Superconformal Index and Gravitational Path Integral

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Ofer Aharony, Ohad Mamroud, Paolo Milan — to appear

Quantum gravity ?

⋆ Flat space: STRING THEORY perturbative definition (+ some non-pert. objects)

⋆ Asymptotically Anti-de-Sitter space: AdS/CFT ⇒ non-perturbative definition in terms of boundary ordinary QFT

Parameter map of AdS/CFT

- Large AdS$_D$ compared with Planck scale $\Rightarrow$ QFT with large "central charge" (large N)

$$\frac{\ell_{\text{AdS}}^{D-2}}{G_N} \sim \text{"c.c."}$$

[Brown, Henneaux 86]

- Large AdS$_D$ compared with higher derivative corrections to Einstein gravity (*e.g.*, massive string or higher-spin modes) $\Rightarrow$ **QFT is strongly coupled**

E.g., in string theory: $$\frac{\ell_{\text{AdS}}^{4}}{\alpha'^2} \sim \lambda$$

$\uparrow$ PROBLEM!

! Take advantage of modern non-perturbative methods !

Black holes & Entropy

$$S_{\mathsf{BH}} = \frac{\mathsf{Area}}{4G_N\hbar/c^3}$$

[Bekenstein 72, 73, 74; Hawking 74, 75]

Black hole $=$ Ensemble of states in quantum gravity $\stackrel{\mathsf{AdS/CFT}}{=}$ Ensemble of states in boundary QFT

$$S_{\mathsf{micro}} = \log N_{\mathsf{micro}} = \frac{\mathsf{Area}}{4G_N} + \log \mathsf{Area} + \ldots$$

(pert. and non-pert.)

Can we

- reproduce the Bekenstein-Hawking entropy?
- compute corrections?
- determine the *exact integer number* N_{micro}?

Black holes in flat space

⋆ String theory reproduces the Bekenstein-Hawking entropy of BPS black holes in asymptotically flat spacetime [Strominger, Vafa 96]

- With enough SUSY, corrections can be computed as well

[I. Bena, G. Compere, A. Dabholkar, F. Denef, R. Dijkgraaf, J. Gomes, J. A. Harvey, M. Henneaux, F. Larsen, J. Maldacena, R. Minasian, G. Moore, S. Murthy, B. Pioline, V. Reys, A. Sen, A. Strominger, E. Verlinde, H. Verlinde, E. Witten, D. Zagier, . . .]

- Given the lack of non-perturbative definition of string theory, to what extent can this be done with lower SUSY?

$\longrightarrow$ Motivation to study black hole entropy in AdS space

Black holes in AdS

- ⋆ BTZ black holes in AdS_3 in terms of CFT_2 understood long ago [Strominger, Vafa 96; ...]
- ⋆ By now we understand AdS black holes in 4 and more dimensions as well:
  - Dual QFT at large N reproduces the Bekenstein-Hawking entropy [FB, Hristov, Zaffaroni 15]
  - Many (BPS) examples in different setups and dimensions, some corrections

[Ardehali, Azzurli, FB, Bobev, Cabo-Bizet, Cassani, Choi, Crichigno, Gang, Honda, Hong, Hosseini, Hristov, Hwang, Jain, Jeon, Kim, Lal, Lezcano, Liu, Martelli, Min, Murthy, Nedelin, Pando Zayas, Passias, Pilch, Rathee, Tachikawa, Toldo, Willett, Yaakov, Zaffaroni, Zhao, ...]

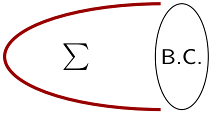
- ⋆ **This talk:** analyze charged rotating BPS black holes in AdS_5
  - remained as a puzzle for a long time [Kinney, Maldacena, Minwalla, Raju 05]
  - very detailed computations might be at reach

⋆ **Strategy:** count states in the boundary QFT
employing a grand canonical partition function

$$\mathcal{I}(y) = \sum_{\text{states}} y^Q$$

Difficult (but well-defined) problem at strong coupling $\longrightarrow$ exploit SUSY

⋆ Partition function = Euclidean gravitational path-integral with fixed boundary conditions



[Witten 98][Dijkgraaf, Maldacena, Moore, E. Verlinde 00][Maloney, Witten 07]

SUSY & localization $\rightsquigarrow$ detailed analysis and a lesson

1. complex saddles play important role
2. not all of them contribute (criterium)

BPS black holes in AdS_5

Setup:

Type IIB string theory on $\text{AdS}_5 \times S^5$ $\longleftrightarrow$ 4d $SU(N)$ $\mathcal{N} = 4$ Super-Yang-Mills

BPS black hole solutions have been constructed in 5d gauged supergravity [Gutowski, Reall 04; ...]

- ⋆ Does $\mathcal{N} = 4$ SYM contain BPS states that reproduce the black hole entropy?
- ⋆ Can one extract hints of new physics, beyond Bekenstein-Hawking?

Rotating & electrically-charged $\frac{1}{16}$-BPS black holes in AdS_5

[Gutowski, Reall 04]
[Chong, Cvetic, Lu, Pope 05][Kunduri, Lucietti, Reall 06]

- Two angular momenta: J_1, J_2

  Three electric charges $U(1)^3 \subset SO(6)$: R_1, R_2, R_3

- BPS & Extremal, 1 complex supercharge $\mathcal{Q}$

  BPS relation: $2M = 2J_1 + 2J_2 + R_1 + R_2 + R_3$

  Extremality: non-linear relation among 5 charges $\rightarrow$ 4 parameters

  [Cabo-Bizet, Cassani, Martelli, Murthy 18; Cassani, Papini 19]

- Near horizon: fibration $\mathrm{AdS}_2 \rightarrow$ squashed S^3

  B-H entropy: $S_{\mathrm{BH}} = \frac{\mathrm{Area}}{4G_N} = \pi\sqrt{R_1R_2 + R_1R_3 + R_2R_3 - 2N^2(J_1 + J_2)}$

- Angular momenta, charges and entropy scale $\sim N^2$

Superconformal index

[Romelsberger 05; Kinney, Maldacena, Minwalla, Raju 05]

⋆ Counts (with sign) BPS states on S^3 $=$ protected operators on flat space

Index of $\mathcal{N}=4$ SYM:

$$\mathcal{I}(p,q,y_1,y_2) = \mathrm{Tr}\,(-1)^F\, e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}\, p^{J_1+\frac{1}{2}R_3}\, q^{J_2+\frac{1}{2}R_3}\, y_1^{\frac{1}{2}(R_1-R_3)}\, y_2^{\frac{1}{2}(R_2-R_3)}$$

Write: $\quad p = e^{2\pi i\tau} \qquad q = e^{2\pi i\sigma} \qquad y_a = e^{2\pi i\Delta_a} \qquad F = R_3 = 2J_1 = 2J_2 \mod 2$

SUSY $\Rightarrow$ at most 4 independent fugacities

⋆ *Exact* integral formula

[Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk 03]
[Sundborg 99][Romelsberger 05][Kinney, Maldacena, Minwalla, Raju 05]

$$\mathcal{I} = \kappa_N \oint_{\mathbb{T}^{\mathrm{rk}(G)}} \prod_{i=1}^{\mathrm{rk}(G)} \frac{dz_i}{2\pi i z_i} \times \frac{\prod_{a=1}^3 \prod_{\rho\in\mathfrak{R}_{\text{adj}}} \widetilde{\Gamma}\big(\rho(u)+\Delta_a;\tau,\sigma\big)}{\prod_{\alpha\in\mathfrak{g}} \widetilde{\Gamma}\big(\alpha(u);\tau,\sigma\big)}$$

with $\quad \Delta_1+\Delta_2+\Delta_3-\tau-\sigma\in\mathbb{Z}\,, \quad z=e^{2\pi i u}$ and (elliptic gamma)

$$\kappa_N = \frac{(p;p)_\infty^{\mathrm{rk}(G)}(q;q)_\infty^{\mathrm{rk}(G)}}{|\mathcal{W}_G|} \qquad \widetilde{\Gamma}(u;\tau,\sigma) = \prod_{m,n=0}^\infty \frac{1-p^{m+1}q^{n+1}/z}{1-p^m q^n z}$$

⋆ The index encodes (*weighted*) degeneracies:

$$\mathcal{I} = 1 + \#\, y + \#\, y^2 + \ldots + d(Q)\, y^Q + \ldots$$

To extract the degeneracies:

$$d(Q) = \frac{1}{2\pi i} \oint \frac{dy}{y^{Q+1}}\, \mathcal{I}(y) = \oint d\Delta \;\; e^{\log \mathcal{I}(\Delta) - 2\pi i Q \Delta}$$

Assuming large degeneracies, saddle-point approximation → Legendre transform

$$\text{entropy} = \log d(Q) \;\simeq\; \log \mathcal{I}(\Delta) - 2\pi i Q \Delta \,\Big|_{\Delta\,=\,\text{extremum}}$$

- We are interested in $Q \sim N^2$ in the large N limit. Tricky!

⋆ $\mathcal{I}$-extremization: superconformal index captures true (unweighted) degeneracies at leading order. [FB, Hristov, Zaffaroni 16]

Similar to [Sen 09] 's argument in flat space

Approaches to large N matrix model

- Direct saddle point approx. [Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk 03] [Kinney, Maldacena, Minwalla, Raju 05]
- Cardy limit $\tau \to 0$ [Choi, J. Kim, S. Kim, Nahmgoong 18] [M. Honda 19; Ardehali 19; Ardehali, Hong, Liu 19] [J. Kim, S. Kim, Song 19; Cabo-Bizet, Cassani, Martelli, Murthy 19] [Amariti, Garozzo, Lo Monaco 19; David, Nian, Pando Zayas 20]
- Bethe Ansatz formulation [FB, Milan 18; FB, Colombo, Soltani, Zaffaroni, Zhang 20] [Lanir, Nedelin, Sela 19; Lezcano, Pando Zayas 19] [Ardehali, Hong, Liu 19]
- Saddle point approx of non-analytic extension [Cabo-Bizet, Murthy 19] [Cabo-Bizet, Cassani, Martelli, Murthy 20]
- Gross-Witten-Wadia-like expansion [Copetti, Grassi, Komargodski, Tizzano 20]

Bethe Ansatz formula for the superconformal index

Alternative *exact* formula. For special case $\tau = \sigma$:

[Closset, Kim, Willett 17]
[FB, Milan 18]

$$\mathcal{I} = \kappa_N \sum_{\hat{u} \in \mathfrak{M}_{\text{BAE}}} \mathcal{Z}(\hat{u}; \Delta, \tau, \tau)\, H(\hat{u}; \Delta, \tau)^{-1}$$

1. $\mathfrak{M}_{\text{BAE}}$ are solutions to "Bethe Ansatz Equations" for $\mathrm{rk}(G)$ complexified holonomies $[\hat{u}_i]$ living on a complex torus T^2_τ of modular parameter τ:

$$Q_i = \prod_{\rho_a \in \mathfrak{R}} P\Big(\rho_a(u) + \Delta_a; \tau\Big)^{\rho_a^i} \qquad P(u;\tau) = \frac{e^{-\pi i \frac{u^2}{\tau} + \pi i u}}{\theta_0(u;\tau)}$$

$$\mathfrak{M}_{\text{BAE}} = \Big\{ [\hat{u}_i] \in T^2_\tau \;\Big|\; Q_i(u) = 1 \Big\}$$

2. κ_N and $\mathcal{Z}$ are the same prefactor and integrand as in the integral formula

3. H is a Jacobian: $\qquad H = \det_{ij} \left(\frac{\partial Q_i}{\partial u_j} \right)$

Bethe Ansatz Equations for $\mathcal{N} = 4$ SYM

Specialize to 4d $SU(N)$ $\mathcal{N} = 4$ SYM:

$$Q_i \equiv e^{-2\pi i \sum_j u_{ij}} \prod_{a=1}^{3} \prod_{j=1}^{N} \frac{\theta_0(u_{ji} + \Delta; \tau)}{\theta_0(u_{ij} + \Delta; \tau)} = (-1)^{N-1}$$

$$u_{ij} = u_i - u_j$$

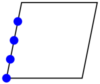
Equations are defined on T^2_τ and are invariant under $SL(2, \mathbb{Z})$

★ Class of exact solutions at finite N [Hosseini, Nedelin, Zaffaroni 16; Hong, Liu 18]

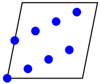
Classified by subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$ of order N

Labelled by $\{m, n, r\}$ with $m \cdot n = N$ and $0 \leq r < n$

- BASIC SOLUTION $\{1, N, 0\}$: $u_j \sim \frac{\tau}{N}\, j$

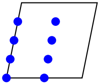


- T-TRANSFORMED $\{1, N, r\}$: $u_j \sim \frac{\tau + r}{N}\, j$ with $0 \leq r < N$



- $SL(2,\mathbb{Z})$-TRANSFORMED SOL's

- More general subgroups, $m > 1$ $\gcd(m, n, r) > 1$

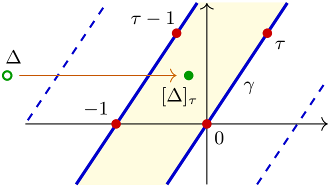


★ Conjectured existence of *continuous* families of solutions [Ardehali, Hong, Liu 19]

Contribution of BASIC SOLUTION at large N

Large N limit is a discontinuous analytic function: Stokes phenomenon

$$[\Delta]_\tau \equiv \Delta + n \quad \text{s.t. } \in \text{STRIP}$$



Contribution of BASIC SOLUTION at large N:

$$\lim_{N\to\infty} \log \mathcal{I}(\tau, \Delta_1, \Delta_2)\Big|_{\substack{\text{BASIC}\\ \text{SOLUTION}}} = \qquad (1^{\text{st}} \text{ and } 2^{\text{nd}} \text{ case})$$

$$= \begin{cases} -i\pi N^2 \dfrac{[\Delta_1]_\tau [\Delta_2]_\tau [\Delta_3]_\tau}{\tau^2} & \text{if } [\Delta_1]_\tau + [\Delta_2]_\tau \in \text{STRIP} \\ \qquad \text{with } \sum_a [\Delta_a]_\tau = 2\tau - 1 & \\ \quad \ldots & \text{if } [\Delta_1]_\tau + [\Delta_2]_\tau + 1 \in \text{STRIP} \end{cases}$$

Black hole entropy

Extract entropy from $\log \mathcal{I}\big|_{\text{BASIC SOLUTION}}$ (1st case)

⋆ Set $X_1 = [\Delta_1]_\tau \quad X_2 = [\Delta_2]_\tau$. Obtain "entropy function":

$$\boxed{\log \mathcal{I} = -i\pi N^2 \, \frac{X_1 X_2 X_3}{\tau^2}} \qquad \text{with} \qquad \sum_{a=1}^{3} X_a = 2\tau - 1$$

Its (constrained) Legendre transform *exactly* gives the black hole entropy:

[Hosseini, Hristov, Zaffaroni 17]

$$S_{\text{BH}} = \log \mathcal{I} - 2\pi i \left(\sum X_a \tfrac{R_a}{2} + 2\tau J \right) \Big|_{\text{constrained extremum}}$$

$$= \pi \sqrt{\textstyle\sum_{a<b} R_a R_b - 4N^2 J}$$

This can be generalized to $\tau \neq \sigma$ [FB, Colombo, Soltani, Zaffaroni, Zhang 20]

⋆ What is the role of all other Bethe Ansatz solutions?

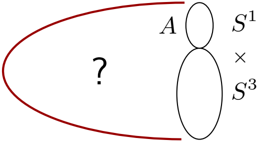
Gravitational path-integral

- Superconformal index is computed by Euclidean partition function in QFT

$$\mathcal{I}_{\mathsf{SCFT}} = Z_{S^3\times S^1} \qquad \text{(with suitable regularization)}$$

- Holographically: $Z_{S^3\times S^1}$ = string theory path-integral $\simeq$ classical saddles + corrections

Fill-in bulk geometry
for given boundary conditions
[Witten 98; Dijkgraaf, Maldacena, Moore, Verlinde 00]



- *Only* SUSY configurations contribute to SUSY observables (localization)

- Euclidean rotation of Lorentzian BPS black hole has $\beta = \infty$ (extremal, $T = 0$)

$\Rightarrow$ Look for complex Euclidean SUSY solutions

Complex Euclidean solutions

[Cabo-Bizet, Cassani, Martelli, Murthy 18]

- ⋆ $\exists$ a 6-parameter family of non-SUSY black hole solutions [Chong, Cvetic, Lu, Pope 05] [Cvetic, Gibbons, Lu, Pope 05] [Wu 11]

  Generic *complex* values of parameters

  $\Rightarrow$ *complex* metric and gauge fields

  Impose SUSY but *not* extremality

- *E.g.*: case of 2 equal angular momenta & 3 equal charges (& chem. potentials)

$$ds^2 = -\frac{1+r^2}{\Xi_a}dt^2 - \frac{2q}{\Xi_a(r^2+a^2)}\nu\varpi + \frac{f(r)}{\Xi_a^2(r^2+a^2)^2}\varpi^2 \qquad \Xi_a = 1-a^2 \qquad \varpi = dt - \nu$$

$$+\frac{r^2+a^2}{\Delta(r)}dr^2 + \frac{r^2+a^2}{\Xi_a}\left(d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\psi^2\right) \qquad \nu = a\left(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi\right)$$

$$A = \frac{3q}{2(r^2+a^2)\Xi_a}\varpi$$

SUSY parameters: (a, r_+)

$$\begin{cases} \text{Charges:} & E = 2J + \frac{3}{2}Q \\ \text{Potentials:} & T = \frac{1}{\beta}\,, \quad \frac{\beta}{2\pi i}(1 + 2\Omega - 2\Phi) = \pm 1 \end{cases}$$

Two branches of complex solutions: $q = -a^2 + (1+2a)r_+^2 \mp i r_+ \sqrt{(r_+^2 - 2a - a^2)^2}$

- Boundary metric: $$ds^2_{\text{bdy}} = \underbrace{dt_{\text{E}}^2}_{S^1} + \underbrace{d\hat{\theta}^2 + \sin^2\hat{\theta}\, d\phi^2 + \cos^2\hat{\theta}\,\psi^2}_{S^3}$$

with $\quad (t_{\text{E}}, \phi, \psi) \cong (t_{\text{E}} + \beta,\ \phi - i\Omega_1\beta,\ \psi - i\Omega_2\beta)$ (from regularity at the horizon)

ϕ, ψ defined mod 2π $\Rightarrow$ all $\Omega_{1,2} + \frac{2\pi i}{\beta}\mathbb{Z}$ give *same* boundary metric

- Boundary gauge field: $$\exp\Big\{ i \oint_{S^1(\text{bdy})} A_a \Big\} = \exp\Big\{ -\beta\,\Phi_a \Big\}$$

Holonomy is gauge inv. $\Rightarrow$ all $\Phi_a + \frac{2\pi i}{\beta}\mathbb{Z}$ give *same* boundary gauge bundle

⋆ B.C.'s only fix (constrained) complex potentials up to $\mathbb{Z}$ shifts!

- To make contact with index, define: $$\tau^{\text{g}}, \sigma^{\text{g}} = \frac{\beta}{2\pi i}(\Omega_{1,2} - 1) \qquad \Delta_a^{\text{g}} = \frac{\beta}{2\pi i}(\Phi_a - 1)$$

SUSY constraint among potentials: $\sum_a \Delta_a^{\text{g}} = \tau^{\text{g}} + \sigma^{\text{g}} \mp 1$ [Cabo-Bizet, Cassani Martelli, Murthy 18]

Match with Bethe Ansatz formula?

⋆ Large N contribution of $\{m, n, r\}$ solutions, fixed m, r: (1st case)

$$\log \mathcal{I}_{\{m,n,r\}} = -\frac{i\pi N^2}{m} \frac{[m\Delta_1]_{\check{\tau}}[m\Delta_2]_{\check{\tau}}[m\Delta_3]_{\check{\tau}}}{(m\tau + r)^2} + \log N + \mathcal{O}(1) + \mathcal{O}(Ne^{-N})$$

where $\sum_a [m\Delta_a]_{\check{\tau}} = 2\check{\tau} - 1$ and $\check{\tau} = m\tau + r$

⋆ On-shell action of complex Euclidean SUSY solutions: (for $\tau = \sigma$)

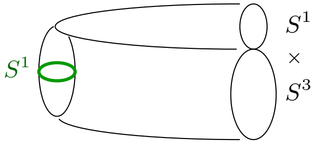
$$I_{\text{grav}} = -i\pi N^2 \frac{(\Delta_1^{\text{g}} - n_1)\,(\Delta_2^{\text{g}} - n_2)\,(\Delta_3^{\text{g}} - n_3)}{(\tau^{\text{g}} + n_4)\,(\tau^{\text{g}} + n_5)}$$

with $\sum_a \Delta_a^{\text{g}} = 2\tau^{\text{g}} - 1$ and $\sum_{\alpha=1}^{5} n_\alpha = 0$

- Gravity reproduces leading contribution for $m = 1$,
  i.e. T-TRANSFORMED SOL's $\{1, N, r\}$ … but it has *too many* solutions!

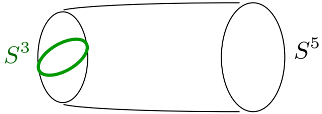
Wrapped D3-branes

Non-perturbative corrections from Euclidean SUSY D3-branes wrapped on 10d geometry at the horizon



- Two possible $S^1 \subset S^3$:

  $w_j = 0 \quad \text{in} \quad |w_1|^2 + |w_2|^2 = 1$



- Three possible $S^3 \subset S^5$:

  $z_a = 0 \quad \text{in} \quad |z_1|^2 + |z_2|^2 + |z_3|^2 = 1$

On-shell action:

$$S_{\text{D3}} = 2\pi N \, \frac{\Delta_a^{\text{g}}}{\tau^{\text{g}}} \qquad \text{or} \qquad S_{\text{D3}} = 2\pi N \, \frac{\Delta_a^{\text{g}}}{\sigma^{\text{g}}} \qquad\qquad (1^{\text{st}} \text{ branch})$$

Non-perturbative corrections: generic *positive* linear combinations of those

(Caveat: analyzed cases of 2 equal J *or* 3 equal Q)

⋆ Effect of D3-brane corrections:

$$\mathcal{I} = Z_{S^3\times S^1} \;\simeq\; e^{I_{\text{grav}}} + \sum_k e^{I_{\text{grav}}}\, e^{ikS_{\text{D3}}} \;\simeq\; \exp\Big\{ \underbrace{I_{\text{grav}}}_{\mathcal{O}(N^2)} + \sum_k \underbrace{e^{ikS_{\text{D3}}}}_{\mathcal{O}(e^{-N})} \Big\}$$

Criterium to retain a complex saddle:

$$\boxed{\mathbb{I}\text{m}\, S_{\text{D3}} > 0 \qquad \text{for all (SUSY) D3-brane embeddings}}$$

Violation implies "D3-brane condensation" towards some other saddle point.

Expected to signal that complex saddle point does *not* contribute to integral.

⋆ Take $\tau = \sigma$ in QFT $\quad \Rightarrow \quad \begin{cases} \tau^{\text{g}} = \sigma^{\text{g}} = \tau + r & \text{for any } r \\ \Delta_a^{\text{g}} = [\Delta_a]_{\tau+r} \end{cases}$

Precise match between gravitational saddles and T-TRANSFORMED SOL's

QFT computation of exponents of non-perturbative corrections for $\{m, n, r\}$ sol's:

$$\exp\left\{\frac{2\pi i N}{m}\ \frac{[m\Delta_a]_{\check{\tau}}}{\check{\tau}}\right\} \qquad (1^{\text{st}} \text{ case})$$

and products thereof.

- $\mathcal{O}(e^{-N})$ non-perturbative corrections, exponentially small
- For $m = 1$, coincide with D3-brane actions around "stable" saddles

Q: Can we perform a more precise QFT evaluation, including the prefactor?

Does it match with quantization of D3-branes?

Orbifold geometries: $m > 1$

The $\{m, n, r\}$ solutions with $m > 1$ correspond to freely-acting SUSY orbifolds of 10d lift of previous solutions

- Take a SUSY complex solution with $\widetilde{\beta} = m\,\beta\,, \quad \widetilde{\Omega}_j\,, \quad \widetilde{\Phi}$

  10d lift: $ds^2(S^5) = ds^2(\mathbb{CP}^2) + (d\eta + \mathcal{A} + A)^2$ [Cvetic *et al.* 99]

  Orbifold:

$$(t_{\rm E},\ \phi,\ \psi,\ \eta)\ \cong\ \Big(t_{\rm E} + \tfrac{\widetilde{\beta}}{m}\,,\ \ \phi - i\widetilde{\Omega}_1\tfrac{\widetilde{\beta}}{m}\,,\ \ \psi - i\widetilde{\Omega}_2\tfrac{\widetilde{\beta}}{m}\,,\ \ \eta - i\widetilde{\Phi}\tfrac{\widetilde{\beta}}{m} + \pi i\,\tfrac{m-1}{m}\Big)$$

Reading off boundary data: $\widetilde{\tau}^{\rm g} = m\tau^{g} + \mathbb{Z}\,, \qquad \widetilde{\Delta}^{\rm g} = m\Delta^{\rm g} + \mathbb{Z}$

"Stability" of Euclidean D3-branes $\Rightarrow$ $\begin{cases} \widetilde{\tau}^{\rm g} = \widetilde{\sigma}^{\rm g} = m\tau + r \equiv \check{\tau} \\ \widetilde{\Delta}^{\rm g}_a = [m\Delta_a]_{\check{\tau}} \end{cases}$

On-shell action reduced by $\frac{1}{m}$ $\rightsquigarrow$ Match with $\log \mathcal{I}_{\{m,n,r\}}$

Conclusions

Summary:

- Careful analysis of superconformal index of $\mathcal{N} = 4$ SYM, using an alternative Bethe Ansatz formulation. Large N: each Bethe Ansatz solution represents a complex saddle point.
- One solution exactly reproduces the Bekenstein-Hawking entropy of BPS black holes in $\text{AdS}_5 \times S^5$.
- Other solutions give corrections from complex gravitational saddles. Criterium: discard complex saddles with diverging D3-instanton corrections.

Some open questions:

- Consequences for Lorentzian physics? Which phases / phase transitions?
- Can we compute corrections more precisely?
- Other Bethe Ansatz solutions?
- Hidden structure? What is the role of $SL(2, \mathbb{Z})$? [cfr. Gadde 20]