy lies in a gap of the spectrum of H.

Definition

Two models (Hamiltonians H_0, H_1) belong to the same topological phase if they can be deformed into each other preserving the gap:

There is a continuous path $H(t) \in A$ of gapped Hamiltonians in some background topological space A linking the two.

In the following we do not talk about a specific Hamiltonians, but only of their homotopy classes in A. We choose A to be a C^*-algebra. Its form has to be physically motivated.

A describes the physics deep inside the material (the bulk).

In the tight binding approx. (configuration space $\mathbb{Z}^d$) A is generated by

1. bounded potentials $C_{pot}(\mathbb{Z}^d) \subset \{V : \mathbb{Z}^d \to \mathbb{C}, bdd\}$
2. translations (possibly twisted by magnetic field Θ) $T_1, \cdots T_d$, $T_i\psi(x) = \psi(x + e_i)$

$$T_i^* T_i = 1 = T_i T_i^* \qquad T_i T_j = e^{\imath \Theta_{ij}} T_j T_i \tag{1}$$

$$T_i V T_i^*(x) = V(x + e_i) \tag{2}$$

3. (finitely many) internal degrees of freedom (spin), internal Hilbert space $\mathbb{C}^N$: A acts on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$

$$A = A' \otimes M_N(\mathbb{C})$$

A' is a twisted crossed product algebra

$$A' = C_{pot}(\mathbb{Z}^d) \rtimes_{\alpha,\Theta} \mathbb{Z}^d$$

Continuous version: $A = C_{pot}(\mathbb{R}^d) \rtimes_{\alpha,\Theta} \mathbb{R}^d \otimes M_N(\mathbb{C})$

Examples of background algebra A

Examples (tight binding)

$$A = C_{pot}(\mathbb{Z}^d) \rtimes_{\alpha,\Theta} \mathbb{Z}^d \otimes M_N(\mathbb{C})$$

1. All bounded potentials are allowed (no symmetry constreint) $C_{pot}(\mathbb{Z}^d) = C_b(\mathbb{Z}^d)$:
   A is the algebra of all local tight binding operators with at most N internal degrees of freedom.
   A is non-separable so has relatively poor K-theory.
2. Crystals: potentials are periodic, internal Hilbert space contains elementary cell: $C_{pot}(\mathbb{Z}^d) = \mathbb{C}$. If no magnetic field:

   $$A = M_N(\mathbb{C}) \rtimes_\alpha \mathbb{Z}^d \cong C(\mathbb{T}^d, M_N(\mathbb{C}))$$

   Most often used in solid state physics (N Bloch bands).
3. Long range order / quasicrystalls: atomic positions described by point set $\mathcal{L} \subset \mathbb{R}^d$, restrict $C_b(\mathcal{L})$ to functions respecting long range order.

   This has been introduced by [Bellissard 1986]: $C_b(\mathcal{L}) \cong C(\Omega)$ where Ω is the hull of the configuration $\mathcal{L}$.

Fix the Fermi energy to be 0 (shift the spectral gap to 0).

- H has gap at 0 $\Leftrightarrow$ H is invertible

Bands below energy 0 are filled, those above unfilled.

Example: $H = 1$ is a Hamiltonian whose bands are completely unfilled. $H = -1$ is a Hamiltonian whose bands are completely filled. Both are topologically trivial.

Definition (same definition)

A topological phase of an insulator in A is a path connected component of the open set

$$GL(A)^{s.a.} = \{H \in A : H^* = H, H^{-1} \in A\}$$

Spectral flattening: every element of $GL(A)^{s.a.}$ is homotopic to a self-adjoint unitary $H^* = H = H^{-1}$.

Definition (rough)

Let A be a C^*-algebra with a unit. $K_0(A)$ is $GL(A)^{s.a.} / \sim_{homotopy}$ turned into an abelian group.

K_0-group of a unital C^*-algebra, van Daele's picture

$K_0(A)$ is $GL(A)^{s.a.}/\sim_{homotopy}$ turned into an abelian group means the following:

1. Stabilise: $GL(M_n(A))^{s.a.} \ni x \mapsto x \oplus 1 \in GL(M_{n+1}(A))^{s.a.}$ (adding unfilled bands).
   $V(A) = \bigcup_{n\geq 1} GL(M_n(A))^{s.a.}/\sim_{homotopy}$ is an abelian semigroup

$$[x] + [y] = [x \oplus y] = [y \oplus x]$$

2. Turn into a group: $K_0(A) = V(A) \times V(A)/\sim_{Grothendieck}$ (adding filled bands).

Definition (slightly weaker definition)

A topological phase of an insulator (slightly weaker sense) in A is an element of $K_0(A)$.

We have allowed adding of unfilled bands and of filled bands (stacking):

Two Hamiltonians are in the same topological phase (in a slightly weaker sense) if, after adding unfilled and filled bands they can be deformed into each other inside $M_n(A)$ without closing the gap.

The Hamiltonian H may be subject to symmetry conditions:

- Ordinary symmetries given by a group G representation ρ: $\mathbb{C}$-linear $\rho_g \in \mathrm{Aut}_{\mathbb{R}}(A)$ such that $\rho_g(H) = H$.
  Restrict A to G-invariant elements.
- Quasiperiodicity (long range order): restrict $C_{pot}(\mathbb{Z}^d)$ to quasiperiodic functions.

The Hamiltonian H may be subject to symmetry conditions:

- Ordinary symmetries given by a group G representation ρ: $\mathbb{C}$-linear $\rho_g \in \mathrm{Aut}_{\mathbb{R}}(A)$ such that $\rho_g(H) = H$.
  Restrict A to G-invariant elements.
- Quasiperiodicity (long range order): restrict $C_{pot}(\mathbb{Z}^d)$ to quasiperiodic functions.

Extra ordinary symmetries $\mathcal{E}$:

- Time reversal symmetry TRS
  anti-linear $\mathfrak{t} \in \mathrm{Aut}_{\mathbb{R}}(A)$ of order 2 and $\mathfrak{t}(H) = H$ (real structure)
- Charge conjugation (particle hole symmetry) PHS
  anti-linear $\mathfrak{c} \in \mathrm{Aut}_{\mathbb{R}}(A)$ of order 2 and $\mathfrak{c}(H) = -H$ (real structure)
- Chiral symmetry CS
  $\mathbb{C}$-linear $\gamma \in \mathrm{Aut}_{\mathbb{C}}(A)$ of order 2 and $\gamma(H) = -H$ (balanced grading)

Definition

A topological phase of an insulator in A with protecting symmetry $\mathcal{E}$ is a path connected component of the open set

$$GL(A,\mathcal{E})^{s.a.} = \{H \in A : H^* = H, H^{-1} \in A, H \text{ satisfies } \mathcal{E}\}$$

or, slightly weaker, an element of $K(A,\mathcal{E})$, obtained as above by turning $GL(A,\mathcal{E})^{s.a.}$ into a group.

Definition

A topological phase of an insulator in A with protecting symmetry $\mathcal{E}$ is a path connected component of the open set

$$GL(A,\mathcal{E})^{s.a.} = \{H \in A : H^* = H, H^{-1} \in A, H \text{ satisfies } \mathcal{E}\}$$

or, slightly weaker, an element of $K(A,\mathcal{E})$, obtained as above by turning $GL(A,\mathcal{E})^{s.a.}$ into a group.

More precisely: Call A balanced graded if it contains an odd self-adjoint unitary e (OSU). e plays the role of a basepoint, or trivial insulator.

Definition (van Daele '84)

Let (A,γ) be a balanced graded real or complex C^*-algebra. Choose e.
Van Daele K-group $DK_e(A,\gamma)$ is obtained from $GL(A,\{\gamma\})^{s.a.}$ as above except $GL(M_n(A),\{\gamma\})^{s.a.} \ni x \mapsto x \oplus e \in GL(M_{n+1}(A),\{\gamma\})^{s.a.}$

$$K(A,\mathcal{E}) = DK_e(A,\gamma) \text{ or } DK_e(A_{\mathbb{R}},\gamma) \quad \text{if } \mathcal{E} \text{ contains a chiral symmetry } \gamma$$

up to isom. $DK_e(A,\gamma)$ does not depend on the choice of e

Recall $A = C_{pot}(\mathbb{Z}^d) \rtimes_\alpha \mathbb{Z}^d \otimes M_N(\mathbb{C})$

Suppose $\mathcal{E}$ acts on internal degrees of freedom, i.e. on $M_N(\mathbb{C})$.

If $\mathcal{E}$ contains chiral sym. γ then $(M_N(\mathbb{C}), \gamma) \cong (M_n(\mathbb{C}) \otimes \mathbb{C}l_2, \mathrm{Ad}_{\sigma_3})$, $N = 2n$.

Up to equivalence, there are 4 real structures $\mathfrak{r}$ on $M_N(\mathbb{C})$ commuting with γ.

even TRS: $M_N(\mathbb{C})^{\mathfrak{r}} = M_N(\mathbb{R}) \cong M_n(\mathbb{R}) \otimes Cl_{1,1}$ (σ_3 is real)

even PHS: $M_N(\mathbb{C})^{\mathfrak{r}} \cong M_n(\mathbb{R}) \otimes Cl_{2,0}$ (σ_3 is imag.)

odd TRS: $M_N(\mathbb{C})^{\mathfrak{r}} \cong M_k(\mathbb{R}) \otimes \mathbb{H} \otimes Cl_{1,1} \cong M_k(\mathbb{R}) \otimes Cl_{0,4}$ (σ_3 real), $N = 4k$

odd PHS: $M_N(\mathbb{C})^{\mathfrak{r}} \cong M_n(\mathbb{R}) \otimes Cl_{0,2}$ (σ_3 is imag.)

All Clifford algebras with even number of generators appear up to Morita equiv..

Definition (higher K-groups)

(A, γ) a real or complex balanced graded algebra.
$K_{1+s-r}(A, \gamma) := DK_e(A \otimes Cl_{r,s}, \gamma \otimes st)$.

- $K(A, \mathcal{E}) \cong KU_1(A)$ if no real symetries (complex K-theorie, Bott 2-periodic)
- $K(A, \mathcal{E}) \cong KO_{1+s-r}(A_{\mathbb{R}})$ with real symmetries (real K-theorie, Bott 8-periodic)

$\mathcal{E}$ does not contain a chiral sym. Trick: add an outer one.

Replace A by $A \otimes \mathbb{C}l_1$, with outer grading γ. Replace $\mathcal{E}$ by $\mathcal{E} \cup \{\gamma\}$.

Up to equivalence, there are 4 real structures $\mathfrak{r}$ on $M_N(\mathbb{C}) \otimes \mathbb{C}l_1$ commuting with γ.

- even TRS: $(M_N(\mathbb{C}) \otimes \mathbb{C}l_1)^{\mathfrak{r}} \cong M_N(\mathbb{R}) \otimes Cl_{1,0}$
- even PHS: $(M_N(\mathbb{C}) \otimes \mathbb{C}l_1)^{\mathfrak{r}} \cong M_N(\mathbb{R}) \otimes Cl_{0,1}$
- odd TRS: $(M_N(\mathbb{C}) \otimes \mathbb{C}l_1)^{\mathfrak{r}} \cong M_n(\mathbb{R}) \otimes \mathbb{H} \otimes Cl_{1,0} \cong M_n(\mathbb{R}) \otimes Cl_{0,3}$
- odd PHS: $(M_N(\mathbb{C}) \otimes \mathbb{C}l_1)^{\mathfrak{r}} \cong M_n(\mathbb{R}) \otimes \mathbb{H} \otimes Cl_{0,1} \cong M_n(\mathbb{R}) \otimes Cl_{3,0}$

All Clifford algebras with odd number of gen. appear up to Morita equiv..

By definition $K(A, \emptyset) := K(A \otimes \mathbb{C}l_1, \{\gamma\}) =: K_0(A)$.

With real symmetry $\mathcal{E}$

$$K(A, \mathcal{E}) := K(A \otimes \mathbb{C}l_1, \mathcal{E} \cup \{\gamma\}) \cong KO_1(A_{\mathbb{R}} \otimes Cl_{r,s}) =: KO_{1+s-r}(A_{\mathbb{R}})$$

Bulk invariants: First summary and comments

- Physical considerations leads to the choice of the algebra A whose elements describe the physics in the bulk.
- The symmetry protected topological phases can be identified with the elements of $K(A, \mathcal{E})$. These are referred to as the bulk invariants.
- Different extra-ordinary symmetry types correspond to the different higher classical K-groups of A.

Bulk invariants: First summary and comments

- Physical considerations leads to the choice of the algebra A whose elements describe the physics in the bulk.
- The symmetry protected topological phases can be identified with the elements of $K(A, \mathcal{E})$. These are referred to as the bulk invariants.
- Different extra-ordinary symmetry types correspond to the different higher classical K-groups of A.

- If A is commutative then the K-groups can be described by vectorbundles with symmetries (twisted K-theory) [Freed, Moore 2013]
- For $A = \mathbb{C} \rtimes \mathbb{R}^d$ (constant potential, cont. translations) one obtains the famous Kitaev table [Kitaev 2007].

Near the boundary the physics of material is described by operators on halfspace $\mathbb{Z}^{d-1} \times \mathbb{N}$.

Relevant algebra $\hat{A}$ differs from bulk algebra A in one respect: restricting the action of perpendicular translation T_d to $\ell^2(\mathbb{Z}^{d-1} \times \mathbb{N})$ "runs into the wall": T_d^* no longer inject. $\hat{A}$ is generated by

1. bounded potentials $C_{pot}(\mathbb{Z}^d) \subset \{V : \mathbb{Z}^d \to \mathbb{C}, bdd\}$
2. translations (possibly twisted by magnetic field Θ) $\hat{T}_1, \cdots \hat{T}_d$, $\hat{T}_i\psi(x) = \psi(x + e_i)$

$$\hat{T}_i^* \hat{T}_i = 1 \quad \hat{T}_i \hat{T}_i^* = 1 - P_0 \qquad \hat{T}_i \hat{T}_j = e^{\imath \Theta_{ij}} \hat{T}_j \hat{T}_i$$
$$\hat{T}_i V \hat{T}_i^*(x) = V(x + e_i)$$

   P_0 is a nontrivial projection (projection onto $\ell^2(\mathbb{Z}^{d-1} \otimes \{0\})$)
3. internal degrees of freedom (spin, pseudo spin): $\hat{A}$ acts on $\ell^2((\mathbb{Z}^{d-1} \times \mathbb{N}) \otimes \mathbb{C}^N$

$$\hat{A} = \hat{A}' \otimes M_N(\mathbb{C})$$

$A' = \mathcal{T}(A)$ is the Toeplitz extension of the crossed product algebra A.

As $\mathcal{E}$ acts on the internal degrees of freedom, it also acts on $\hat{A}$ and q intertwines this actions.

Exact sequence underlying the Bulk-Boundary Correspondence

[Richter, Schulz-Baldes +K. 2002]

Shifting the boundary to $+\infty$ corresponds to the surjective algebra homomorphism

$$q : \hat{A} \to A, \qquad q(\hat{T}_i) = T_i$$

Its kernel is generated by P_0

$$J = \ker q = \hat{A} P_0 \hat{A}$$

thus consists of operators localized at the boundary.

K is a homological functor: the exact sequence

$$J \overset{i}{\hookrightarrow} \hat{A} \overset{q}{\twoheadrightarrow} A$$

induces an isomorphism, the K-theoretical bulk-boundary correspondence

$$K_i(A, \mathcal{E})/\mathrm{im} q_* \overset{\delta}{\cong} K_{i-1}(J, \mathcal{E}) \cap \ker i_*$$

Q: How can we understand physically the elements of $K_{i-1}(J)$?

Bulk versus boundary invariants, the rough picture

[Alldridge, Max, Zirnbauer 2019; Bourne, Rennie, K. 2020]

There is a linear homomorphism $A \ni T_i \mapsto \hat{T}_i \in \hat{A}$ (not multiplicative!).

Consider a Hamiltonian $H \in GL^{s.a.}(A)$.

- $\hat{H}$ is H with a choice of boundary conditions at the boundary.
- The class of H in $DK(A, \mathcal{E})$ (van Daele) is the bulk invariant.
- The class of $\hat{H}$ in $KK(\mathbb{C}, J, \mathcal{E})$ (Kasparov) is the boundary invariant.
- The Cayley transform induces an isomorphism between van Daele's and Kasparov's picture of K-theory

- van Daele's picture of K-theory is topological bulk physics
- Kasparov's picture of K-theory is topological boundary physics

Kasparov's picture of $K_{-1}(B)$

Let B be a (σ-unital) C^*-algebra.

1. A (right) B Hilbert-C^*-module X_B is a "Hilbert space" with scalars replaced by B.
   - the scalar product is B-valued
   - $X_B = B$ is a B Hilbert-C^*-module of rank 1 (like $\mathbb{C}$ is a one-dim. Hilbert space)
   - B-compact linear maps $T : X_B \to X_B$ are defined as closure of finite rank operators
   - the adjoint T^* of a linear map $T : X_B \to X_B$ is defined with B-valued scalar product

$$\langle x, T y\rangle = \langle T^* x, y\rangle$$

2. An endomorphism of X_B is a linear map T which admits an adjoint T^*
3. A Kasparov $\mathbb{C} - B$-cycle (X_B, F) is a B Hilbert-C^*-module together with a self-adjoint endomorphism F such that $F^2 - 1$ is B-compact. They can be added up by direct sum.
4. A $KK(\mathbb{C}, B)$-cycle (X_B, F) is degenerate if $F^2 = 1$.
5. The set of equivalence classes of Kasparov $\mathbb{C} - B$-cycles modulo homotopy, unitary equivalence and addition of degenerate KK-cycles is $KK(\mathbb{C}, B)$. Direct sum induces abelian group structure. $KK(\mathbb{C}, B)$ is isomorphic to $K_{-1}(B)$.

Kasparov's picture of $K_{-1}(B)$

Let B be a (σ-unital) C^*-algebra.

1. A (right) B Hilbert-C^*-module X_B is a "Hilbert space" with scalars replaced by B.
   - the scalar product is B-valued
   - $X_B = B$ is a B Hilbert-C^*-module of rank 1 (like $\mathbb{C}$ is a one-dim. Hilbert space)
   - B-compact linear maps $T : X_B \to X_B$ are defined as closure of finite rank operators
   - the adjoint T^* of a linear map $T : X_B \to X_B$ is defined with B-valued scalar product
   $$\langle x, T y \rangle = \langle T^* x, y \rangle$$
2. An endomorphism of X_B is a linear map T which admits an adjoint T^*
3. A Kasparov $\mathbb{C} - B$-cycle (X_B, F) is a B Hilbert-C^*-module together with a self-adjoint endomorphism F such that $F^2 - 1$ is B-compact. They can be added up by direct sum.
4. A $KK(\mathbb{C}, B)$-cycle (X_B, F) is degenerate if $F^2 = 1$.
5. The set of equivalence classes of Kasparov $\mathbb{C} - B$-cycles modulo homotopy, unitary equivalence and addition of degenerate KK-cycles is $KK(\mathbb{C}, B)$. Direct sum induces abelian group structure. $KK(\mathbb{C}, B)$ is isomorphic to $K_{-1}(B)$.

Extra ordinary symmetries $\mathcal{E}$ can be treated in a similar way as for DK:

1. If $\mathcal{E}$ has a chiral symmetry, hence B carries a balanced grading then X_B is required to have a compatible grading and F to be an odd operator.
2. If $\mathcal{E}$ has a real symmetry (TRS or PHS / even or odd), hence B carries a real structure then X_B is required to carry a compatible real structure and F to be real or imaginary.

Theorem (Bourne, Rennie, K. 2020)

Let (B,γ) be a balanced graded real or complex C^*-algebra with choice of basepoint e and $H \in B$ and odd self-adjoint unitary. The map (Cayley transform)

$$x \mapsto \left(\overline{(x-e)B}, \frac{1}{2}e(x+e)(x-e)^{-1}|x-e|\right)$$

induces an isomorphism between $DK_e(B,\gamma)$ and $KK(\mathbb{C}, B)$.

Theorem (Alldridge, Max, Zirnbauer 2019; Bourne, Rennie, K. 2020)

Let H be a Hamiltonian $H \in GL^{s.a.}(A)$ with spectral gap Δ at 0. Let $P_\Delta(\hat{H})$ be the spectral projection of $\hat{H}$ to Δ.

$$(J, P_\Delta(\hat{H})\hat{H})$$

is a KK-cycle whose class corresponds to the class of H under the bulk-boundary correspondence:

$$\delta([H]) = [J, P_\Delta(\hat{H})\hat{H}].$$

Recall: H has a spectral gap Δ at the Fermi energy (which we moved to 0).

$$\text{bulk K-group} \ni [H] \stackrel{\delta}{\to} [J, P_\Delta(\hat{H})\hat{H}] \in \text{boundary K-group}$$

and $\ker \delta = \mathrm{im} q_*$.

1. If $[H] \notin \mathrm{im} q_*$ then the spectrum of $\hat{H}$ must cover the gap Δ at 0. Therefore there must be states (resonances) which are localized at the boundary.
2. These resonances cannot propagate into the bulk.
3. These resonances cannot be destroyed by bending, denting the boundary or by adding disorder to it. They are stable against perturbation of the boundary.

Proof:

1. If the spectrum of $\hat{H}$ has a gap in Δ then $P_\Delta(\hat{H})\hat{H}$ is homotopic to an operator whose square is 1 thus defining a degenerate KK-cycle. This would mean that $[J, P_\Delta(\hat{H})\hat{H}] = 0$, a contradiction to $[H] \notin \ker \delta$.
2. There is no bulk spectrum in the gap.
3. A perturbation which is restricted to the boundary does not affect the bulk invariant. Alternative argument: A perturbation which is restricted to the boundary can change $\hat{H}$ only up to a J-compact operator.

Bulk-Boundary Correspondence: second summary and questions

- We have established a correspondence between bulk invariants and boundary invariants for topological insulators.
- Both are derived naturally from the Hamiltonian of the system.
- The correspondence arises from algebraic topology (it is not an algebra homomorphism).

Bulk-Boundary Correspondence: second summary and questions

- We have established a correspondence between bulk invariants and boundary invariants for topological insulators.
- Both are derived naturally from the Hamiltonian of the system.
- The correspondence arises from algebraic topology (it is not an algebra homomorphism).

Questions:

- How can we detect that the bulk invariant is not trivial, $[H] \notin \mathrm{im} q_*$?
- Can one measure the topological invariants?

Bulk-Boundary Correspondence: second summary and questions

- We have established a correspondence between bulk invariants and boundary invariants for topological insulators.
- Both are derived naturally from the Hamiltonian of the system.
- The correspondence arises from algebraic topology (it is not an algebra homomorphism).

Questions:

- How can we detect that the bulk invariant is not trivial, $[H] \notin \mathrm{im} q_*$?
- Can one measure the topological invariants?

Answer: Numerical topological invariants

- Any functional $K(A, \mathcal{E}) \to \mathbb{C}$ or $\mathbb{Z}_2$ is a numerical invariant for topol. phases.
- A numerical invariant serves to distinguish K-group elements and can detect the topological non-triviality of a material.
- A numerical invariant may be physically measurable (Hall conductivity).

There are two approaches to numerical invariants (partly related through index theory):

Via *KK*-theory: the dual of K-theory $KK(\mathbb{C}, A)$ is K-homology $KK(A, \mathbb{C})$ with $\mathbb{R}$ in place of $\mathbb{C}$ if we have real symmetries.
The duality pairing is the Kasparov product

$$KK_i(\mathbb{C}, A) \times KK_j(A, \mathbb{C}) \to KK_{i-j}(\mathbb{C}, \mathbb{C})$$

and $KK_{j-i}(\mathbb{C}, \mathbb{C})$ or $KK_{j-i}(\mathbb{R}, \mathbb{R})$ is $\mathbb{Z}$, $\mathbb{Z}_2$ or 0. This gives a numerical invariant (an index). [Grossmann, Schulz-Baldes 2016]
For our algebras (crossed product) exists a fundamental class $[\lambda_d] \in KK_d(A, \mathbb{C})$ (purely geometric data: Dirac operator in momentum space). The dual boundary map $\delta^* : KK_i(\mathbb{C}, J) \to KK_{i+1}(\mathbb{C}, A)$ maps $[\lambda_{d-1}]$ to $[\lambda_d]$.
This yields a numerical bulk-boundary correspondance:

$$[H] \times [\lambda_d] = [(J, P_\Delta(\hat{H})\hat{H})] \times [\lambda_{d-1}]$$

[Bourne,Carey,Rennie,+K 2017].

Via cyclic cohomology: cyclic cohomology is a generalisation of de Rham cohomology to algebras.

A K-group element may be combined with a cyclic cocycle to obtain a Chern number generalising the integral of a Chern class.
Leads to a numerical bulk-boundary correspondence which is, however, trivial for $\mathbb{Z}_2$-invariants [Richter, Schulz-Baldes +K. 2002]

Close to linear response theory [TKNN, Bellissard-Connes] for IQHE

Direct approach fails for $\mathbb{Z}_2$-invariants, these need secondary pairings with cyclic cocycles [K. 2019].