La physique aux temps du corona

Rencontres Théoriciennes, LPTHE - Paris
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Holomorphic Modular Bootstrap for 2d CFT

Sunil Mukhi

IISER PUNE tifr

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Based on:

"Rational CFT With Three Characters: The Quasi-Character Approach",
Sunil Mukhi, Rahul Poddar and Palash Singh
arXiv:2002.01949.

"Contour Integrals and the Modular S-Matrix",
Sunil Mukhi, Rahul Poddar and Palash Singh
arXiv:1912.04298.

"Curiosities above $c = 24$",
A. Ramesh Chandra and Sunil Mukhi,
arXiv:1812.05109.

"Towards a Classification of Two-Character Rational Conformal Field Theories",
A. Ramesh Chandra and Sunil Mukhi,
arXiv:1810.09472.

And previous work:

"On 2d Conformal Field Theories with Two Characters",
Harsha Hampapura and Sunil Mukhi,
JHEP 1601 (2106) 005, arXiv: 1510.04478.

"Cosets of Meromorphic CFTs and Modular Differential Equations",
Matthias Gaberdiel, Harsha Hampapura and Sunil Mukhi,
JHEP 1604 (2016) 156, arXiv: 1602.01022.

"Two-dimensional RCFT's without Kac-Moody symmetry",
Harsha Hampapura and Sunil Mukhi,
JHEP 1607 (2016) 138, arXiv: 1605.03314.

"Universal RCFT Correlators from the Holomorphic Bootstrap",
Sunil Mukhi and Girish Muralidhara,
JHEP 1802 (2018) 028, arXiv: 1708.06772.

Related work:

"Hecke Relations in Rational Conformal Field Theory",
Jeffrey A. Harvey and Yuxiao Wu,
arXiv: 1804.06860.

And older work:

"Differential equations for correlators and characters in arbitrary rational conformal field theories",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Nucl. Phys. B312 (1989) 15.

"On the classification of rational conformal field theories",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Phys. Lett. B213 (1988) 303.

"Reconstruction of conformal field theories from modular geometry on the torus",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Nucl. Phys. B318 (1989) 483.

"Contour Integral Representations for the Characters of Rational Conformal Field Theories",
Sunil Mukhi, Sudhakar Panda and Ashoke Sen,
Nucl. Phys. B326 (1989) 351.

"Differential equations for rational conformal characters",
S. Naculich,
Nucl. Phys. B 323 (1989) 423.

Outline

- The partition function of a 2D CFT is:

$$Z(\tau,\bar{\tau}) = \mathrm{tr}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}, \quad q = e^{2\pi i \tau}$$

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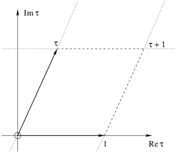
- Operator/path-integral equivalence implies $Z(\tau,\bar\tau)$ = path integral on torus with modular parameter τ:



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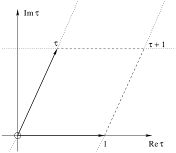

- The torus has global diffeomorphisms given by PSL(2,Z):

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL(2,Z)}$$

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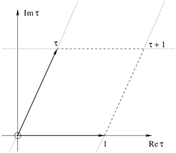

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- Hence the partition function must be modular invariant:

$$Z(\gamma\tau, \gamma\bar{\tau}) = Z(\tau, \bar{\tau})$$

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- We can have superconformal algebras too, but the rules change due to double-valuedness of fermion fields. I will not discuss them here, though they are very interesting.
- When there is a larger chiral algebra, the definition of a primary state is suitably modified. For example in a Kac-Moody-Virasoro-$W^{(3)}$ theory, a primary satisfies:

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- If there is a larger chiral algebra then generically there will be a smaller number of primary fields under it.

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- Their partition function is of the form:

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where $\chi_i(\tau)$ is the character associated to a given primary:

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- Often, multiple primaries have the same character. For example in the SU(3) Kac-Moody algebra, primaries in the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations have the same character.

- Characters are holomorphic in the interior of moduli space but can diverge on the boundary $\tau \to i\infty$ $(q \to 0)$:

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- To have a modular-invariant partition function, the characters must be vector-valued modular functions:

$$\chi_i(\gamma\tau) = \sum_{j=0}^{p-1} M_{ij}(\gamma)\chi_j(\tau), \quad \gamma \in \mathrm{SL(2,Z)}$$

with $M^\dagger M = 1$.

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- They are also interesting to mathematicians:
  - RCFT characters are vector-valued modular functions with non-negative integral coefficients.
  - These theories often have exotic discrete symmetries (Monster, Baby Monster, Mathieu...).

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- Decoupling null vectors, [Belavin-Polyakov-Zamolodchikov 1984] found RCFT's for specific values of the central charge:

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  where p, q are co-prime positive integers.
- The resulting theories, called Virasoro minimal models, have $\frac{1}{2}(p-1)(q-1)$ primary fields and are solvable: critical exponents, correlators, partition function.

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- Similarly the $W^{(p)}$ algebras etc. have their own minimal series.
- Finally using the coset construction on Kac-Moody algebras, one can obtain vast families of RCFT.

- Though useful, this approach has its limitations. For example, the number of characters in each minimal series grows rapidly and only the first few are of practical interest.

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- Thus, we ask how to classify RCFT with one character, two characters, etc.
- This requires a completely different starting point.

- This is the case of one character. The only primary is the identity, so no critical exponents! The partition function has the form:

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- It is a well-known mathematical fact that this is only possible if:

$$\chi(\tau) = j^{\frac{\alpha}{3}} P(j)$$

  where $\alpha \in (0, 1, 2)$, $P(j)$ is a polynomial, and $j(q)$ is the Klein j-invariant:

$$j(q) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

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$c = 8:$	$\chi = j^{\frac{1}{3}}$	E_8 (unique)
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$c = 24:$	$\chi = j + \mathcal{N}$	free boson, Niemeier lattice
$c = 32:$	$\chi = j^{\frac{1}{3}}(j + \mathcal{N})$	free boson, even unimodular 32d lattice

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- The above examples correspond to c free bosons compactified on a torus $\mathbb{R}^c/\Gamma$, where Γ is an even, unimodular lattice – but there are more general possibilities when $c \geq 24$.

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- Most values of $\mathcal{N}$ do not give a CFT. One can prove this by extracting the dimension of the current algebra (which is $\mathcal{N} + 744$) and noting that for too large $\mathcal{N}$, the central charge cannot be as low as 24.
- For $c > 24$, there are huge numbers of meromorphic RCFT but no complete classification.

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- It has no Kac-Moody algebra and does not belong to any minimal series.
- However its states fall into representations of the largest discrete group, the Monster group, of order:

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- A hypothetical class of generalisations of the Monster CFT was proposed in [Witten 2007] to be holographically dual to pure gravity in AdS_3. However the proposed theories may not exist, and this proposal is also no longer believed to hold.

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- For the one-character case, Problem (I) was effectively solved by Klein in the 19th century. Problem (II) is solved only for $c \leq 24$.
- At $c = 32$ there are already around 10^{10} even unimodular lattices and there should be a corresponding number of orbifolds etc.

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  1. The partition function is modular invariant, but not holomorphic.
  2. The characters are holomorphic, but not modular invariant.
  3. However, the characters solve a Modular Linear Differential Equation on moduli space that is both holomorphic and modular invariant.

- It is easily shown that any p-component vector-valued modular function $\chi(q)$ satisfies an MLDE of the form:

$$\Big(\mathcal{D}_\tau^p + \phi_1(q)\mathcal{D}_\tau^{p-1} + \cdots + \phi_p(q)\Big)\chi(q) = 0$$

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- Here r is the weight of the modular form on which it acts, and $E_2(\tau)$ is the second Eisenstein series, which transforms as a connection.

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$$\Big(\mathcal{D}_\tau^p + \phi_1(q)\mathcal{D}_\tau^{p-1} + \cdots + \phi_p(q)\Big)\chi(q) = 0$$

  where $\mathcal{D}_\tau = \frac{\partial}{\partial\tau} - \frac{i\pi r}{6}E_2(\tau)\cdots$ is a covariant derivative.
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- In general they can be meromorphic, although the characters themselves are holomorphic.

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- For given ℓ there is a finite basis of functions of the Eisenstein series E_4, E_6 from which the ϕ_j are built. Hence the differential equation has finitely many parameters.
- Thus the general case is labelled (p, ℓ) where p is the number of characters and ℓ is the Wronskian index.

- The Riemann-Roch theorem gives an important relation between the critical exponents, the number p of characters and the Wronskian index ℓ:

$$\sum_{i=0}^{p-1}\left(-\frac{c}{24}+h_i\right)=\frac{p(p-1)}{12}-\frac{\ell}{6}$$

- For two and three characters, the MLDE takes the form:

$$\Big(\mathcal{D}_\tau^2 + \phi_2(\tau)\mathcal{D}_\tau + \phi_4(\tau)\Big)\chi = 0$$
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- The next simplest case is $\ell = 2$ where, for example, $\phi_2 \sim \frac{E_6}{E_4}$.

- Solutions of the differential equation are, by construction, vector-valued modular functions, and have an expansion of the form:

$$\chi_i(\tau) = q^{-\frac{c}{24}+h_i}(a_0^i + a_1^i q + a_2^i q^2 + \cdots)$$

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- The coefficients a_n^i are rational functions of the parameters in the equation (e.g. μ, μ') but generically they are not integers.
- The methodology to find "admissible" characters is then:
  (i) Vary the parameters in the equation until the first few coefficients a_n^i are non-negative integers.
  (ii) Verify that the a_n^i continue to be non-negative integers to very high orders in q. Then we have an "admissible character".

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m_1	c	h	Identification
1	$\frac{2}{5}$	$\frac{1}{5}$	$c=-\frac{22}{5}$ minimal model $(c \leftrightarrow c-24h)$
3	1	$\frac{1}{4}$	$k=1$ SU(2) WZW model
8	2	$\frac{1}{3}$	$k=1$ SU(3) WZW model
14	$\frac{14}{5}$	$\frac{2}{5}$	$k=1$ G_2 WZW model
28	4	$\frac{1}{2}$	$k=1$ SO(8) WZW model
52	$\frac{26}{5}$	$\frac{3}{5}$	$k=1$ F_4 WZW model
78	6	$\frac{2}{3}$	$k=1$ E_6 WZW model
133	7	$\frac{3}{4}$	$k=1$ E_7 WZW model
190	$\frac{38}{5}$	$\frac{4}{5}$	?
248	8	$\frac{5}{6}$	$\supset k=1$ E_8 WZW model

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- Thus in this case, Problems (I) and (II) are both solved.
- These theories are simple and well-known. However they occurred all together for the first time here. Several years later [Pierre Deligne 1996] observed that the same Lie algebras form a series with special properties.

C. R. Acad. Sci. Paris, t. 322, Série I, p. 321-326, 1996
Algèbres de Lie/*Lie Algebra*

La série exceptionnelle de groupes de Lie

Pierre DELIGNE

School of Mathematics, Institute for Advanced Study,
Princeton, NJ 08540, USA.
e-mail: deligne@math.ias.edu

Résumé. Numérologie des groupes exceptionnels et une interprétation conjecturale.

The exceptional series of Lie groups

Abstract. *Numerology of exceptional Lie groups and a conjectural explanation.*

Soit G^0 le groupe déployé adjoint de l'un des types suivant : A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_8. On fixe un épinglage de G^0. On note G le groupe des automorphismes de G^0. Pour Γ le groupe des

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- On interchanging the two characters, $c = -\frac{22}{5}, h = -\frac{1}{5}$. This is the famous non-unitary Lee-Yang edge singularity CFT.
- The second-last line with $c = \frac{38}{5}$ and 190 currents, also has negative fusion rules. This time on exchanging the two characters we get a 57-fold degenerate identity character. Therefore we rejected this case in 1988.

- However after Deligne's work, a "hole" was found by [Landsberg-Manivel 2004] between E_7 and E_8. Notably the dimension of this "intermediate Lie algebra" is 190.

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THE SEXTONIONS AND $E_{7\frac{1}{2}}$

J. M. LANDSBERG[1], L. MANIVEL

ABSTRACT. We fill in the "hole" in the exceptional series of Lie algebras that was observed by Cvitanovic, Deligne, Cohen and deMan. More precisely, we show that the intermediate Lie algebra between $\mathfrak{e}_7$ and $\mathfrak{e}_8$ satisfies some of the decomposition and dimension formulas of the exceptional simple Lie algebras. A key role is played by the *sextonions*, a six dimensional algebra between the quaternions and octonions. Using the sextonions, we show simliar results hold for the rows of an expanded Freudenthal magic chart. We also obtain new interpretations of the adjoint variety of the exceptional group G_2.

- Thereafter it was proposed that there are generalised CFT's called "Intermediate Vertex Operator Algebras" whose identity is degenerate. The first two examples are the ones at $c = \frac{2}{5}, \frac{22}{5}$.

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The Intermediate Vertex Subalgebras
of the Lattice Vertex Operator Algebras

KAZUYA KAWASETSU
Department of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153-8914, Japan. e-mail: kawasetu@ms.u-tokyo.ac.jp

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Abstract. A notion of *intermediate vertex subalgebras* of lattice vertex operator algebras is introduced, as a generalization of the notion of *principal subspaces*. Bases and the graded dimensions of such subalgebras are given. As an application, it is shown that the characters of some modules of an intermediate vertex subalgebra between E_7 and E_8 lattice vertex operator algebras satisfy some modular differential equations. This result is an analogue of the result concerning the "hole" of the *Deligne dimension formulas* and the *intermediate Lie algebra* between the simple Lie algebras E_7 and E_8.

- $(p = 2, \ell = 2)$. Again, just 9 solutions [Naculich 1989, Hampapura-Mukhi 2015].

m_1	c	h
410	$\frac{82}{5}$	$\frac{6}{5}$
323	17	$\frac{5}{4}$
234	18	$\frac{4}{3}$
188	$\frac{94}{5}$	$\frac{7}{5}$
140	20	$\frac{3}{2}$
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This series has central charges $24 - c$ and conformal dimensions $2 - h$ relative to the $(p = 2, \ell = 0)$ series.

- $(p=2,\ell=2)$. Again, just 9 solutions [Naculich 1989, Hampapura-Mukhi 2015].

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For nearly three decades it remained unclear whether these admissible characters were really CFT's ("Problem (II)").

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- Ising model $\mathcal{M}_{3,4}$, $\mathcal{M}_{2,7}$, $\mathcal{M}_{2,5} \times \mathcal{M}_{2,5}$, $\mathrm{SU}(2)_2$, $\mathrm{SU}(4)_1$, $\mathrm{SU}(5)_1$, $\mathrm{SO(N)}_1$ are the straightforward examples.

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- Their list is just the above series, plus a series of "exotic" theories with:

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- As I will shortly explain, all these theories (plus a few more) were already found by us in 2016 using the Novel Coset Construction.

- The cases $(p = 2, \ell \geq 4)$ have been classified [Chandra-Mukhi 2018a] and I will discuss those later. This gives a complete classification of admissible characters for $p = 2$.

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- For $p \geq 6$ the number of modular forms grows faster than the degree of the equation. The MLDE method has not been so useful in these cases (though no one has really tested it).

Outline

- In [Gaberdiel-Hampapura-Mukhi 2016], we proposed a novel coset construction to address a particular problem: the existence of $(p = 2, \ell = 2)$ admissible characters.

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- This explains why the two sets have the same number of members, and the matching pattern of central charges and conformal dimensions.
- More importantly it provides a definition of the $(p = 2, \ell = 2)$ characters as genuine RCFT.

	$(p=2,\ell=0)$				$(p=2,\ell=2)$				
No.	c	h	m_1	Algebra	$\tilde{c}$	$\tilde{h}$	$\tilde{m}_1$	$m_1+\tilde{m}_1$	Example KM algebra
1	1	$\frac{1}{4}$	3	$A_{1,1}$	23	$\frac{7}{4}$	69	72	$(A_{1,1})^{23}$
2	2	$\frac{1}{3}$	8	$A_{2,1}$	22	$\frac{5}{3}$	88	96	$(A_{5,2})^2C_{2,1}A_{2,1}$
3	$\frac{14}{5}$	$\frac{2}{5}$	14	$G_{2,1}$	$\frac{106}{5}$	$\frac{8}{5}$	106	120	$E_{6,3}(G_{2,1})^2$
4	4	$\frac{1}{2}$	28	$D_{4,1}$	20	$\frac{3}{2}$	140	168	$(D_{4,1})^5$
5	$\frac{26}{5}$	$\frac{3}{5}$	52	$F_{4,1}$	$\frac{94}{5}$	$\frac{7}{5}$	188	240	$E_{7,2}B_{5,1}$
6	6	$\frac{2}{3}$	78	$E_{6,1}$	18	$\frac{4}{3}$	234	312	$A_{11,1}D_{7,1}$
7	7	$\frac{3}{4}$	133	$E_{7,1}$	17	$\frac{5}{4}$	323	456	$D_{10,1}E_{7,1}$

Table: Characters with $\ell=0$ and $\ell=2$.

- The $\ell=2$ theories are simple but not previously known from any other construction. As seen in the table, they have non-simple Kac-Moody algebras but are not WZW models.

- If we take the coset of a meromorphic theory with $c = 8N$ by a (p, ℓ) theory with exponents (c, h_i), the resulting theory has $(p, \ell^{\mathcal{C}})$ and exponents $(c^{\mathcal{C}}, h_i^{\mathcal{C}})$ where:

$$c^{\mathcal{C}} = 8N - c$$

$$\ell^{\mathcal{C}} = p^2 + (2N-1)p - 6\sum_{i=1}^{p} n_i - \ell$$

where $n_i = h_i + h_i^{\mathcal{C}}$.

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  where $n_i = h_i + h_i^{\mathcal{C}}$.
- Next we apply the coset construction to $(p = 3, \ell = 0)$ theories. We find some very interesting coset pairs relative to the $c = 24$ meromorphic CFT's.

- If we take the coset of a meromorphic theory with $c = 8N$ by a (p, ℓ) theory with exponents (c, h_i), the resulting theory has $(p, \ell^{\mathcal{C}})$ and exponents $(c^{\mathcal{C}}, h_i^{\mathcal{C}})$ where:

$$c^{\mathcal{C}} = 8N - c$$

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- Next we apply the coset construction to $(p = 3, \ell = 0)$ theories. We find some very interesting coset pairs relative to the $c = 24$ meromorphic CFT's.
- From the above formula, the coset duals in this case also have $\ell = 0$. Here is a table:

	$\mathcal{D}$					$\mathcal{C}$					
No.	c	h_1	h_2	m_1	Algebra	$\tilde{c}$	$\tilde{h}_1$	$\tilde{h}_2$	$\tilde{m}_1$	$m_1 + \tilde{m}_1$	Schellekens No.
1	$\frac{3}{2}$	$\frac{3}{16}$	$\frac{1}{2}$	3	$\mathfrak{a}_{1,2}$	$\frac{45}{2}$	$\frac{29}{16}$	$\frac{3}{2}$	45	48	$5, 7, 8, 10$
2	$\frac{5}{2}$	$\frac{5}{16}$	$\frac{1}{2}$	10	$\mathfrak{c}_{2,1}$	$\frac{43}{2}$	$\frac{27}{16}$	$\frac{3}{2}$	86	96	$25, 26, 28$
3	3	$\frac{3}{8}$	$\frac{1}{2}$	15	$\mathfrak{a}_{3,1}$	21	$\frac{13}{8}$	$\frac{3}{2}$	105	120	$30, 31, 33-35$
4	$\frac{7}{2}$	$\frac{7}{16}$	$\frac{1}{2}$	21	$\mathfrak{b}_{3,1}$	$\frac{41}{2}$	$\frac{25}{16}$	$\frac{3}{2}$	123	144	$39, 40$
5	4	$\frac{2}{5}$	$\frac{3}{5}$	24	$\mathfrak{a}_{4,1}$	20	$\frac{8}{5}$	$\frac{7}{5}$	120	144	$37, 40$
6	$\frac{9}{2}$	$\frac{9}{16}$	$\frac{1}{2}$	36	$\mathfrak{b}_{4,1}$	$\frac{39}{2}$	$\frac{23}{16}$	$\frac{3}{2}$	156	192	$47, 48$
7	5	$\frac{5}{8}$	$\frac{1}{2}$	45	$\mathfrak{d}_{5,1}$	19	$\frac{11}{8}$	$\frac{3}{2}$	171	216	49
8	$\frac{11}{2}$	$\frac{11}{16}$	$\frac{1}{2}$	55	$\mathfrak{b}_{5,1}$	$\frac{37}{2}$	$\frac{21}{16}$	$\frac{3}{2}$	185	240	53
9	6	$\frac{3}{4}$	$\frac{1}{2}$	66	$\mathfrak{d}_{6,1}$	18	$\frac{5}{4}$	$\frac{3}{2}$	198	264	$54, 55$
10	$\frac{13}{2}$	$\frac{13}{16}$	$\frac{1}{2}$	78	$\mathfrak{b}_{6,1}$	$\frac{35}{2}$	$\frac{19}{16}$	$\frac{3}{2}$	210	288	56
11	7	$\frac{7}{8}$	$\frac{1}{2}$	91	$\mathfrak{d}_{7,1}$	17	$\frac{9}{8}$	$\frac{3}{2}$	221	312	59
12	$\frac{17}{2}$	$\frac{17}{16}$	$\frac{1}{2}$	136	$\mathfrak{b}_{8,1}$	$\frac{31}{2}$	$\frac{15}{16}$	$\frac{3}{2}$	248	384	62
13	$\frac{31}{2}$	$\frac{15}{16}$	$\frac{3}{2}$	248	$\mathfrak{e}_{8,2}$	$\frac{17}{2}$	$\frac{17}{16}$	$\frac{1}{2}$	136	384	62
14	9	$\frac{9}{8}$	$\frac{1}{2}$	153	$\mathfrak{d}_{9,1}$	15	$\frac{7}{8}$	$\frac{3}{2}$	255	408	63
15	10	$\frac{5}{4}$	$\frac{1}{2}$	190	$\mathfrak{d}_{10,1}$	14	$\frac{3}{4}$	$\frac{3}{2}$	266	456	64

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- Thus in particular we found new theories with central charges $\frac{31}{2}, \frac{35}{2}, \frac{37}{2}, \frac{39}{2}, \frac{41}{2}, \frac{43}{2}, \frac{45}{2}$. Moreover in a subsequent paper [Hampapura-Mukhi 2016], we found a $c = \frac{47}{2}$ CFT associated to the Baby Monster group. This accounts for most of the Franc-Mason exotic list. We also found $c = 14, 15, 17, 19, 20, 21$ theories which seem to be missing from their list.

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- We showed how to construct infinitely many admissible characters in all these cases, and proved that our method is complete [thanks to Ashoke Sen, ISM 2018].
- We also proposed a strategy to solve Problem (II), and showed how it works for a number of examples with $\ell = 6$.

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- One can think of admissible characters as special cases of quasi-characters.
- Example: at $c = 25$, there is a pair of quasi-characters whose identity character looks like:

$$\chi_0 = q^{-\frac{25}{24}}(1 - 245q + 142640q^2 + 18615395q^3 + 837384535q^4 + \cdots)$$

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  and all higher coefficients are positive.
- Clearly quasi-characters cannot directly describe a CFT: what sense does a degeneracy of -245 make?

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- The sum has $c = 25, h = \frac{5}{4}$ from which one finds $\ell = 6$.

No.	c	h	Character sum
1	$\frac{122}{5}$	$\frac{6}{5}$	$\chi_{LY}^{n=10} + N_1\chi_{LY}^{n=0}$
2	25	$\frac{5}{4}$	$\chi_{A_1}^{n=4} + N_1\chi_{A_1}^{n=0}$
3	26	$\frac{4}{3}$	$\chi_{A_2}^{n=6} + N_1\chi_{A_2}^{n=0}$
4	$\frac{134}{5}$	$\frac{7}{5}$	$\chi_{LY}^{n=11} + N_1\chi_{LY}^{n=1}$
5	28	$\frac{3}{2}$	$\chi_{D_4}^{n=2} + N_1\chi_{D_4}^{n=0}$
6	$\frac{146}{5}$	$\frac{8}{5}$	$\chi_{LY}^{n=12} + N_1\chi_{LY}^{n=2}$
7	30	$\frac{5}{3}$	$\chi_{A_2}^{n=7} + N_1\chi_{A_2}^{n=1}$
8	31	$\frac{7}{4}$	$\chi_{A_1}^{n=5} + N_1\chi_{A_1}^{n=1}$
9	$\frac{158}{5}$	$\frac{9}{5}$	$\chi_{LY}^{n=13} + N_1\chi_{LY}^{n=3}$

Table: $\ell = 6$ pairs obtained by addition of quasi-characters

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\begin{array}{ll}
c = 6r + 1, & A_1 \text{ class} \\
c = 4r + 2,\ r \neq 2 \bmod 3 & A_2 \text{ class} \\
c = 8r + 4 & D_4 \text{ class} \\
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  Similar formulae hold for $\ell = 2, 4$.
- By repeated addition of the $\ell = 0, 2, 4$ quasi-characters we generate admissible characters for all $\ell = 6m, 6m + 2, 6m + 4$.
- Thus our procedure generates all admissible characters for all even $\ell \geq 6$. Completeness can be proved by a simple inductive argument.

- Given these new infinite families of admissible characters, which of them are actual CFT? (Problem (II)).

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- By a general formula that we saw earlier, if we take the coset of Kervaire lattice theories by a $(p = 2, \ell = 0)$ CFT, we find $(p = 2, \ell = 6)$ CFT and we can characterise them.

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- By a general formula that we saw earlier, if we take the coset of Kervaire lattice theories by a $(p = 2, \ell = 0)$ CFT, we find $(p = 2, \ell = 6)$ CFT and we can characterise them.
- The coset construction lets us compute the undetermined constant N_1 in the previous table and ensures the existence of a CFT for that value of N_1.

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Label	c	h_1	h_2	Remark
$\mathcal{M}_{3,4}$	$23r + \frac{1}{2}$	$\frac{15}{8}r + \frac{1}{16}$	$r + \frac{1}{2}$	
$A_{1,2}$	$21r + \frac{3}{2}$	$\frac{13}{8}r + \frac{3}{16}$	$r + \frac{1}{2}$	
$C_{2,1}$	$19r + \frac{5}{2}$	$\frac{11}{8}r + \frac{5}{16}$	$r + \frac{1}{2}$	
$B_{n,1}$	$(23 - 2n)r + \frac{2n+1}{2}$	$\frac{(15-2n)}{8}r + \frac{2n+1}{16}$	$r + \frac{1}{2}$	
$\mathcal{M}_{2,7}$	$\frac{304}{7}r - \frac{68}{7}$	$\frac{20}{7}r - \frac{3}{7}$	$\frac{9}{7}r - \frac{2}{7}$	$r \neq 4 \bmod 7$
$D_{n,1}$	$2(12 - n)r + n$	$\frac{(8-n)}{4}r + \frac{n}{8}$	$r + \frac{1}{2}$	$n = 2 \bmod 4$
$\mathcal{M}_{2,5} \otimes \mathcal{M}_{2,5}$	$\frac{208}{5}r - \frac{44}{5}$	$\frac{14}{5}r - \frac{2}{5}$	$\frac{12}{5}r - \frac{1}{5}$	$n \neq 3 \bmod 5$
$D_{n,1}$	$2(12 - n)r + n$	$\frac{(8-n)}{4}r + \frac{n}{8}$	$r + \frac{1}{2}$	$n \neq 2 \bmod 4$

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- The status of $\ell = 6m + 1, 6m + 2, 6m + 3, 6m + 4, 6m + 5$ remains unclear for 3-character theories. Possibly some are ruled out. Currently we don't have a single example of admissible characters belonging to any of these sets.
- We also don't (yet) have a proof that our $\ell = 6m$ classification is complete, but it is probably true and should not be hard to prove.

- As we have seen, characters are vector-valued modular functions. Hence in particular:

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- Via the Verlinde formula, this matrix determines the fusion rules of the CFT:

$$N_{ijk} = \sum_{m=0}^{p-1} \frac{\mathcal{S}_{im}\mathcal{S}_{jm}\mathcal{S}^*_{km}}{\mathcal{S}_{0m}}$$

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  which are a key input to reconstruct the full CFT from its characters.

- However the MLDE method does not help us find the modular $\mathcal{S}$-matrix. This is because $\tau \to -\frac{1}{\tau}$ does not have a term-by-term action on the q-series.

- It is well-known that the moduli space of the torus can be parametrised in terms of a variable λ defined as:

$$\lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} = 16\, q^{1/2}(1 - 8q^{1/2} + 44q + \ldots)$$

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- For $(p = 2, \ell = 0)$, the characters of an RCFT can be written as a pair of hypergeometric functions:

$$\chi_0(\tau) = \mathcal{N}_0\ (\lambda(1-\lambda))^{\frac{1}{6}-h}\, {}_2F_1\Big(\tfrac{1}{2} - h, \tfrac{1}{2} - 3h\ \Big|\ 1 - 2h\ \Big|\ \lambda\Big)$$

$$\chi_1(\tau) = \mathcal{N}_1\ (\lambda(1-\lambda))^{\frac{1}{6}+h}\, {}_2F_1\Big(\tfrac{1}{2} + h, \tfrac{1}{2} + 3h\ \Big|\ 1 + 2h\ \Big|\ \lambda\Big)$$

where $\mathcal{N}_A$ are normalisations.

- We can use the integral representation of these functions to write:

$$\chi_0(\tau) = N_0\ (\lambda(1-\lambda))^{\frac{2}{3}-a} \int_1^\infty dt\, \Big[t(t-1)(t-\lambda)\Big]^a$$

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- Now we can explicitly compute the $\mathcal{S}$-matrix. We send $\lambda \to 1-\lambda$ and then deform contours avoiding the branch cuts. Each character comes back to a linear combination of both characters, with:

$$\mathcal{S}_{AB} = \begin{pmatrix} \frac{\sin \pi a}{\sin 2\pi a} & -\frac{\sin \pi a}{\sin 2\pi a} \\ -\frac{\sin 3\pi a}{\sin 2\pi a} & -\frac{\sin \pi a}{\sin 2\pi a} \end{pmatrix}$$

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- Now we can explicitly compute the $\mathcal{S}$-matrix. We send $\lambda \to 1-\lambda$ and then deform contours avoiding the branch cuts. Each character comes back to a linear combination of both characters, with:

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- From this one can compute degeneracies, multiplicities and fusion rules for all the cases where a corresponds to a CFT.

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- From this we constructed $\mathcal{S}$ for all $(p = 3, \ell = 0)$ theories. This enabled us to determine the fusion rules, correlation functions etc.
- A surprising result, noted in [Mukhi-Panda-Sen 1989], was that even for $p \geq 4$ one can find contour-integral representations whose $\lambda \to 0$ behaviour fits the critical exponents of large families of known RCFT.

- This led to the conjecture that the following $n+1$ contour integrals [Dotsenko-Fateev 1984,1985] are the characters of large classes of $\ell = 0$ RCFT:

$$\hat{J}_A(\lambda) = N_A \left(\lambda(1-\lambda)\right)^{\alpha} \int_1^{\infty} dt_n \int_1^{\infty} dt_{n-1} \cdots \int_0^{\lambda} dt_A \cdots \int_0^{\lambda} dt_1$$

$$\prod_{i=1}^{A} \left[t_i(1-t_i)(\lambda - t_i)\right]^a \prod_{i=A+1}^{n} \left[t_i(t_i-1)(t_i-\lambda)\right]^a \prod_{0\leq k<i\leq n} (t_i - t_k)^{2\rho}$$

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- Nonetheless for many theories, including minimal models $\mathcal{M}_{2,r}$ and $\mathrm{SU}(2)_k$ for all k, the exponents of the known theory can be reproduced by these contour integrals.
- Even so, it does not follow that the contour integrals describe the corresponding characters in general. This does follow for $p \le 5$ (proof via MLDE) but not for $p \ge 6$.

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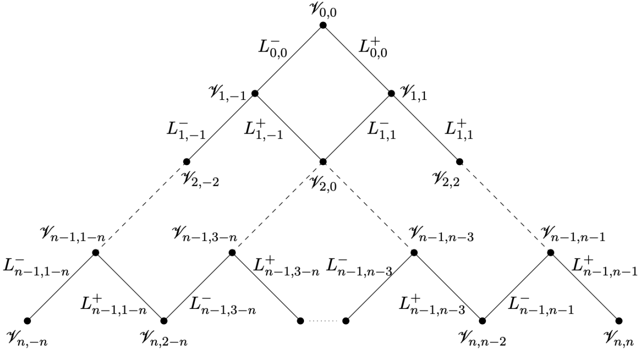
- In [Mukhi-Poddar-Singh 2019] we investigated this conjecture further.
- We developed a “sum over paths” algorithm to compute the modular $\mathcal{S}$-matrix for these contour integrals. This enabled us to compute $\mathcal{S}_{AB}$ for many cases even up to 20 characters, and verify the conjecture in every case.



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- A great deal of useful information emerges, which would not be accessible by the more standard methods of minimal series and null vectors.
- These theories provide some lessons for generic (not rational) CFT: finding a modular invariant partition function does not imply a CFT exists!

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- Quasi-characters are interesting in themselves, but also make a useful tool to build admissible characters with arbitrary Wronskian index ℓ.
- The Contour Integral Representation describes the characters of large families of known theories, but may also be a classification tool. We really don't know why it works at all.

Thank you

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3. Constrain which subset of the admissible characters are actual CFT by additional consistency requirements.

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- The only complete classification so far is the set of admissible characters for two-character RCFT.