Rational spin chains at higher rank:

new tools to solve, completeness, and separation of variables

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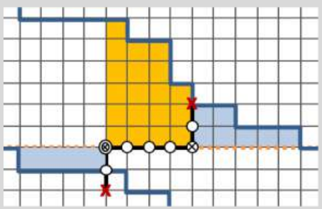
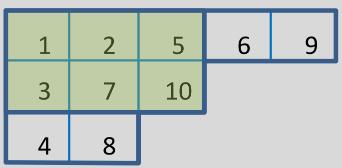
LPTHE, Paris, 24/01/2020



1608.06504 w/C.Marboe
1712.01811 w/M.Günaydin



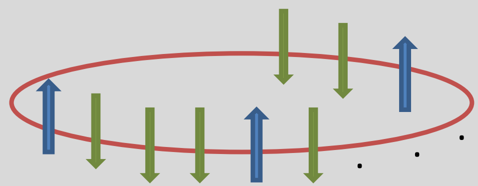
2002.xxxxx w/D.Chernyak & S.Leurent

$$\Psi = \prod \det Q_i\,(x_j)$$

1810.10996 w/P.Ryan
1910.13442 w/N.Gromov, F.Levkovich-Maslyuk, P.Ryan
2002.xxxxx w/P.Ryan

Heisenberg XXX spin chain

$$H = \sum_{\alpha=1}^{L} \vec{\sigma}_{\alpha} \cdot \vec{\sigma}_{\alpha+1}$$

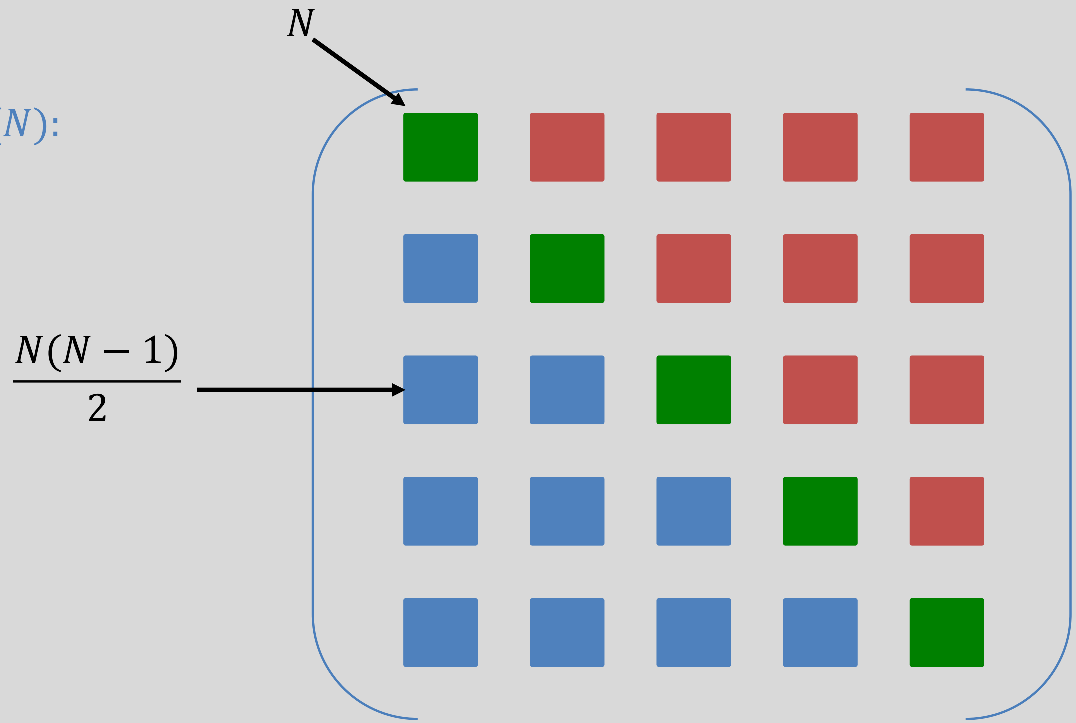


[Bethe '1931]

We are studying generalisations to higher ranks

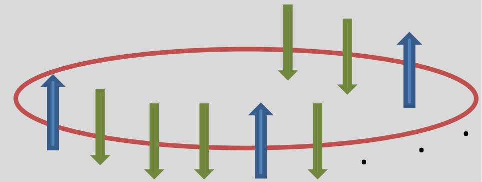
Why higher rank is not simply "more indices"?

$\mathfrak{gl}(N)$:

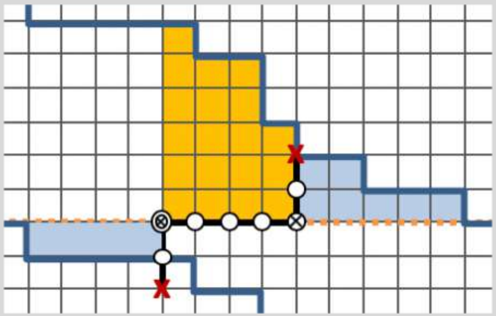


- $N = 1$ – trivial
- $N = 2$ – well understood
- $N = 3$ – can be often brute-forced
- $N \geq 4$ – requires new techniques

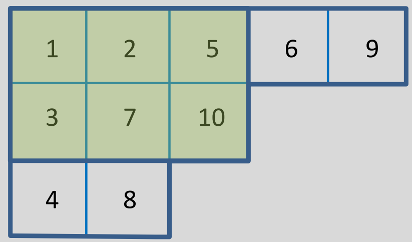
Numbers → Young tableaux



- Young diagrams/tableaux is a combinatorial tool that is used when…



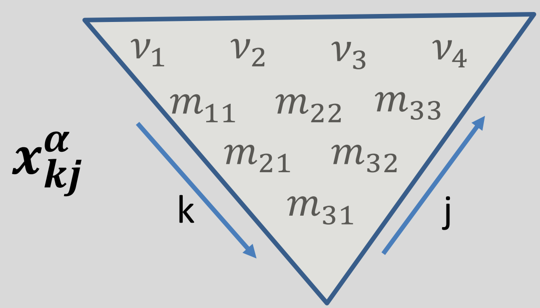
…finding spectrum



…counting solutions

$$\Psi = \prod \det Q_i\,(x_j)$$

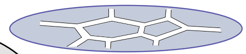
…separating variables



Part I. Motivation

Motivation N1: AdS/CFT

PLANAR N=4 SYM



Gauge theory

String theory

AdS/CFT integrability

0 ∞

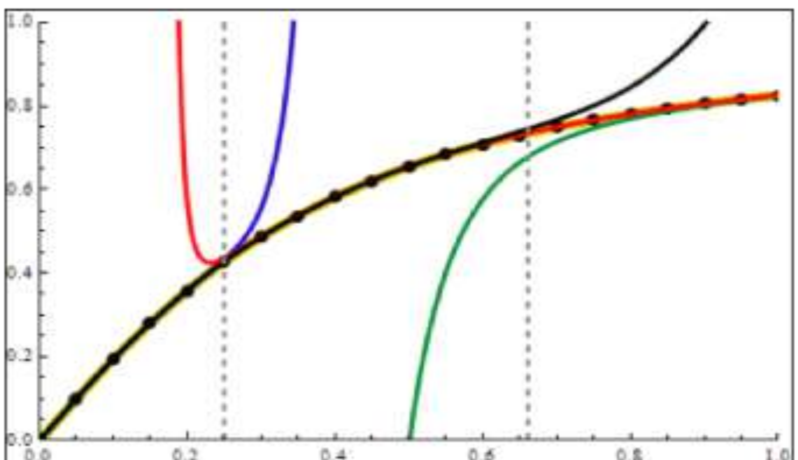
$\mathcal{N}=4$ SYM

$$g^2 = \frac{g_{YM}^2 N_c}{16\pi^2} = \frac{\lambda}{16\pi^2}$$

IIB, $AdS_5 x S^5$

Weak coupling

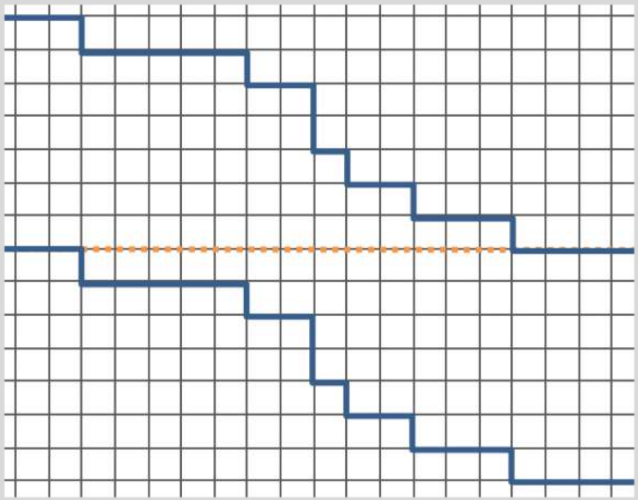
Strong coupling



Motivation N2: Representation theory

$SU(p,q|m)$, extended and non-compact Young diagrams

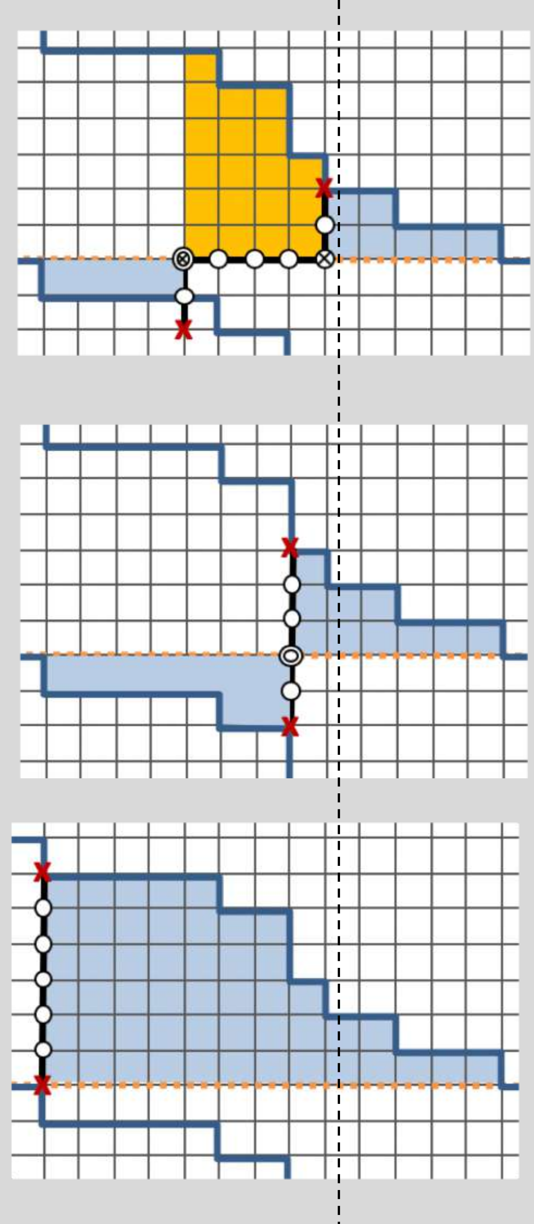
[Günaydin, D.V. '17]
[Marboe, D.V.'17]



$SU(2,2|4)$

$SU(2,3)$

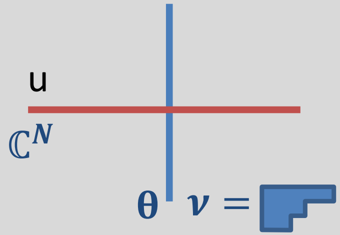
$SU(6)$



Motivation N3: Fundamental questions about integrable systems

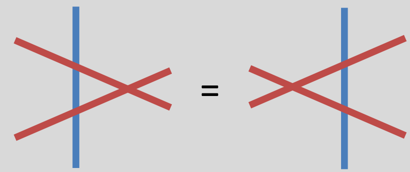
Rational compact GL(N) generalization of Heisenberg spin chain:

Lax operator: L(u) =

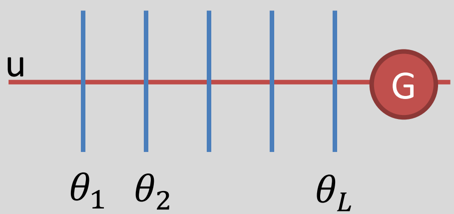


$$= (\boldsymbol{u} - \boldsymbol{\theta})\mathbf{1} - \hbar \sum_{i,j=1}^{N} \boldsymbol{E}_{ij} \otimes \boldsymbol{\pi}_{\nu}(\boldsymbol{E}_{ji})$$

Yangian: L(u)L(v)R(u-v) = R(u-v) L(v) L(u)

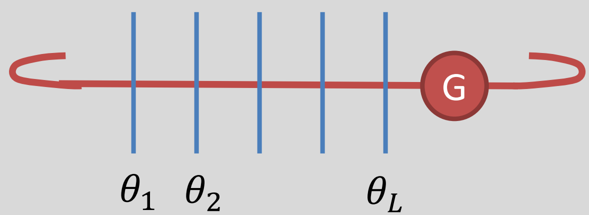


Monodromy matrix: M(u) =



u – *spectral parameter/rapidity*
θ_α - inhomogeneity ($\alpha = 1,2,\ldots,L$)
G – twist
z_i - twist eigenvalues (i $= 1,2,\ldots,N$)

Transfer matrix: T(u)=Tr M(u)=



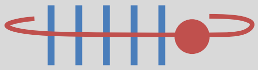
→ *[T(u),T(v)]=0, Bethe Ansatz to diagonalise …*

Motivation N2: Fundamental questions about integrable systems

$$\left(\frac{u_k + \frac{\hbar}{2}}{u_k - \frac{\hbar}{2}}\right)^L = \prod_{j \neq k}^{M} \frac{u_k - u_j + \hbar}{u_k - u_j - \hbar}$$

- Did we found all [independent] commuting charges?
- How many d.o.f. is there?
- Are Bethe equations complete (and what does it mean)?
- Ok, so can we actually solve Bethe equations? How efficient compared to …?
- What are the wave functions (and do we care)?
- How to compute observables beyond spectrum?

Motivation N2: Fundamental questions about integrable systems



$$\left(\frac{u_k + \frac{\hbar}{2}}{u_k - \frac{\hbar}{2}}\right)^L = \prod_{j \neq k}^{M} \frac{u_k - u_j + \hbar}{u_k - u_j - \hbar}$$

Completeness →
- Did we found all [independent] commuting charges?

SoV →
- How many d.o.f. is there?

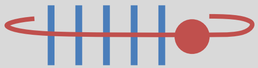
Completeness →
- Are Bethe equations complete (and what does it mean)?

New techniques to solve →
- Ok, so can we actually solve Bethe equations? How efficient compared to …?

SoV →
- What are the wave functions (and do we care)?

SoV ⇢
- How to compute observables beyond spectrum?

Motivation N2: Fundamental questions about integrable systems



$$\left(\frac{u_k+\frac{\hbar}{2}}{u_k-\frac{\hbar}{2}}\right)^L=\prod_{j\neq k}^{M}\frac{u_k-u_j+\hbar}{u_k-u_j-\hbar}$$

$$\boldsymbol{det\, Q_i(u+\hbar(1-j))=\prod_{\alpha=1}^{L}(u-\theta_\alpha)}$$

Completeness → • Did we found all [independent] commuting charges?

SoV → • How many d.o.f. is there?

Completeness → • Are Bethe equations complete (and what does it mean)?

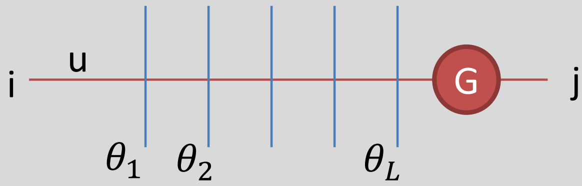
New techniques to solve → • Ok, so can we actually solve Bethe equations? How efficient compared to …?

SoV → • What are the wave functions (and do we care)?

SoV ⋯→ • How to compute observables beyond spectrum?

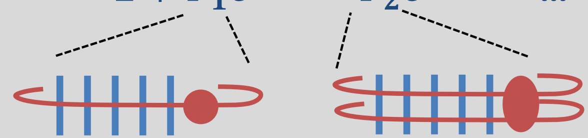
Part II. Bethe Algebra

$M_{i,j}(\mathrm{u}) =$



Bethe algebra:

$$\mathbf{det}\left(\mathbf{1} + \boldsymbol{M}(\boldsymbol{u})\boldsymbol{e}^{-\hbar\partial_u}\right) = \sum_{a=1}^{N} \boldsymbol{T}_a(\boldsymbol{u})\boldsymbol{e}^{-a\,\hbar\partial_u} = \mathbf{1} + \boldsymbol{T}_1\boldsymbol{e}^{-\hbar\partial_u} + \boldsymbol{T}_2\boldsymbol{e}^{-2\hbar\partial_u} + \ldots$$



- Maximal commutative subalgebra of Yangian (if G is generic) [Nazarov, Olshanetski '93]
- Has simple spectrum (at generic point) in spin chain representation [Mukhin, Tarasov, Varchenko '13] [cf. completeness discussion]

Parameterisation of the Bethe algebra

- Baxter Q-operators:

[Bazhanov, Staudacher,'10-11] – any representation

[Kazakov, Leurent, Tsuboi'10] – defining representation

$$Q_1, Q_2, \dots, Q_N$$

$\updownarrow$

- Wronskian relations:

$$T_a = \frac{det_{1\le i,j\le N}\, Q_i(u+\hbar([j>a]+1-j))}{det_{1\le i,j\le N}\, Q_i(u+\hbar(1-j))}$$

$\rightleftarrows$

$$det\, Q_i(u+\hbar(1-j)) = \prod_{\alpha=1}^{L}(u-\theta_\alpha)$$

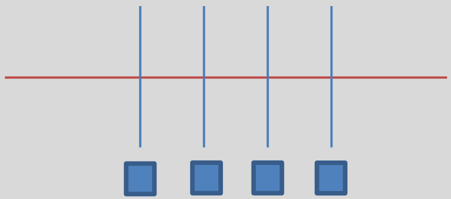
$\updownarrow$

- Baxter equation:

[Krichever, Lipan, Wiegmann, Zabrodin'96]

[Talalayev'04]

$$\det\left(1+M(u)e^{-\hbar\partial_u}\right) Q_i = 0$$

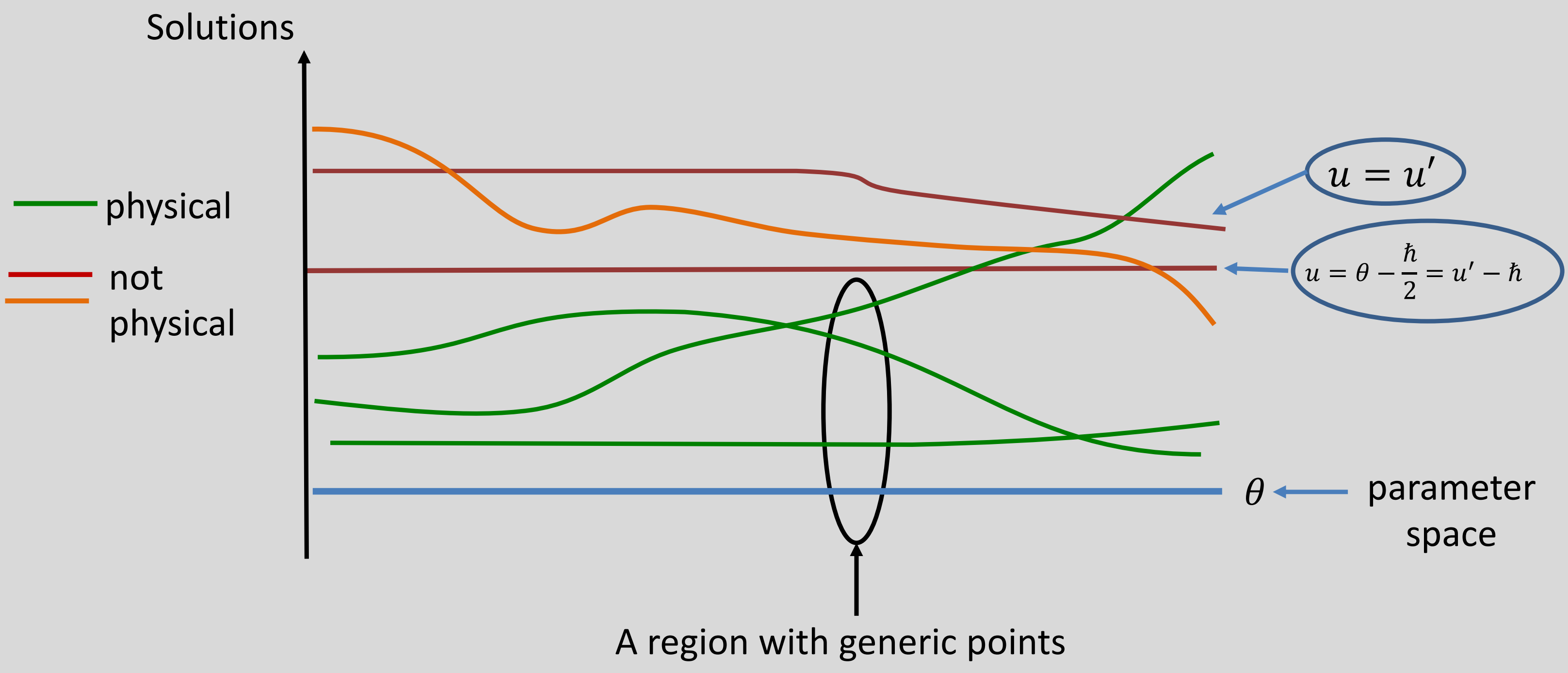


Part III. Completeness

(only defining representation, but supersymmetric case)

Structure of solutions of Bethe equations

$$\left(\frac{u_k+\frac{\hbar}{2}}{u_k-\frac{\hbar}{2}}\right)^L = \prod_{j\neq k}^{M}\frac{u_k-u_j+\hbar}{u_k-u_j-\hbar} \quad\longrightarrow\quad \prod_{\alpha=1}^{L}\frac{u_k-\theta_\alpha+\frac{\hbar}{2}}{u_k-\theta_\alpha-\frac{\hbar}{2}} = \prod_{j\neq k}^{M}\frac{u_k-u_j+\hbar}{u_k-u_j-\hbar} \Big/ S_M$$



[Krichever, Lipan, Wiegmann, Zabrodin'96]

$$\prod_{\alpha=1}^{L} \frac{u_k - \theta_\alpha + \frac{\hbar}{2}}{u_k - \theta_\alpha - \frac{\hbar}{2}} = \prod_{j \neq k}^{M} \frac{u_k - u_j + \hbar}{u_k - u_j - \hbar}$$

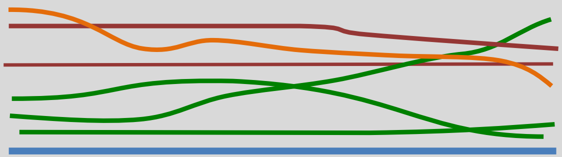
S_M

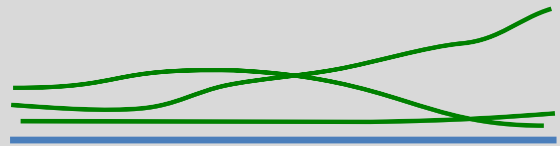
Replace by Wronskian quantisation condition

$$\boldsymbol{det\, Q_i(u + \hbar(1 - j)) = \prod_{\alpha=1}^{L} (u - \theta_\alpha)}$$

$$Q_N = \prod_{i=1}^{M} (u - u_i)$$

[Mukhin, Tarasov, Varchenko'13]



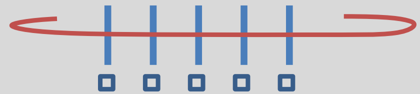


- At **any** point the polynomial equation above is isomorphic to Bethe algebra

$$\boldsymbol{Q_1, Q_2, \ldots, Q_N} \rightleftarrows \boldsymbol{det\, Q_i(u + \hbar(1 - j)) = \prod_{\alpha=1}^{L} (u - \theta)}$$

- Can now construct eigenvectors

$$|v_i> = \prod_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} |\Omega>$$

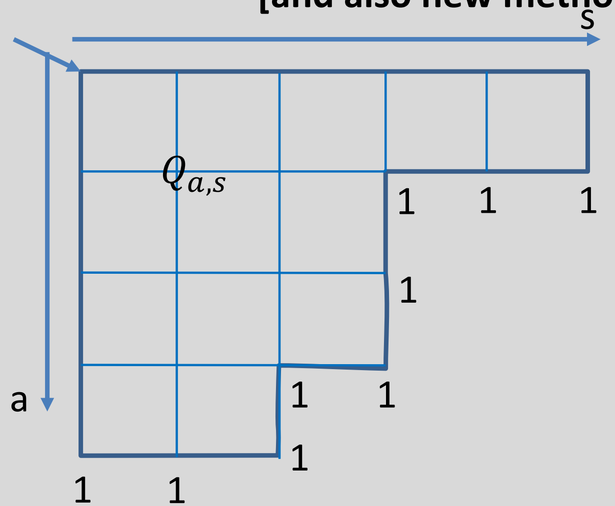


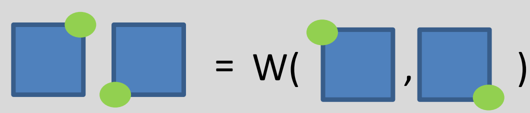
Bosonic (i.e. non-susy) case

$$\det Q_i(u+\hbar(1-j)) = \prod_{\alpha=1}^{L}(u-\theta_\alpha)$$

Supersymmetric case (G=1)
[and also new method for Bosonic]

$$\prod_{\alpha=1}^{L}(u-\theta_\alpha)$$

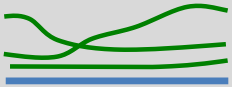




$$Q_{a,s+1}(u)Q_{a+1,s}(u) = \begin{vmatrix} Q_{a,s}(u) & Q_{a,s}(u-\hbar) \\ Q_{a+1,s+1}(u) & Q_{a+1,s+1}(u-\hbar) \end{vmatrix}$$

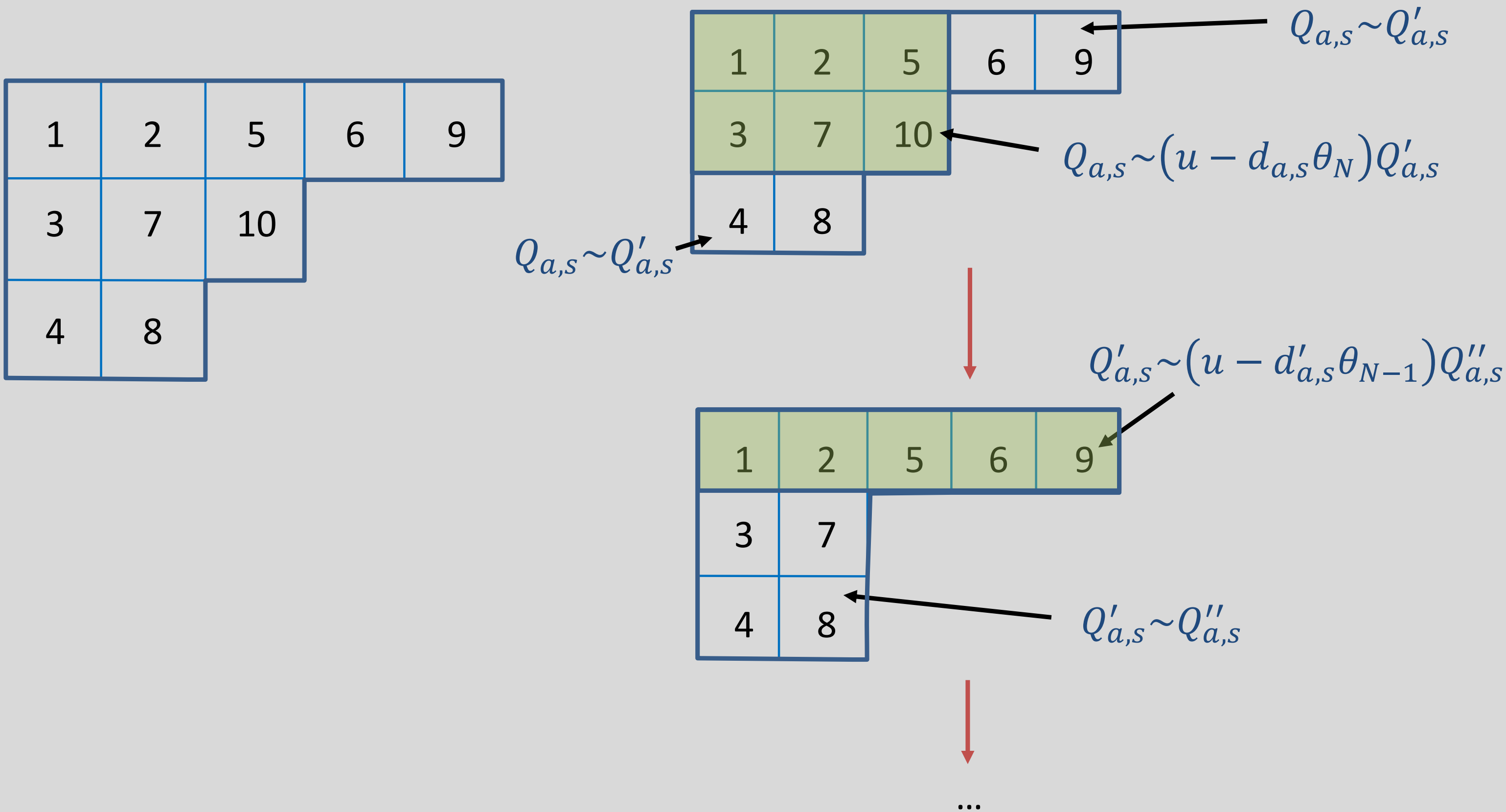
[Chernyak, Leurent, DV'20 –to appear]

- Isomorphism between Bethe algebra and Q − system on Young diagram



New parameterisation of solutions:

- the regime $\theta_N = \Lambda\, \theta_{N-1} = \Lambda^2 \theta_{N-2} = \ldots$ is generic for large enough Λ.
- In this regime solutions are labelled (one-to-one) by Standard Young Tableaux.



Young diagram Q-system:

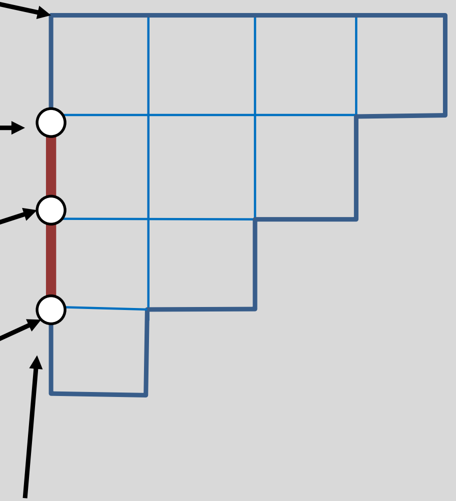
[Marboe, DV'16]

$$\prod_{\alpha=1}^{L}(\boldsymbol{u}-\boldsymbol{\theta}_\alpha)$$

$$\prod_{j=1}^{M_1}(u-u_j^{(1)})$$

$$\prod_{j=1}^{M_2}(u-u_j^{(2)})$$

$$\prod_{j=1}^{M_2}(u-u_j^{(3)})$$



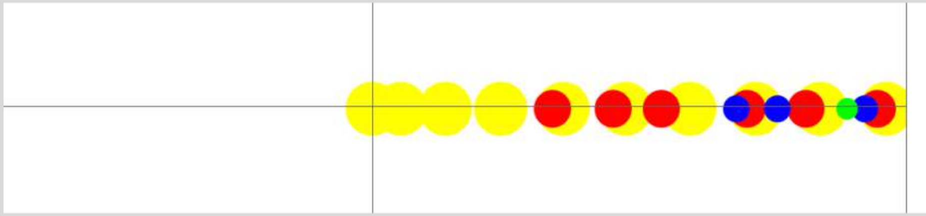
Dynkin diagram = Cartan Matrix

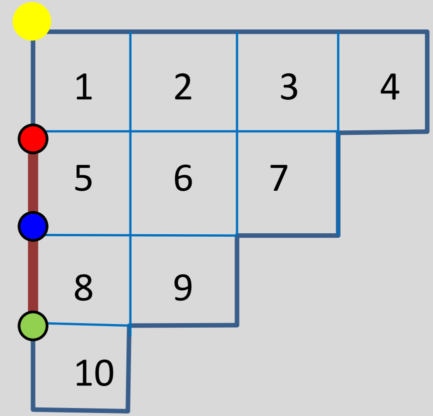
Nested Bethe equations:

$$\prod_{\alpha=1}^{L}\frac{u_k^{(a)}-\theta_\alpha+\hbar\left(\nu_a-\frac{a}{2}\right)}{u_k^{(a)}-\theta_\alpha+\hbar\left(\nu_{a+1}-\frac{a}{2}\right)}=-\prod_{b=1}^{N-1}\prod_{j=1}^{M_b}\frac{u_k^{(a)}-u_j^{(b)}+\frac{\hbar}{2}c_{ab}}{u_k^{(a)}-u_j^{(b)}-\frac{\hbar}{2}c_{ab}}$$

[Chernyak, Leurent, DV'20 –to appear]

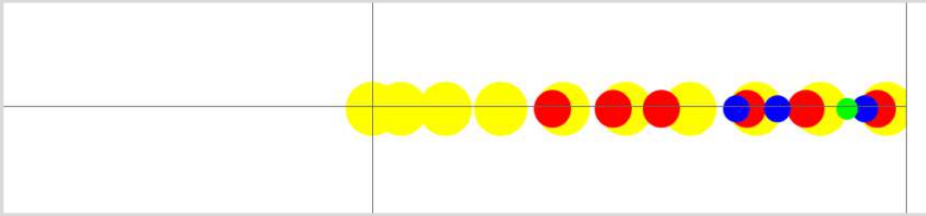
$\theta_N \gg \theta_{N-1} \gg \theta_{N-2} \gg \ldots$ (plot in logarithmic scale)

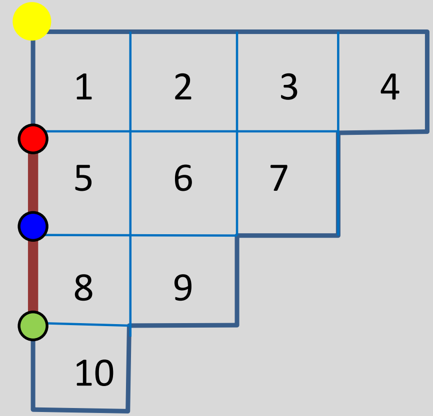




$\theta_N \gg \theta_{N-1} \gg \theta_{N-2} \gg \ldots$ (plot in logarithmic scale)

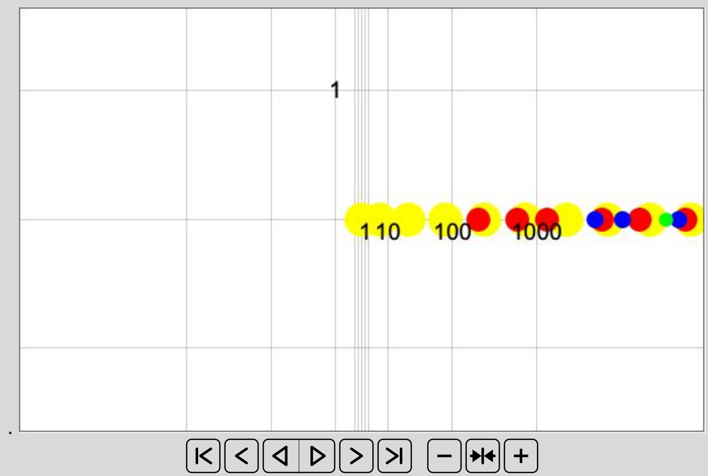
[Chernyak, Leurent, DV'20 –to appear]





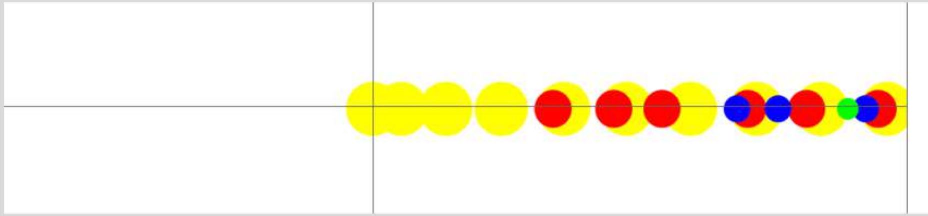
Movie time….

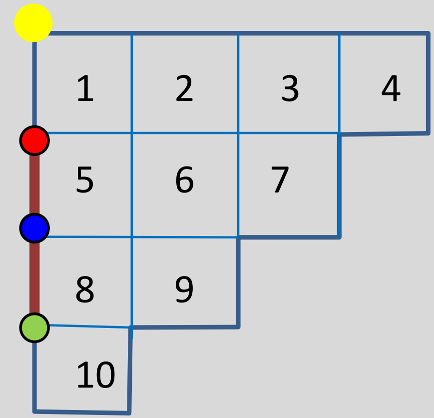
(should work in Acrobat Reader)



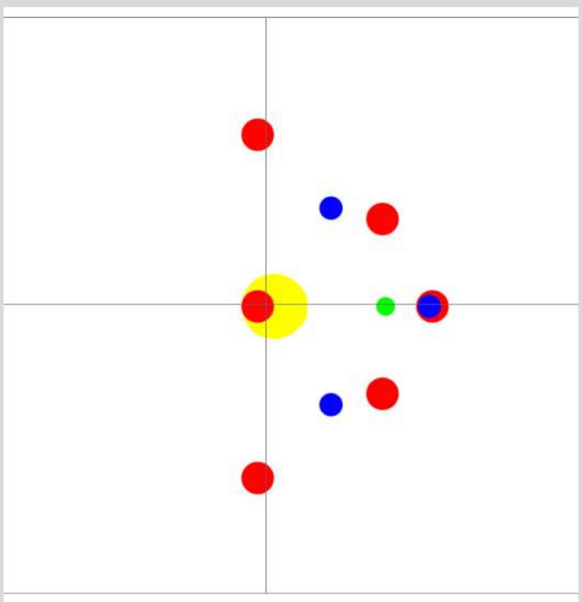
[Chernyak, Leurent, DV'20 –to appear]

$\theta_N \gg \theta_{N-1} \gg \theta_{N-2} \gg \ldots$ (plot in logarithmic scale)





$\theta_N = \theta_{N-1} = \theta_{N-2} = \ldots = 0$



- Continuation between two regimes is unambiguous because spectrum is non-degenerate for real θ
- Conjecture: the defined [precise] mapping between standard Young tableaux and solutions of BAE is precisely the same as Kerov-Kirillov-Reshetikhin bijection (formulated under assumption of [inprecise] string hypothesis]

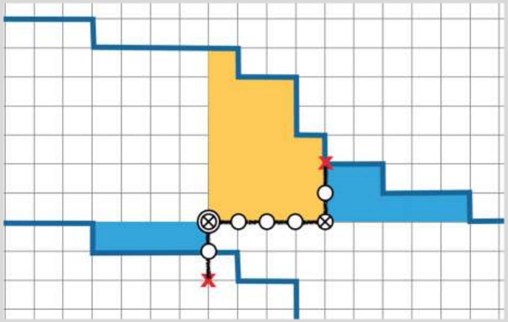
Application to AdS/CFT

$$\langle \mathcal{O}_a(x)\overline{\mathcal{O}}_b(y)\rangle = \frac{\delta_{ab}}{|x-y|^{2\Delta(g)}}$$

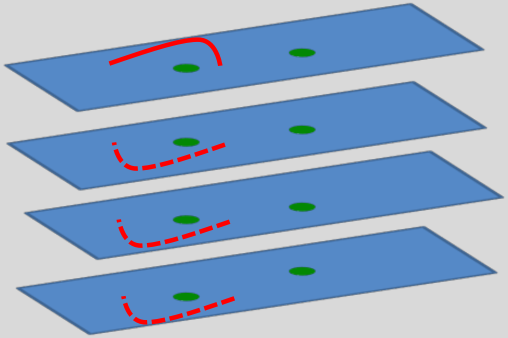
[Gromov, Kazakov, Leurent, D.V. '13-14]

- Solved by Quantum Spectral Curve

$$\mathbf{det}\left(\mathbf{1} + \boldsymbol{M}(\boldsymbol{u})\boldsymbol{e}^{-\hbar\partial_u}\right)\boldsymbol{Q}(\boldsymbol{u}) = \mathbf{0}$$

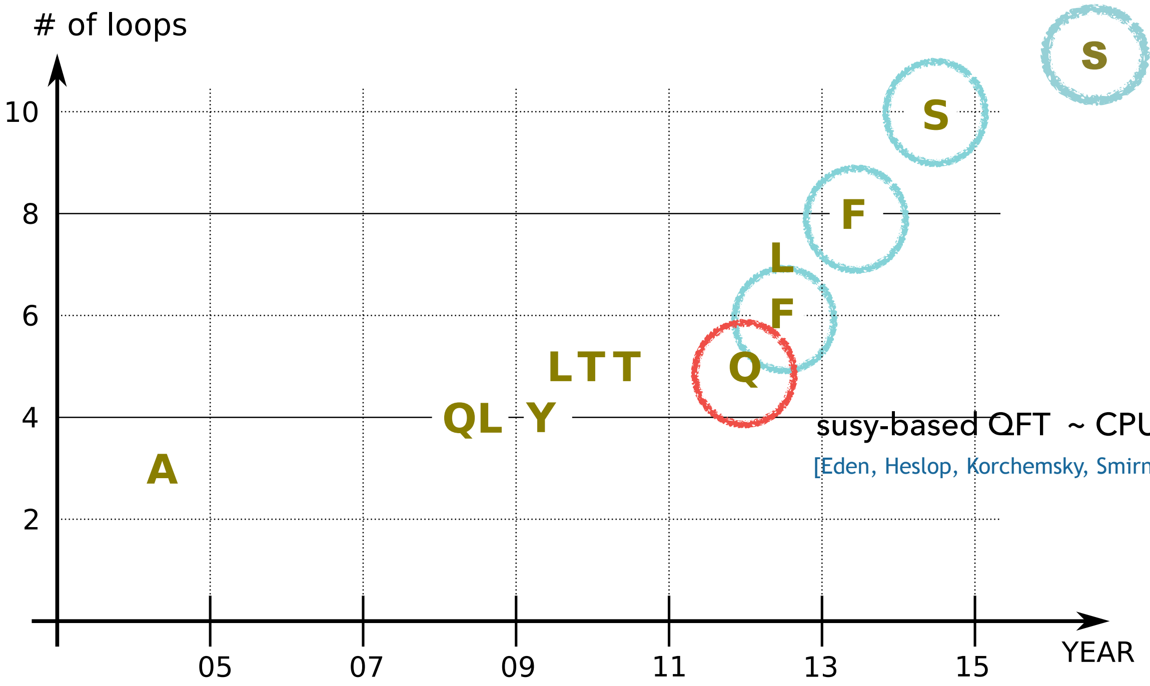


[Gunaydin, D.V. '17]
[Marboe, D.V.'17]



QUANTUM SPECTRAL CURVE

Konishi anomalous dimension:



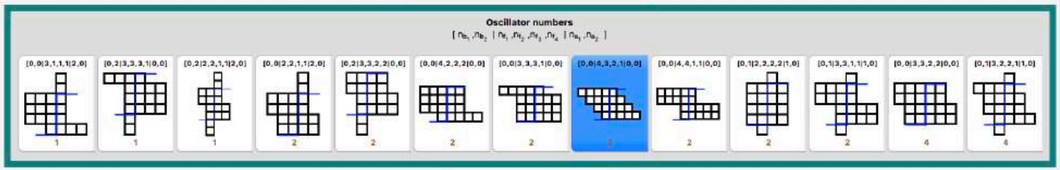
QSC approach

loop	time
5	14 s
6	43 s
7	2.5 m
8	11 m
9	53 m
10	5.5 h
11	34 h

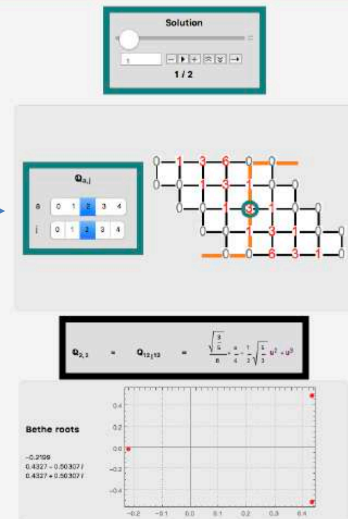
ALL COMBINED… FULL PERTURBATIVE SPECTRUM OF ADS/CFT

[C.Marboe, D.V.'18]

8000+ lowest-lying states computed!

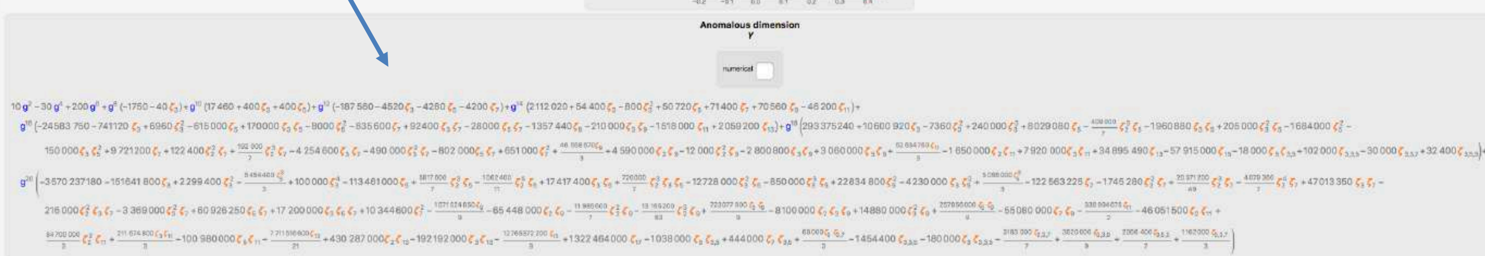


Choose multiplet



Choose soultion of Q-system

9 to 11-loop result
(depending on complexity)



Example: 11-loop Konishi anomalous dimension: [Marboe, D.V.'13-18]

$$\begin{aligned}
\gamma_{11} \;=\;& -242508705792 + 107663966208\zeta_3 + 70251466752\zeta_3^2 - 12468142080\zeta_3^3 \\
&+1463132160\zeta_3^4 - 71663616\zeta_3^5 + 180173002752\zeta_5 - 16655486976\zeta_3\zeta_5 \\
&-24628230144\zeta_3^2\zeta_5 - 2895575040\zeta_3^3\zeta_5 + 19278176256\zeta_5^2 - 9619845120\zeta_3\zeta_5^2 \\
&+2504494080\zeta_3^2\zeta_5^2 + \frac{882108048384}{175}\zeta_5^3 + 45602231040\zeta_7 + 14993482752\zeta_3\zeta_7 \\
&-12034759680\zeta_3^2\zeta_7 + 1406730240\zeta_3^3\zeta_7 + 30605033088\zeta_5\zeta_7 + 21217637376\zeta_3\zeta_5\zeta_7 \\
&-\frac{1309941061632}{275}\zeta_5^2\zeta_7 - 13215327552\zeta_7^2 - 4059901440\zeta_3\zeta_7^2 - 69762034944\zeta_9 \\
&+23284599552\zeta_3\zeta_9 - 3631889664\zeta_3^2\zeta_9 - 11032374528\zeta_5\zeta_9 - 6666706944\zeta_3\zeta_5\zeta_9 \\
&-23148129024\zeta_7\zeta_9 - 10024051968\zeta_9^2 - 54555179184\zeta_{11} + \frac{10048541184}{5}\zeta_3\zeta_{11} \\
&-726029568\zeta_3^2\zeta_{11} - 8975463552\zeta_5\zeta_{11} - 22529041920\zeta_7\zeta_{11} - \frac{1437993422496}{175}\zeta_{13} \\
&+\frac{1504385419392}{35}\zeta_3\zeta_{13} - 30324602880\zeta_5\zeta_{13} - \frac{151130039581392}{875}\zeta_{15} - 41375093760\zeta_3\zeta_{15} \\
&-\frac{196484147423712}{275}\zeta_{17} + 309361358592\zeta_{19} - 1729880064 Z_{11}^{(2)} - \frac{1620393984}{5}\zeta_3 Z_{11}^{(2)} \\
&-131383296\zeta_5 Z_{11}^{(2)} + \frac{138107420928}{175} Z_{13}^{(2)} + \frac{3543865344}{35}\zeta_3 Z_{13}^{(2)} - \frac{5716780416}{7} Z_{13}^{(3)} \\
&-\frac{674832384}{7}\zeta_3 Z_{13}^{(3)} + \frac{48227088384}{175} Z_{15}^{(2)} + \frac{3581880576}{25} Z_{15}^{(3)} + 754974720 Z_{15}^{(4)} \\
&-\frac{854924544}{11} Z_{17}^{(2)} + \frac{4963244544}{55} Z_{17}^{(3)} + \frac{818159616}{275} Z_{17}^{(4)} + \frac{175363688448}{1925} Z_{17}^{(5)}\,.
\end{aligned}$$

$$\begin{aligned}
Z_{11}^{(2)} &= -\zeta_{3,5,3} + \zeta_3\,\zeta_{3,5} \\
Z_{13}^{(2)} &= -\zeta_{5,3,5} + 11\,\zeta_5\,\zeta_{3,5} + 5\,\zeta_5\,\zeta_8 \\
Z_{13}^{(3)} &= -\zeta_{3,7,3} + \zeta_3\,\zeta_{3,7} + 12\,\zeta_5\,\zeta_{3,5} + 6\,\zeta_5\,\zeta_8 \\
&\ldots
\end{aligned}$$

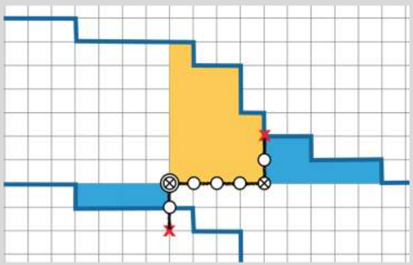
⟵ Single-valued MZV's

[Broadhurst, Kreimer'95]
[Schnetz'14-18]

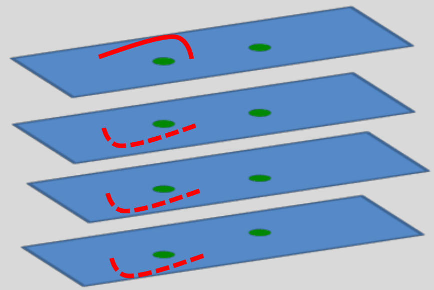
Application to AdS/CFT

$$\langle \mathcal{O}_a(x)\overline{\mathcal{O}}_b(y)\rangle = \frac{\delta_{ab}}{|x-y|^{2\Delta(g)}}$$

- Solved by Quantum Spectral Curve

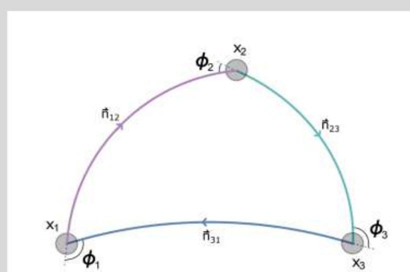


+



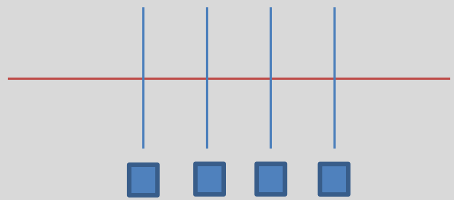
$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = \frac{1}{N_c}\frac{C_{123}(g)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}$$

- Need certain access to wave functions

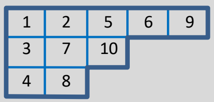


[Cavaglia, Gromov, Levkovich-Maslyuk'18]

$$\boxed{C_{123}^{\bullet\bullet\circ} = \frac{\langle q_1\, q_2\, e^{-\phi_3 u}\rangle}{\sqrt{\langle q_1^2\rangle\langle q_2^2\rangle}}}\;,$$

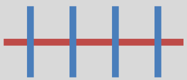


Completeness
(spectrum)



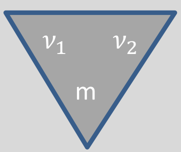
Standard YT

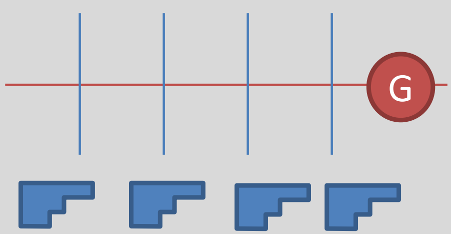
Part IV.
Separation of variables



$GL(N) \times \mathbb{C}[S_L]$

Semi-standard YT
(GT patterns)





SoV
(basis)

Eigensate of spin chain Hamiltonian

$$\langle \mathbf{x} | \Psi \rangle = \prod_{\sigma=1}^{D} \psi(x_\sigma)$$

An SoV basis

D should be thought as number of d.o.f.

- A good SoV basis:

$$\langle \mathbf{x}| = \langle \boldsymbol{\Omega}| \prod \det \widehat{\boldsymbol{Q}}_{\circ}(\boldsymbol{x})$$

[Ryan, D.V. '18]
[Ryan, D.V. '20-to appear]

- Why is this a basis that separates variables?

$$\langle \mathbf{x}| = \langle \boldsymbol{\Omega}| \prod_{\boldsymbol{\sigma}} \hat{T}\,(\boldsymbol{x}_{\boldsymbol{\sigma}})$$

[Maillet, Nicolli '18]

$$\langle \mathbf{x}|\Psi\rangle = \langle \boldsymbol{\Omega}| \prod_{\boldsymbol{\sigma}} \hat{T}\,(\boldsymbol{x}_{\boldsymbol{\sigma}})|\boldsymbol{\Psi}\rangle = \langle \boldsymbol{\Omega}| \prod_{\boldsymbol{\sigma}} \boldsymbol{t}_{\boldsymbol{\tau}}\,(\boldsymbol{x}_{\boldsymbol{\sigma}})|\boldsymbol{\Psi}\rangle = \prod_{\boldsymbol{\sigma}} \boldsymbol{t}_{\boldsymbol{\tau}}\,(\boldsymbol{x}_{\boldsymbol{\sigma}})\langle \boldsymbol{\Omega}|\Psi\rangle$$

- In which sense the proposal we make is good?

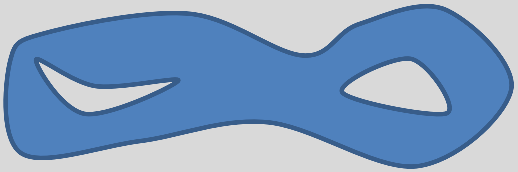
Classical Lax-type integrability:

- M – NxN matrix, coefficients are degree L polynomials in u

$$\{M(u) \overset{\otimes}{,} M(v)\} = [\frac{P}{u-v}, M(u) \otimes M(v)]$$

- Classical spectral curve (encodes spectrum)

$$\det\big(\lambda - M(u)\big) = 0$$



Classical Lax-type integrability:

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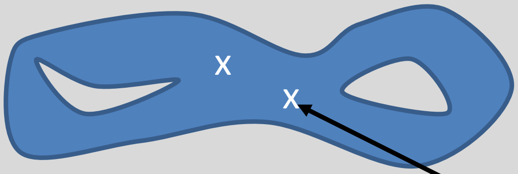


- Need dynamical divisor to describe dynamics - $L\frac{N(N-1)}{2}$ marked points
- Hamiltonian system:

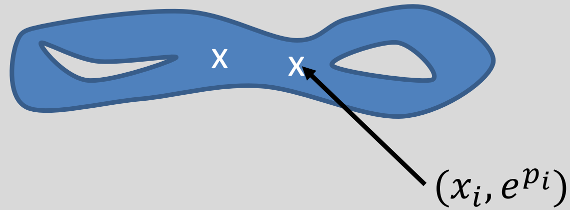
$$\det\big(e^{p_i} - M(x_i)\big) = 0, \qquad \{x_i, p_j\} = \delta_{ij}$$

- Separation of Variables: HJ equation: $\det\left(e^{\frac{\partial S}{\partial x_i}} - M(x_i)\right) = 0$

- Expected naive quantisation: $\det(e^{\hbar\partial_x} - M(x))\psi(x) = 0$

$$\Psi(x) = \prod_{\sigma} \psi(x_\sigma)$$

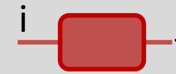
- x_i are zeros of B:



[Scott'94]
[Gekhtman'95]

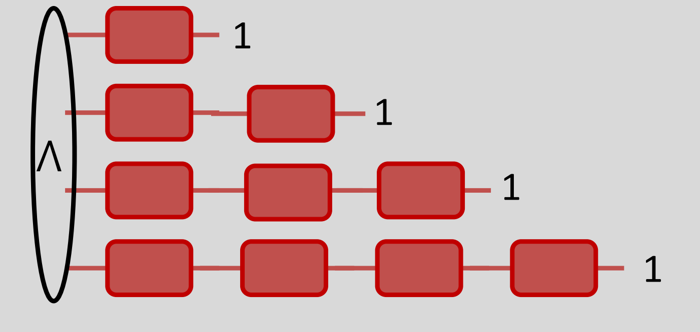
$$B(u) = \frac{det|M(u)\mathbf{v}, M(u)^2\mathbf{v}, \dots, M(u)^N\mathbf{v}|}{\det M} = \prod_i (u - x_i)$$

$$M_{ij} =$$



$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

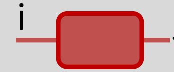
$B(u) =$



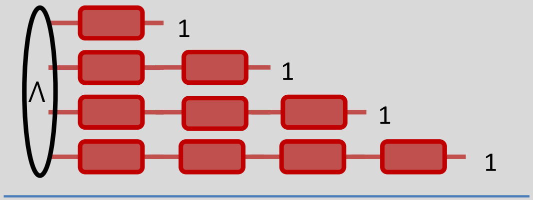
det M

- **Classical B:**

$$M_{ij} =$$



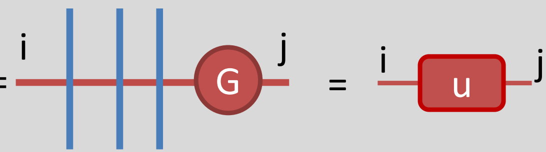
$B(u) =$



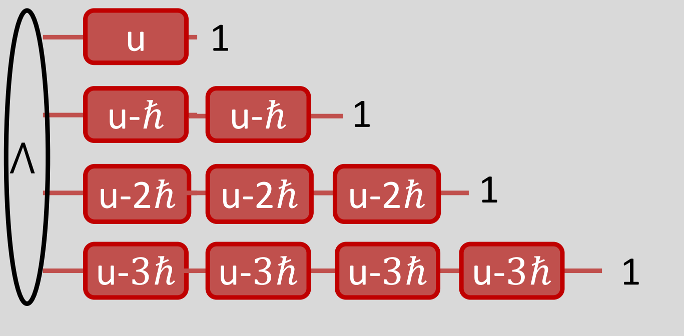
det M

- **Quantisation of B:**

$M_{ij} =$



$B(u) =$



qdet M

$$Q_N = \prod_{r=1}^{M} (u - u_r)$$

- $N = 2$: B of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ **[Sklyanin'89]**
- $N = 3$: **[Sklyanin'92]**
- $any\ N$: **[Smirnov'01]**

[Gromov, Levkovich-Maslyuk, Sizov '16]

Numerical evidence →

For spin chain in fundamental representation

$$|\Psi\rangle = \prod_{r=1}^{M} B(u_r)|\Omega'\rangle$$

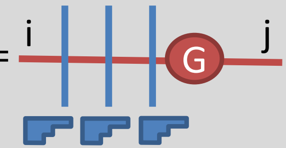
Diagonalisation of quantum B

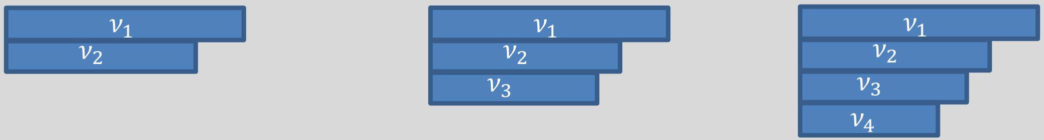
[Ryan, D.V. ’18]
[Ryan, D.V. ’20-to appear]

- Embedding morphism

$$\varphi\colon\ Y(gl_{m-1}) \hookrightarrow Y(gl_m)$$

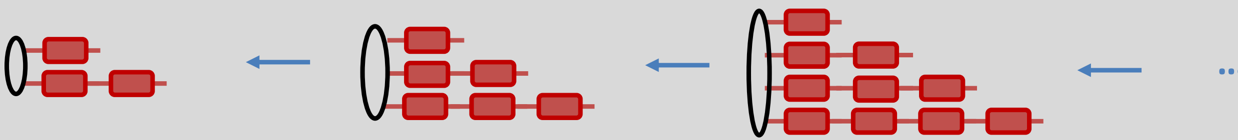
$$\varphi\colon\ M_{i,j} \longmapsto \begin{vmatrix} M_{i,j} & M_{m,j} \\ M_{i,m} & M_{m,m} \end{vmatrix}$$

$M_{ij} =$ 



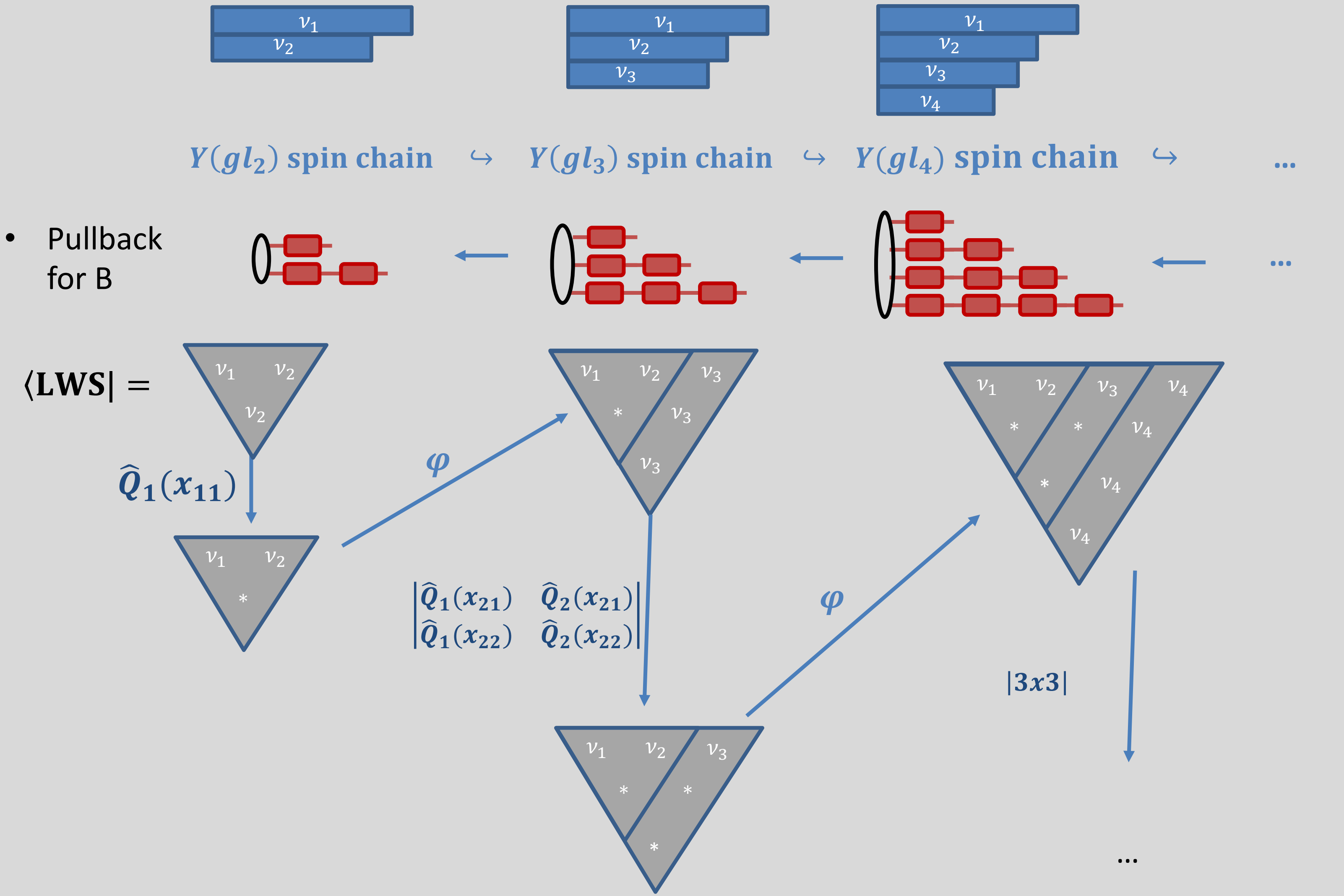
$Y(gl_2)$ spin chain $\hookrightarrow$ $Y(gl_3)$ spin chain $\hookrightarrow$ $Y(gl_4)$ spin chain $\hookrightarrow$ ...

- Pullback for B

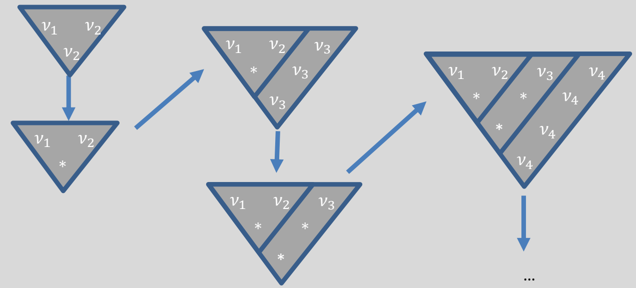


[Ryan, D.V. ’18]
[Ryan, D.V. ’20-to appear]

Diagonalisation of quantum B



- The recursive procedure to construct eigenvectors of B:

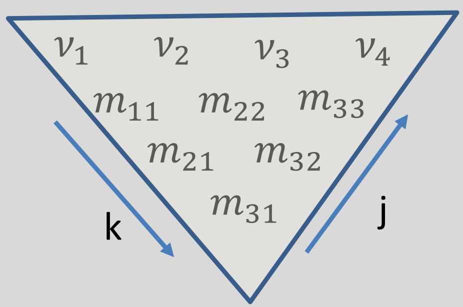


$$\langle \mathbf{x} | = \langle \boldsymbol{\Omega} | \prod_{\alpha=1}^{L} \prod_{k=1}^{N-1} \boldsymbol{det}_{1\le i,j\le k} \hat{Q}_i(\boldsymbol{x}_{kj}^{\alpha})$$

$$\langle \mathbf{x} | = \langle \boldsymbol{\Omega} | \prod_{\alpha=1}^{L} \hat{Q}_1(x_{11}^{\alpha}) \begin{vmatrix} \hat{Q}_1(x_{21}^{\alpha}) & \hat{Q}_2(x_{21}^{\alpha}) \\ \hat{Q}_1(x_{22}^{\alpha}) & \hat{Q}_2(x_{22}^{\alpha}) \end{vmatrix} \cdots$$

- Eigenvalues of $\boldsymbol{X}_{kj}^{\alpha}$

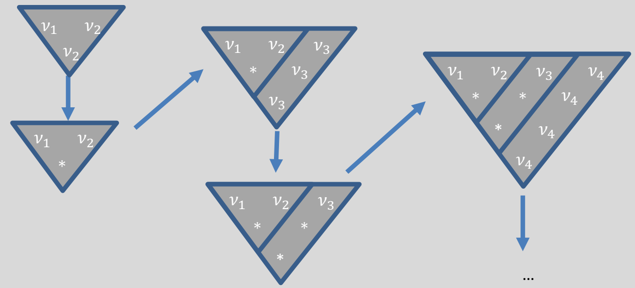
$$\boldsymbol{x}_{kj}^{\alpha} = \boldsymbol{\theta}_{\alpha} + \hbar(\boldsymbol{m}_{kj}^{\alpha} + \mathbf{1} - \boldsymbol{j})$$



$\boldsymbol{m}_{kj}$ satisfy branching rules of GT patterns

- $\langle \mathbf{x} |$ form a Basis

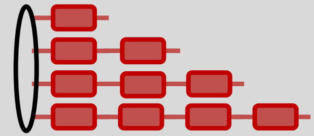
[Ryan, D.V. '18]
[Ryan, D.V. '20-to appear]



- A good SoV basis is constructed using the following formula:

$$\langle \mathbf{x}| = \langle \mathbf{\Omega}| \prod \det \widehat{Q}_{\circ}(\boldsymbol{x})$$

- It diagonalises quantum $\hat{B}(u)$

B = 

qdet M

$$= \prod(\boldsymbol{u} - \boldsymbol{X})$$

- It separates variables, and wave functions are naturally Baxter Q-functions – solutions of Baxter equation:

$$\mathbf{det}\left(\mathbf{1} + \boldsymbol{M}(\boldsymbol{u})\boldsymbol{e}^{-\hbar \partial_u}\right)\boldsymbol{Q}_{\boldsymbol{i}} = \mathbf{0}$$

- Recipe for construction of Bethe eigenstates:

$$|\mathbf{\Psi}\rangle = \prod \det Q_{\circ}(\boldsymbol{X})|\mathbf{\Omega}'\rangle$$

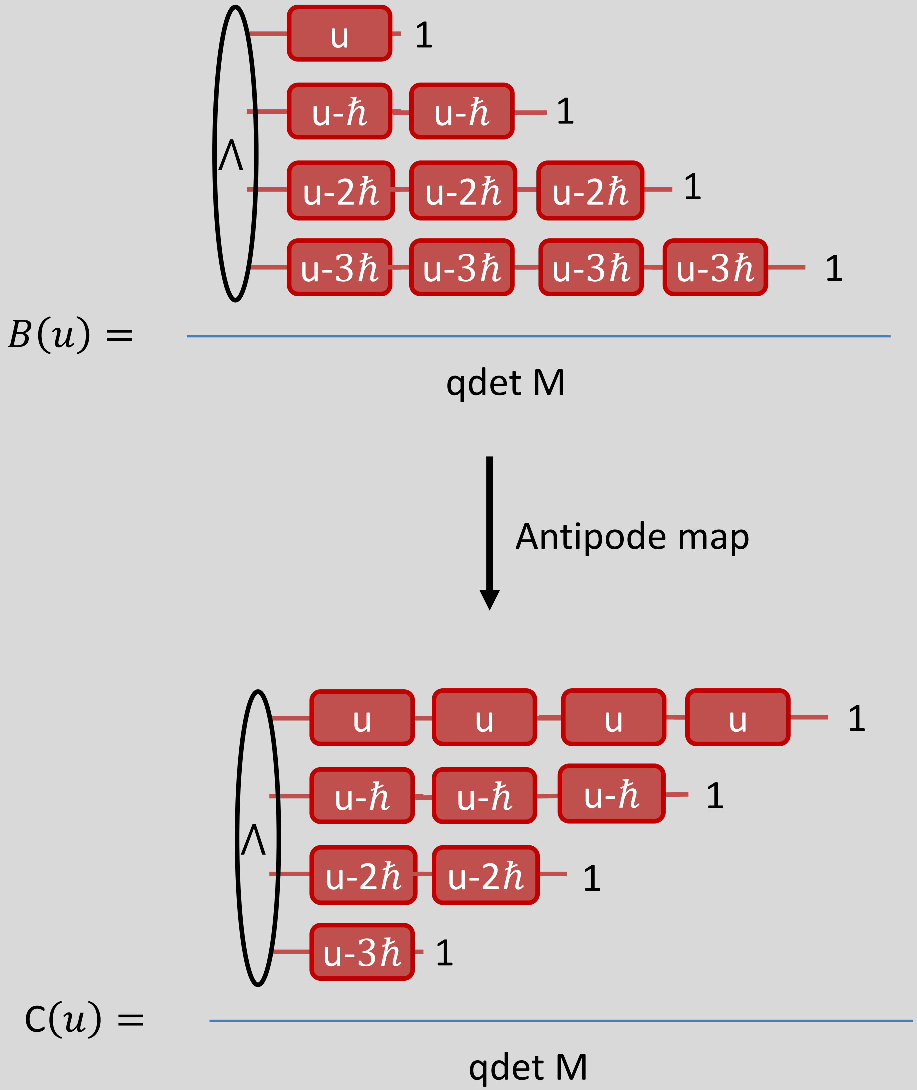
Fundamental representation

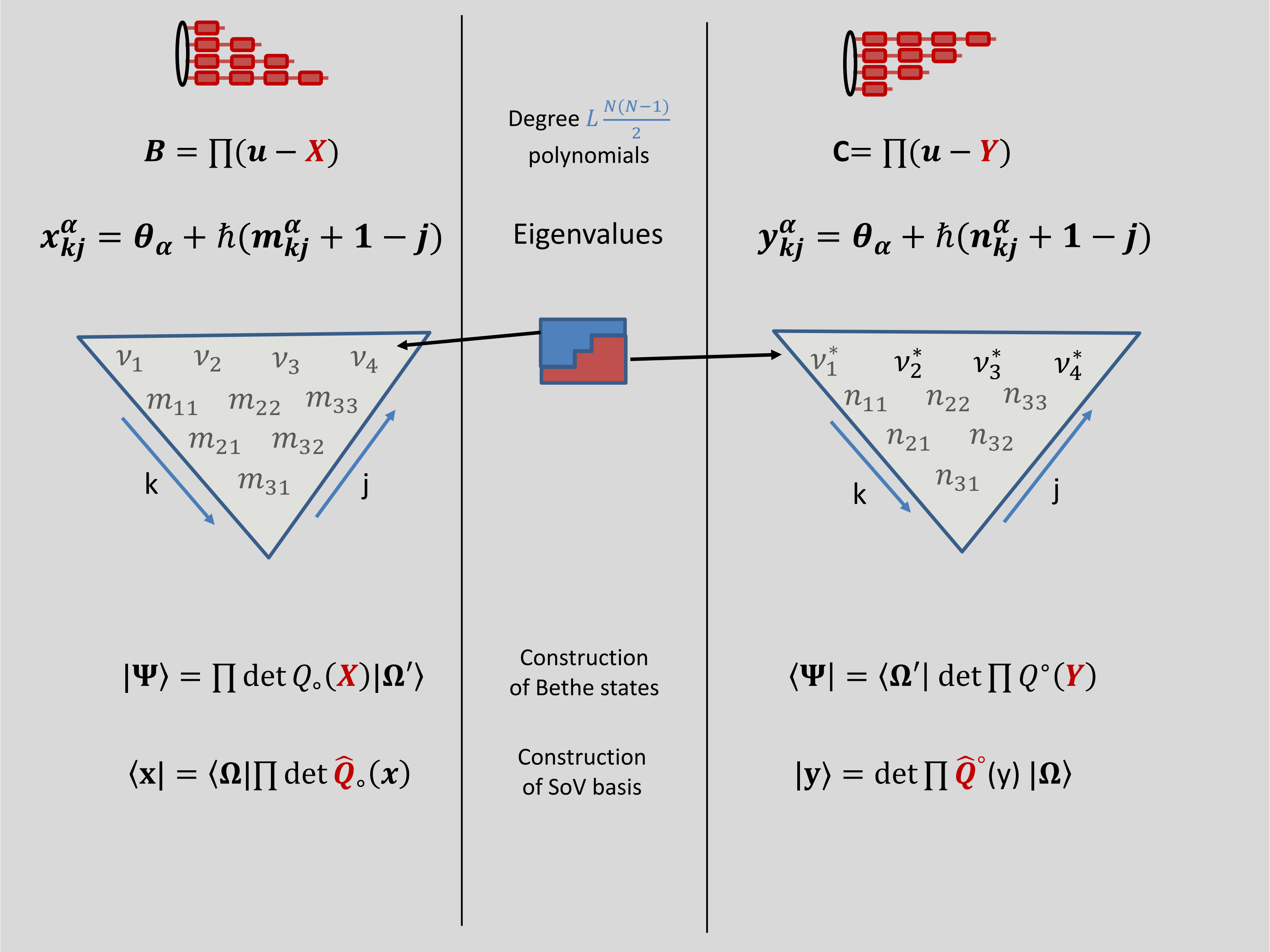
$$|\mathbf{\Psi}\rangle = \prod_{r=1}^{M} \boldsymbol{B}(\boldsymbol{u}_r)|\mathbf{\Omega}'\rangle$$

Part VI. Scalar product

Two ways to quantise B:

[Gromov, Levkovich-Maslyuk, Ryan, DV '19]




Degree $L^{\frac{N(N-1)}{2}}$
polynomials
$B = \prod(u - X)$
$C = \prod(u - Y)$
$x_{kj}^{\alpha} = \theta_{\alpha} + \hbar(m_{kj}^{\alpha} + 1 - j)$
Eigenvalues
$y_{kj}^{\alpha} = \theta_{\alpha} + \hbar(n_{kj}^{\alpha} + 1 - j)$
ν_1 ν_2 ν_3 ν_4
m_{11} m_{22} m_{33}
m_{21} m_{32}
m_{31}
k
j
ν_1^* ν_2^* ν_3^* ν_4^*
n_{11} n_{22} n_{33}
n_{21} n_{32}
n_{31}
k
j
$|\Psi\rangle = \prod \det Q_{\circ}(X)|\Omega'\rangle$
Construction
of Bethe states
$\langle\Psi| = \langle\Omega'| \det \prod Q^{\circ}(Y)$
$\langle x| = \langle\Omega|\prod \det \widehat{Q}_{\circ}(x)$
Construction
of SoV basis
$|y\rangle = \det \prod \widehat{Q}^{\circ}(y)\, |\Omega\rangle$

$$|\boldsymbol{\Psi}\rangle = \prod \det Q_\circ(\boldsymbol{X})|\boldsymbol{\Omega}'\rangle \qquad \langle\boldsymbol{\Psi}| = \langle\boldsymbol{\Omega}'|\prod \det Q^\circ(\boldsymbol{Y})$$

$$\det\left(1 + M(u)\overrightarrow{e^{-\hbar\partial_u}}\right) Q_i = 0 \qquad Q^i \det\left(1 + M(u)\overleftarrow{e^{-\hbar\partial_u}}\right) = 0$$

- This observation allows to fix. $< \Psi_A|\Psi_B >$ up to normalisation:

(using trick of [Cavaglia, Gromov, Levkovich-Maslyuk '19], classical limit similar to measure of [Smirnov,Zeitilin'02])

$$M_{(a,\alpha),(b,\beta)} = \oint \frac{q_B^{1+b}(u)u^{\alpha-1}q_1^A(u - \frac{3i}{2} + ia)}{Q_\theta(u + \frac{i}{2})Q_\theta(u - \frac{i}{2})} e^{2\pi u\beta} du \, .$$

$$\det M(A, B) = \langle\Psi_A|\Psi_B\rangle$$

← Result for GL(3)

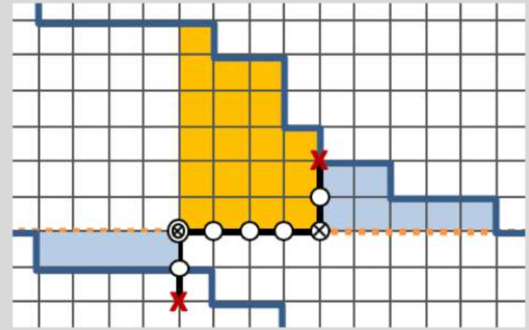
[Gromov, Levkovich-Maslyuk, Ryan, DV '19]

- Can use to e.g. compute diagonal form factors of M_{ij}

Part VI. Conclusions

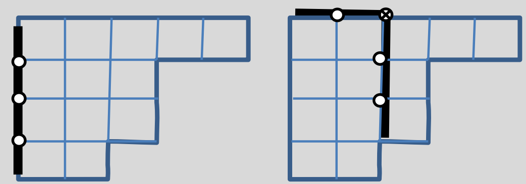
- We need higher ranks, e.g. for AdS/CFT integrability

To develop: calculus of non-compact tableaux.

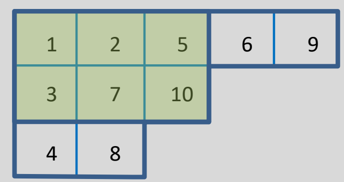


- Nested BAE -> QQ relations on Young diagram

$$\left(\frac{u_k^{(a)} + \hbar\left(\nu_a - \frac{a}{2}\right)}{u_k^{(a)} + \hbar\left(\nu_{a+1} - \frac{a}{2}\right)}\right)^L = -\prod_{b=1}^{N-1}\prod_{j=1}^{M_b}\frac{u_k^{(a)} - u_j^{(b)} + \frac{\hbar}{2}c_{ab}}{u_k^{(a)} - u_j^{(b)} - \frac{\hbar}{2}c_{ab}}$$

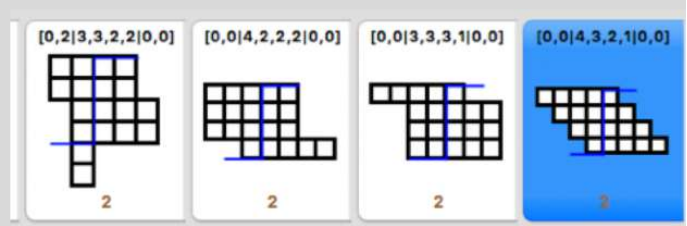


- Completeness and systematic labelling using SYT

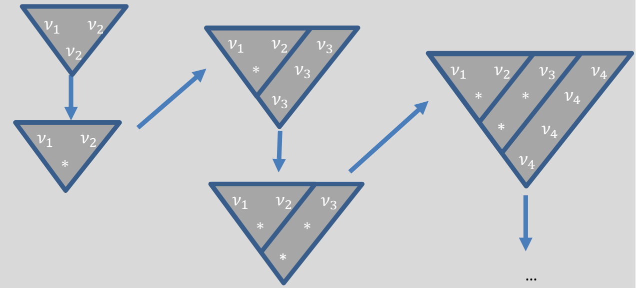


Conjecture: This is the same labelling as Kerov-Kirillov-Reshetikhin bijection (for solutions satisfying string hypothesis). So we can use it as exact defenition of Bethe strings

- Efficient tool to compute N=4 SYM spectrum (8000+ states explicitly, up to 9-11 loops)

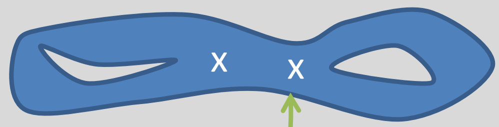


- Construction of SoV basis based on quantum dynamical divisor



$$\langle \mathbf{x}| = \langle \mathbf{\Omega}| \prod \det \hat{Q}_{\circ}(x)$$

- Generation of Bethe eigenvectors:



$$|\Psi\rangle = \prod \det Q_{\circ}(X) |\Omega\rangle$$

In case of symmetric powers of fundamental representation this reduces to

$$|\Psi\rangle = \prod_{r=1}^{M} B(u_r)|\Omega'\rangle$$

- Using Yangian antipode to construct a class of scalar products

$$\det\left(1 + M(u) e^{-\hbar \partial_u}\right) Q_i = 0 \qquad Q^i \det\left(1 + M(u) e^{-\hbar \partial_u}\right) = 0$$

$$M_{(a,\alpha),(b,\beta)} = \oint \frac{q_B^{1+b}(u) u^{\alpha-1} q_1^A(u - \frac{3i}{2} + ia)}{Q_\theta(u + \frac{i}{2}) Q_\theta(u - \frac{i}{2})} e^{2\pi u \beta} du \,.$$

$$\det M(A,B) = \langle \Psi_A | \Psi_B \rangle$$

What next:

- *Supersymmetry for SoV*
- *Noncompact representations*
- *Other groups*
- *Norms*
- *Correlation functions*

Main Target goal:

Solve AdS/CFT integrable systems, derive AdS/CFT corresepondence

Thank you