Free-Fermion entanglement and time and band limiting

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based on work done in collaboration with

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1. Introduction

Free-Fermion chains offer exactly solvable framework to study entanglement in many-body systems - Much studied

Basically one

- Takes the chain in state $|\Psi\rangle\rangle$ (ground state)
- Divides the system in two parts
- Examines how these two parts are coupled in $|\Psi\rangle\rangle$

Goals

- Draw parallel with time and band limiting problems in signal processing
- Show how methods developed in that context allow to obtain tridiagonal matrices that commute with the hopping matrix of the *Entanglement Hamiltonian*

2. Outline

- Free-Fermion Hamiltonian and orthogonal polynomials
- Ground state and correlations
- Reduced density matrix and entanglement Hamiltonian
- Time and band limiting
- Bispectral problems and algebraic Heun operators
- Parallel between entanglement in free-Fermion chains and time and band limiting
- Commuting Jacobi matrices
- The uniform chain
- A non-uniform example
- Concluding remarks

3. Free-Fermion Hamiltonian and orthogonal polynomials

Open quadratic free-Fermion inhomogeneous Hamiltonian

$$\widehat{\mathcal{H}} = \sum_{n=0}^{N-1} J_n (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) - \sum_{n=0}^{N} B_n c_n^\dagger c_n = \sum_{m,n=0}^{N} c_m^\dagger \widehat{H}_{mn} c_n \ ,$$

J_n and B_n real parameters, $\{c_m^\dagger, c_n\} = \delta_{m,n}$

To diagonalize $\widehat{\mathcal{H}}$, first diagonalize $(N+1) \times (N+1)$ matrix $\widehat{H} = |\widehat{H}_{mn}|_{0 \leq m,n \leq N}$

In the canonical orthonormal basis $\{|0\rangle, |1\rangle, \dots, |N\rangle\}$ of $\mathbb{C}^{N+1}$, the position basis, one has

$$\widehat{H}|n\rangle = J_{n-1}|n-1\rangle - B_n|n\rangle + J_n|n+1\rangle \,, \qquad 0 \leq n \leq N$$

with $J_N = J_{-1} = 0$

The spectral problem for $\widehat{H}$ reads

$$\widehat{H}|\omega_k\rangle = \omega_k|\omega_k\rangle \quad with \quad |\omega_k\rangle = \sum_{n=0}^{N} \phi_n(\omega_k)|n\rangle$$

Order the $N+1$ eigenvalues $\omega_0, \omega_1, \dots \omega_N$ such that $\omega_k < \omega_{k+1}$
Normalize $|\omega_0\rangle, |\omega_1\rangle, \dots |\omega_N\rangle$ so they form an orthonormal basis of $\mathbb{C}^{N+1}$ - the momentum basis

The eigenfunctions $\phi_n(\omega_k)$ are real, since the matrix $\widehat{H}$ is real and its eigenvalues are non-degenerate. Hence $|n\rangle = \sum_{k=0}^{N} \phi_n(\omega_k)|\omega_k\rangle$

Thus eigenfunctions satisfy the orthonormality conditions

$$\sum_{n=0}^{N} \phi_n(\omega_k)\phi_n(\omega_p) = \delta_{kp} \quad , \quad \sum_{k=0}^{N} \phi_m(\omega_k)\phi_n(\omega_k) = \delta_{mn}$$

From tridiagonal action of $\widehat{H}$ in position basis, deduce that $\phi_n(\omega_k)$ satisfy the recurrence relation

$$\omega_k\phi_n(\omega_k) = J_n\phi_{n+1}(\omega_k) - B_n\phi_n(\omega_k) + J_{n-1}\phi_{n-1}(\omega_k), \qquad 0 \le n \le N$$

As known this connects chain to orthogonal polynomials:

$$Let \quad \phi_n(\omega_k) = \sqrt{W_k}\, \chi_n(k) \qquad with \quad \chi_0(k) = 1 \quad and \quad \chi_{-1}(k) = 0$$

$$\omega_k \chi_n(k) = J_n \chi_{n+1}(k) - B_n \chi_n(k) + J_{n-1} \chi_{n-1}(k)\,, \qquad 0 \le n \le N$$

$\chi_n(k)$ orthogonal polynomials of discrete variable with orthogonality relation

$$\sum_{k=0}^{N} W_k\, \chi_m(k) \chi_n(k) = \delta_{mn} \qquad\qquad W_k \quad weights$$

Having diagonalized $\widehat{H}$, we see that the Hamiltonian $\widehat{\mathscr{H}}$ can be rewritten as

$$\widehat{\mathscr{H}} = \sum_{k=0}^{N} \omega_k \tilde{c}_k^{\dagger} \tilde{c}_k\,,$$

where the annihilation operators $\tilde{c}_k$ are given by

$$\tilde{c}_k = \sum_{n=0}^{N} \phi_n(\omega_k)\, c_n\,, \qquad c_n = \sum_{k=0}^{N} \phi_n(\omega_k)\, \tilde{c}_k\,,$$

and creation operators $\tilde{c}_k^{\dagger}$ obtained by Hermitian conjugation These obey

$$\{\tilde{c}_k^{\dagger}, \tilde{c}_p\} = \delta_{k,p}\,, \qquad \{\tilde{c}_k^{\dagger}, \tilde{c}_p^{\dagger}\} = \{\tilde{c}_k, \tilde{c}_p\} = 0$$

Eigenvectors of $\widehat{\mathscr{H}}$ given by

$$|\Psi\rangle\rangle = \tilde{c}^{\dagger}_{k_1} \dots \tilde{c}^{\dagger}_{k_r} |0\rangle\rangle \,,$$

with $k_1, \dots, k_r \in \{0, \dots, N\}$ pairwise distinct

Vacuum state $|0\rangle\rangle$ is annihilated by all the annihilation operators

$$\tilde{c}_k |0\rangle\rangle = 0 \,, \qquad k = 0 \,, \dots \,, N$$

The energy eigenvalues are given by

$$E = \sum_{i=1}^{r} \omega_{k_i}$$

Ground state

Shall consider entanglement in ground state $|\Psi_0\rangle\rangle$ constructed by filling the Fermi sea:

$$|\Psi_0\rangle\rangle = \tilde{c}_0^\dagger \dots \tilde{c}_K^\dagger |0\rangle\rangle\,,$$

where $K \in \{0, 1, \dots, N\}$ is the greatest integer below the Fermi momentum, such that

$$\omega_K < 0\,, \qquad \omega_{K+1} > 0\,.$$

K can be modified by adding a constant term in the external magnetic field B_n.

Correlation matrix

The 1- particle correlation matrix $\widehat{C}$ in the ground state is the $(N+1)\times(N+1)$ matrix with entries

$$\widehat{C}_{mn} = \langle\langle\Psi_0|c_m^\dagger c_n|\Psi_0\rangle\rangle\,.$$

Using $c_n = \sum_{k=0}^{N}\phi_n(\omega_k)\,\tilde{c}_k$, definition of ground state, anticommutation relations and property of vacuum:

$$\widehat{C}_{mn} = \sum_{k=0}^{K}\phi_m(\omega_k)\phi_n(\omega_k)\,, \qquad 0\le n,m\le N\,.$$

Since $\phi_n(\omega_k) = \langle n|\omega_k\rangle$ it is seen

$$\widehat{C} = \sum_{k=0}^{K}|\omega_k\rangle\langle\omega_k|\,,$$

i.e. $\widehat{C}$ is projector onto subspace of $\mathbb{C}^{N+1}$ spanned by vectors $|\omega_k\rangle$ with $k=0,\ldots,K$ running over filling labels

4. Reduced density matrix and entanglement Hamiltonian

To discuss entanglement need **bipartition**:

- Part 1: sites $\{0, 1, ..., \ell\}$
- Part 2: sites $\{\ell+1, \ell+2, ..., N\}$

Entanglement properties in ground state provided by reduced density matrix

$$\rho_1 = tr_2 |\Psi_0\rangle\rangle\langle\langle\Psi_0| \qquad (2^{\ell+1} \times 2^{\ell+1})$$

Observation (Peschel, Vidal et al.):
ρ_1 determined by "chopped" correlation matrix C - $(\ell+1) \times (\ell+1)$ submatrix of $\widehat{C}$:

$$C = |\widehat{C}_{mn}|_{0 \le m,n \le \ell}$$

Argument:

1. $|\Psi_0\rangle\rangle$ Slater determinant $\Rightarrow$ all correlations expressed in terms of matrix elements $\widehat{C}_{mn}$
2. When m, n belong to Part 1

$$C_{mn} = tr(\rho_1 \, c_m^\dagger c_n), \qquad m\,, n \in \{0, 1, \ldots, \ell\},$$

3. To have factorization of correlations - given trace formula - by Wick's theorem must have

$$\rho_1 = \kappa \, \exp(-\mathscr{H}),$$

with the *entanglement Hamiltonian* $\mathscr{H}$

$$\mathscr{H} = \sum_{m,n\in\{0,\ldots,\ell\}} h_{mn} \, c_m^\dagger c_n \, .$$

The hopping matrix $h = |h_{mn}|_{0\leq m,n\leq \ell}$ is defined so that 2 holds, one finds

$$h = \log[(1-C)/C] \, .$$

Thus see that ρ_1, and entanglement Hamiltonian $\mathscr{H}$, are obtained from the $(l+1)\times(l+1)$ matrix C.

Introduce the projectors

$$\pi_1 = \sum_{n=0}^{\ell} |n\rangle\langle n| \quad \text{and} \quad \pi_2 = \sum_{k=0}^{K} |\omega_k\rangle\langle\omega_k| = \widehat{C},$$

the chopped correlation matrix can be written as

$$C = \pi_1 \pi_2 \pi_1$$

To calculate entanglement entropies one has to compute the eigenvalues of C

Not easy to do numerically because the eigenvalues of that matrix are exponentially close to 0 and 1

Parallel between study of entanglement properties of finite free-Fermion chains and *time and band limiting problems* will indicate how this can be circumvented

5. An overview of time and band limiting

$f(t)$ signal limited to band of frequencies $[-W, W]$:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-W}^{W} dp\, e^{ipt} F(p) \in B_W$$

Also want signal of finite duration

$$f \neq 0 \quad \text{for} \quad -T < t < T$$

Impossible: $f(t) \in B_W$ is entire in complex t-plane

If $f(t) = 0$ for any interval, it implies

$$f(t) \equiv 0$$

In 1970s Slepian, Landau, Pollak (Bell labs) asked:

Which band-limited signal $\in B_W$ is best concentrated in time interval $-T < t < T$, second best etc.?

i.e. are maximizing

$$\alpha^2(T) = \frac{\int_{-T}^{T} f^2(t)dt}{\int_{-\infty}^{\infty} f^2(t)dt}$$

$$= \frac{\int_{-T}^{T} dt \int_{-W}^{W} dp'' e^{ip''t} F(p'') \int_{-W}^{W} dp' e^{-ip't} F*(p')}{\int_{-W}^{W} dp \mid F(p) \mid^2}$$

$$= 2\frac{\int_{-W}^{W} dp' \int_{-W}^{W} dp'' [\frac{sin(p'-p'')T}{(p'-p'')}] F(p'')F^*(p')}{\int_{-W}^{W} dp' F(p')F^*(p')}$$

Well known that answer provided by eigenfunctions of integral operator

$$GF(p) = \int_{-W}^{W} dp' K(p-p') F(p') = \lambda F(p)$$

with ”sinc kernel”

$$K(p-p') = \frac{\sin(p-p')T}{\pi(p-p')}$$

- In principle, concentration problem solved
- However spec(G) accumulates sharply at origin
- Numerical computations intractable

Amazing result (Slepian, Landau, Pollak)
There exists a 2^{nd} order differential operator D that commutes with integral operator G

D arises in separating Laplacian in prolate spheroidal coordinates

Great because D has common eigenfunctions with G and 2^{nd} order differential operator well behaved numerically

First see how this operator D can be obtained

Consider the projectors

$$\pi_L^x f(x) = \begin{cases} f(x) & -L < x < L \\ 0 & \text{otherwise} \end{cases}$$

$$= [\Theta(x+L) - \Theta(x-L)] f(x)$$

$\Theta(x)$ step function

Let $\mathscr{F} : f(t) \to F(p)$ be the Fourier transform and $\mathscr{F}^{-1}$ the inverse

Take the projectors

π_W^p and the Fourier transformed $\hat{\pi}_T^p = \mathscr{F} \pi_T^t \mathscr{F}^{-1}$

It is straightforward to see that

$$G = \pi_W^p \hat{\pi}_T^p \pi_W^p$$

Indeed

$$\begin{aligned}
\pi_W^p \hat{\pi}_T^t \pi_W^p F(p) &= \pi_W^p \mathscr{F} \pi_T^t \mathscr{F}^{-1} \pi_W^p F(p) \\
&\propto \pi_W^p \int_{-T}^{T} dt e^{-ipt} \int_{-W}^{W} dp' e^{ip't} F(p') \\
&\propto \pi_W^p \int_{-W}^{W} dp' [\frac{sin(p-p')T}{(p-p')}] F(p') \\
&= GF(p)
\end{aligned}$$

With the remaining π_W^p ensuring that G transforms band-limited $F(p)$ onto themselves.

Finding the commuting operator D

Take D to be of the general form

$$D = A(p)\frac{d^2}{dp^2} + B(p)\frac{d}{dp} + C(p)$$

Consider first projector onto the semi-infinite interval $[W, \infty)$

$$\tilde{\pi}_W^p = \Theta(p - W)$$

It is easy to see that

$[D, \tilde{\pi}_W^p] = 2A(p)\delta(p - W)\frac{d}{dp} + (-A'(p) + B(p))\delta(p - W) = 0$ if

$$A(W) = 0 \qquad \text{and} \qquad A'(W) = B(W)$$

Recall that

$$\pi_W^p = \Theta(p+W) - \Theta(p-W)$$

In this case $[D, \pi_W^p] = 0$ requires

$$A(\pm W) = 0 \qquad \text{and} \qquad A'(\pm W) = B(\pm W)$$

Accomodated by

$$\begin{aligned} D &= (p^2 - W^2)\tfrac{d^2}{dp^2} + 2p\tfrac{d}{dp} + C(p) \\ &= \tfrac{1}{2}\{\tfrac{d^2}{dp^2}, p^2\} - W^2\tfrac{d^2}{dp^2} + \tilde{C}(p) \end{aligned}$$

Recall that

$$G = \pi_W^p \hat{\pi}_T^p \pi_W^p$$

so if in addition $[D, \hat{\pi}_T^p] = 0$, this would imply $[D, G] = 0$

$$[D, \hat{\pi}_T^p] = [D, \mathscr{F} \pi_T^t \mathscr{F}^{-1}] = 0 \qquad \text{implies} \qquad [\mathscr{F}^{-1} D \mathscr{F}, \pi_T^t] = 0$$

i.e. Fourier transform $\tilde{D} = \mathscr{F}^{-1} D \mathscr{F}$ of D commutes with a projector in t with parameter T similar to π_W^p

Under Fourier transform:

$$p^2 \leftrightarrow -\frac{d^2}{dt^2} \qquad , \qquad -\frac{d^2}{dp^2} \leftrightarrow t^2$$

Reminder: $D = \frac{1}{2}\{\frac{d^2}{dp^2}, p^2\} - W^2 \frac{d^2}{dp^2} + \tilde{C}(p)$

Take $\tilde{C}(p) = T^2 p^2$, then

$\tilde{D} = \frac{1}{2}\{\frac{d^2}{dt^2}, t^2\} + W^2 t^2 - T^2 \frac{d^2}{dt^2}$

which by comparison satisfies $[\tilde{D}, \pi_T^t] = 0$ and hence

$D = \frac{1}{2}\{\frac{d^2}{dp^2}, p^2\} - W^2 \frac{d^2}{dp^2} + T^2 p^2$

commutes with G

Reasons for existence of commuting D not well understood - been traced to bispectral problems (Grünbaum, V. , Zhedanov)

$\psi(t,p) = e^{ipt}$ in Fourier transforms are solutions of

$$-\frac{d^2}{dt^2}\psi(t,p) = p^2\psi(t,p), \qquad -\frac{d^2}{dp^2}\psi(t,p) = t^2\psi(t,p)$$

Most simple bispectral problem: $\psi(t,p)$ eigenfunctions of operator acting on t with eigenvalues depending on p and vice-versa

Consider two operators in "frequency" representation: $X = -\frac{d^2}{dp^2} \quad Y = p^2$

If we form the "algebraic Heun operator" out of bispectral operators X, Y:

$$D = \{X,Y\} + \tau[X,Y] + \mu X + \nu Y, \qquad \{X,Y\} = XY + YX$$

the conditions for $[D,G] = 0$ on parameters are easy to find.

In present case:

$\tau = 0 \quad \mu = -2W^2 \quad \nu = -2T^2$

This is what we want to apply to free-Fermion entanglement

Why the name Algebraic Heun operator **?**

Heun equation is Fuchsian 2nd order differential equation with four regular singularities

Standard form given by

$$\frac{d^2}{dx^2}\psi(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-d}\right)\frac{d}{dx}\psi(x) + \frac{\alpha\beta x - q}{x(x-1)(x-d)}\psi(x) = 0,$$

where

$$\alpha + \beta - \gamma - \delta + 1 = 0,$$

to ensure regularity of the singular point at $x = \infty$

This equation can be written in the form

$$M\psi(x) = \lambda\psi(x)$$

with M the Heun operator given by

$$M = x(x-1)(x-d)\frac{d^2}{dx^2} + (\rho_2 x^2 + \rho_1 x + \rho_0)\frac{d}{dx} + r_1 x + r_0$$

Bispectral problems - again

Operators X and Y form bispectral pair if they have common eigenfunctions $\psi(x,n)$ such that

$$X\psi(x,n) = \omega(x)\psi(x,n)$$
$$Y\psi(x,n) = \lambda(n)\psi(x,n),$$

with X acting on the variable n and Y, on the variable x

Note: Forming products of these operators must take both in same representation "n" or "x"

All hypergeometric orthogonal polynomials $p_n(x)$ form bispectral problem

$p_n(x)$ **solution to both**

- 3-term recurrence relation $\leftrightarrow X$
- differential or difference equation $\leftrightarrow Y$

x-representation: X acts a multiplication by the variable and Y as the differential or difference operator

n-representation: X acts as a three-term difference operator over n and Y as multiplication by the eigenvalue

Jacobi Polynomials $P_n^{(\alpha,\beta)}(x)$

These polynomials satisfy **differential equation**

$$D_x P_n^{(\alpha,\beta)}(x) = \lambda_n P_n^{(\alpha,\beta)}(x) \quad \text{with} \quad \lambda_n = -n(n+\alpha+\beta+1)$$

where D_x is hypergeometric operator

$$D_x = x(x-1)\frac{d^2}{dx^2} + (\alpha+1-(\alpha+\beta+2)x)\frac{d}{dx}$$

They also satisfy the **three-term recurrence relation**

$$xP_n^{(\alpha,\beta)}(x) = P_{n+1}^{(\alpha,\beta)}(x) + b_n P_n^{(\alpha,\beta)}(x) + u_n P_{n-1}^{(\alpha,\beta)}(x)$$

Bispectral problem with

x representation:

$$X = x \qquad Y = D_x$$

n representation:

$$X = T_n^{+} + b_n \cdot 1 + u_n T_n^{-1} \qquad Y = \lambda_n \qquad \text{where} \qquad T_n^{\pm} f_n = f_{n\pm 1}$$

Algebraic Heun operator

- Forming the most general operator W bilinear in bispectral operators of Jacobi polynomials (in x representation):

  $W = \tau_1\{X, Y\} + \tau_2[X, Y] + \tau_3 X + \tau_4 Y + \tau_0$ with the τ s constants

  Grünbaum, V., Zhedanov found that W is **standard Heun operator**
- W can be defined for any bispectral problem and because of result for Jacobi OP it has been called *Algebraic Heun operator*
- Construct can be carried for all hypergeometric OP families.

  In case of Hahn polynomials it gives a difference version of Heun equation - q-Heun obtained from Big q-Jacobi OPs

6. Parallel between entanglement in free-Fermion chains and time and band limiting

- Picking ground (or other reference) state restricts energies - corresponds to band limiting
  Projector $\pi_2 = \sum_{k=0}^{K} |\omega_k\rangle\langle\omega_k| = \widehat{C}$
- Establishing bipartition "chops" space - akin to time limiting
  Projector $\pi_1 = \sum_{n=0}^{\ell} |n\rangle\langle n|$
- Wish to solve eigenvalue problem for "chopped" correlation matrix
  $C = \pi_1 \pi_2 \pi_1$ which is full matrix
- Will look for tridiagonal Jacobi matrix T - analog of 2nd order differential operator in discrete realm
- Cast in bispectral problem framework
- Form algebraic Heun operator in terms of bispectral pair

6. Parallel between entanglement in free-Fermion chains and time and band limiting (continued)

- Determine conditions for
  $[T,\pi_1]=[T,\pi_2]=0 \qquad \Rightarrow \qquad [T,C]=0$

For open and finite free-Fermion chains, bispectral problems provided by (truncated) hypergeometric polynomials in a discrete variable of Askey scheme

These polynomials $\chi_n(k)$ satisfy **recurrence** relations of the form

$$\omega_k\chi_n(k)=J_n\chi_{n+1}(k)-B_n\chi_n(k)+J_{n-1}\chi_{n-1}(k)\,, \qquad 0\leq n\leq N$$

with $J_N=J_{-1}=0$; as well as **difference** relations of the form

$$\lambda_n\chi_n(k)=\overline{A}_k\chi_n(k+1)-(\overline{A}_k+\overline{C}_k)\chi_n(k)+\overline{C}_k\chi_n(k-1)\,, \qquad 0\leq k\leq N\,,$$

$$\overline{A}_k=\overline{J}_k\sqrt{\frac{W_{k+1}}{W_k}} \qquad \overline{C}_k=\overline{J}_{k-1}\sqrt{\frac{W_{k-1}}{W_k}}$$

with $\overline{J}_{-1}=\overline{J}_N=0$.

7. Commuting Jacobi matrices for chains associated to finite hypergeometric polynomials

Strategy

- Pick J_n and B_n in the Hamiltonian so that polynomials associated via recurrence relation are *bispectral*
- Exploit difference equation to construct T

Since $\phi_n(\omega_k) = \langle n|\omega_k\rangle$ obeys

$$\lambda_n \phi_n(\omega_k) = \overline{J}_k \phi_n(\omega_{k+1}) - \overline{B}_k \phi_n(\omega_k) + \overline{J}_{k-1} \phi_n(\omega_{k-1}), \qquad 0 \leq k \leq N,$$

can define operator $\widehat{X}$ in basis $\{|n\rangle\}$ by

$$\widehat{X}|n\rangle = \lambda_n |n\rangle,$$

which consequently acts as follows in the $\{|\omega_k\rangle\}$ basis

$$\widehat{X}|\omega_k\rangle = \overline{J}_{k-1}|\omega_{k-1}\rangle - \overline{B}_k|\omega_k\rangle + \overline{J}_k|\omega_{k+1}\rangle.$$

We have

$$\widehat{H}|\omega_k\rangle = \omega_k|\omega_k\rangle, \qquad \widehat{H}|n\rangle = J_{n-1}|n-1\rangle - B_n|n\rangle + J_n|n+1\rangle$$

and

$$\widehat{X}|\omega_k\rangle = \bar{J}_{k-1}|\omega_{k-1}\rangle - \bar{B}_k|\omega_k\rangle + \bar{J}_k|\omega_{k+1}\rangle, \qquad \widehat{X}|n\rangle = \lambda_n|n\rangle$$

Introduce algebraic Heun operator:

$$\widehat{T} = \{\widehat{X}, \widehat{H}\} + \mu\widehat{X} + \nu\widehat{H}$$

It is immediate to see that $\widehat{T}$ is tridiagonal in both the position basis

$$\begin{aligned}\widehat{T}|n\rangle &= J_{n-1}(\lambda_{n-1} + \lambda_n + \nu)|n-1\rangle + (\mu\lambda_n - 2B_n\lambda_n - \nu B_n)|n\rangle \\ &\quad + J_n(\lambda_n + \lambda_{n+1} + \nu)|n+1\rangle\ ,\end{aligned}$$

and the momentum basis

$$\begin{aligned}\widehat{T}|\omega_k\rangle &= \bar{J}_{k-1}(\omega_{k-1} + \omega_k + \mu)|\omega_{k-1}\rangle + (\nu\omega_k - 2\bar{B}_k\omega_k - \mu\bar{B}_k)|\omega_k\rangle \\ &\quad + \bar{J}_k(\omega_k + \omega_{k+1} + \mu)|\omega_{k+1}\rangle\ .\end{aligned}$$

Let $\widehat{T}_{mn} = \langle m|\widehat{T}|n\rangle$, and define the "chopped" matrix T by

$$T = |\widehat{T}_{mn}|_{0\leq m,n\leq \ell}\,. \tag{1}$$

T and C will commute

$$[T, C] = 0 \tag{2}$$

if the parameters in $\widehat{T}$ are given by

$$\mu = -(\omega_K + \omega_{K+1}) \quad \text{and} \qquad \nu = -(\lambda_\ell + \lambda_{\ell+1})\,. \tag{3}$$

$$\widehat{T}|n\rangle = \ldots + J_n(\lambda_n + \lambda_{n+1} - \lambda_\ell - \lambda_{\ell+1})|n+1\rangle$$

$$\widehat{T}|\omega_k\rangle = \ldots + \bar{J}_k(\omega_k + \omega_{k+1} - \omega_K - \omega_{K+1})|\omega_{k+1})\rangle$$

$\widehat{T}$ leaves subspace $\{|n\rangle, n = 0, 1, \ldots, \ell\}$ invariant $\quad\Rightarrow\quad [T, \pi_1] = 0$

$\widehat{T}$ leaves the subspace $\{|\omega_k\rangle, k = 0, 1, \ldots, K\}$ invariant $\quad\Rightarrow\quad [T, \pi_2] = 0$

Since $C = \pi_1 \pi_2 \pi_1$ $[T, C] = 0$

- Since T is non-degenerate T and C have a unique set of common eigenvectors
- Since T is tridiagonal, its eigenvectors can be readily computed numerically
- Acting with C on these eigenvectors, eigenvalues of C can be easily obtained
- The eigenvalues of the entanglement Hamiltonian $\mathscr{H}$, and therefore the entanglement entropy of the model, can then also be easily obtained

Main result

The tridiagonal matrix

$$T = \begin{pmatrix} d_0 & t_0 & & & & \\ t_0 & d_1 & t_1 & & & \\ & t_1 & d_2 & t_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & t_{\ell-2} & d_{\ell-1} & t_{\ell-1} \\ & & & & t_{\ell-1} & d_\ell \end{pmatrix}$$

with

$$t_n = J_n(\lambda_n + \lambda_{n+1} - \lambda_\ell - \lambda_{\ell+1})$$

$$d_n = -B_n(2\lambda_n - \lambda_\ell - \lambda_{\ell+1}) - \lambda_n(\omega_K + \omega_{K+1}) \, .$$

commutes with the correlation matrix C and the hopping matrix h of entanglement Hamiltonian $\mathscr{H}$

8. The homogeneous chain

Apply to obtain commuting tridiagonal matrix T for the homogeneous chain with

$$J_0 = \ldots = J_{N-1} = -\frac{1}{2}, \qquad B_n = 0.$$

The wavefunctions $\phi_n(\omega_k) = \langle n|\omega_k\rangle$ satisfy

$$\omega_k \phi_n(\omega_k) = -\frac{1}{2}\phi_{n+1}(\omega_k) - \frac{1}{2}\phi_{n-1}(\omega_k), \qquad 0 \leq n \leq N$$

From recurrence relation

$$2xR_n(x) = R_{n+1}(x) + R_{n-1}(x), \qquad n = 0, 1, \ldots.$$

of Chebyshev polynomials of the second kind

$$R_n(x) = \frac{\sin(\theta(n+1))}{\sin(\theta)}, \qquad x = \cos(\theta), \qquad n = 0, 1, \ldots,$$

See that $\phi_n(\omega_k)$ will be related to discretized Chebyshev polynomials

Imposing the truncation condition $R_{N+1} = 0$ yields

$$2\cos(\theta)\sin((N+1)\theta) = \sin(N\theta)\,,$$

which has solutions

$$\theta = \theta_k = \frac{\pi(k+1)}{N+2} \quad \text{for any integer } k$$

Normalized eigenfunctions are

$$\phi_n(\omega_k) = \sqrt{\frac{2}{N+2}}\sin(\theta_k)R_n(x_k) = \sqrt{\frac{2}{N+2}}\sin\left[\frac{\pi(k+1)(n+1)}{N+2}\right],$$

where

$$\omega_k = -x_k = -\cos(\theta_k)\,, \qquad k = 0, 1, \ldots, N\,.$$

Note that

$$\phi_n(\omega_k) = \phi_k(\omega_n) \quad (n \leftrightarrow k)$$

It is immediate to obtain the difference relation for the wave function $\phi_n(\omega_k)$ from their recurrence relation by using the duality property $\phi_n(\omega_k) = \phi_k(\omega_n)$:

$$\omega_n \phi_n(\omega_k) = -\frac{1}{2}\phi_n(\omega_{k+1}) - \frac{1}{2}\phi_n(\omega_{k-1}),$$

The second bispectral operator $\widehat{X}$ has hence eigenvalues

$$\lambda_n = \omega_n = -\cos(\theta_n)$$

in the basis $\{|n\rangle\}$
The matrix T is then given by with

$$t_n = \frac{1}{2}\left[\cos(\theta_n) + \cos(\theta_{n+1}) - \cos(\theta_\ell) - \cos(\theta_{\ell+1})\right]$$

$$d_n = -\cos(\theta_n)\left[\cos(\theta_K) + \cos(\theta_{K+1})\right]$$

This readily recovers recent results of Eisler & Peschl.

8. An inhomogeneous chain

Take

$$J_n = \frac{1}{2}\sqrt{(N-n)(n+1)}, \qquad B_n = -\frac{N}{2}$$

This defines inhomogeneous chain known to possess Perfect State Transfer (PST) - Christandl, Datta, Dorlas et al.

The wavefunctions $\phi_n(\omega_k) = \langle n|\omega_k\rangle$ satisfy

$$\begin{aligned} \omega_k \phi_n(\omega_k) = & \frac{1}{2}\sqrt{(N-n)(n+1)}\phi_{n+1}(\omega_k) + \frac{N}{2}\phi_n(\omega_k) \\ & + \frac{1}{2}\sqrt{(N-n+1)n}\phi_{n-1}(\omega_k), \qquad 0 \le n \le N \end{aligned}$$

Take

$$\phi_n(\omega_k) = (-1)^n \sqrt{\binom{N}{n}} \sqrt{W_k} R_n(k)$$

See that $R_n(k)$ satisfies

$$-\omega_k R_n(k) = \frac{1}{2}(N-n)R_{n+1} - \frac{N}{2}R_n(k) + \frac{n}{2}R_{n-1}(k)$$

which is recurrence relation of symmetric Krawtchouk polynomials

$$R_n(k) = {}_2F_1\left(\begin{matrix}-n, & -k \\ & -N\end{matrix}; 2\right), \qquad n = 0, 1, \ldots, N,$$

if $\omega_k = k$ (this provides spectrum)

Krawtchouk polynomials known to be orthogonal wrt binomial distribution

$$W_k = (\frac{1}{2})^N \binom{N}{k}$$

Observe also self-duality: $R_n(k) = R_k(n)$

The eigenfunctions $\phi_n(\omega_k)$ are given by

$$\phi_n(\omega_k) = (-1)^n 2^{-\frac{N}{2}} \sqrt{\binom{N}{n}\binom{N}{k}} R_n(k)$$

where $\omega_k = k$

The difference equation

$$\lambda_n \phi_n(\omega_k) = \overline{J}_k \phi_n(\omega_{k+1}) - \overline{B}_k \phi_n(\omega_k) + \overline{J}_{k-1} \phi_n(\omega_{k-1})$$

is obtained from the recurrence relation

$$\begin{aligned} \omega_k \phi_n(\omega_k) =& \frac{1}{2}\sqrt{(N-n)(n+1)}\phi_{n+1}(\omega_k) + \frac{N}{2}\phi_n(\omega_k) \\ &+ \frac{1}{2}\sqrt{(N-n+1)n}\phi_{n-1}(\omega_k), \qquad 0 \le n \le N \end{aligned}$$

by performing $n \leftrightarrow k$ and exploiting $\phi_n(\omega_k) = (-1)^{(n+k)} \phi_k(\omega_n)$

One finds

$$\bar{J}_k = -\frac{1}{2}\sqrt{(N-k)(k+1)}, \qquad \bar{B}_k = -\frac{N}{2}, \qquad \lambda_n = n$$

The commuting tridiagonal matrix T

$$T = \begin{pmatrix} d_0 & t_0 & & & \\ t_0 & d_1 & t_1 & & \\ & \ddots & \ddots & \ddots & \\ & & t_{\ell-2} & d_{\ell-1} & t_{\ell-1} \\ & & & t_{\ell-1} & d_\ell \end{pmatrix}$$

has therefore the entries

$$\begin{aligned} t_n &= J_n(\lambda_n + \lambda_{n+1} - \lambda_\ell - \lambda_{\ell+1}) \\ &= (n-\ell)\sqrt{(N-n)(n+1)} \end{aligned}$$

$$\begin{aligned} d_n &= -B_n(2\lambda_n - \lambda_\ell - \lambda_{\ell+1}) - \lambda_n(\omega_K + \omega_{K+1}) \\ &= \frac{N}{2}(2n-2\ell-1) - n(2K+1) \end{aligned}$$

9. Concluding remarks

- Stressed parallel between study of entanglement in finite free-Fermion chains and time and band limiting problems
- Indicated that methods developed in later context could be helpful in former
- Shown that for chains associated to bispectral orthogonal polynomials, algebraic Heun operator readily provides a tridiagonal matrix that commutes with correlation matrix and hopping matrix of entanglement Hamiltonian
- The approach provides such commuting matrices for the many chains corresponding to finite discrete polynomials of Askey scheme

For details and references see

F.A. Grünbaum, L. Vinet, A. Zhedanov, *Algebraic Heun operator and band-time limiting*, Comm. Math. Phys. (2018) arXiv 1711.07862
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