FROM SYMMETRIC PRODUCT *CFTs* TO AdS_3

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ROADMAP

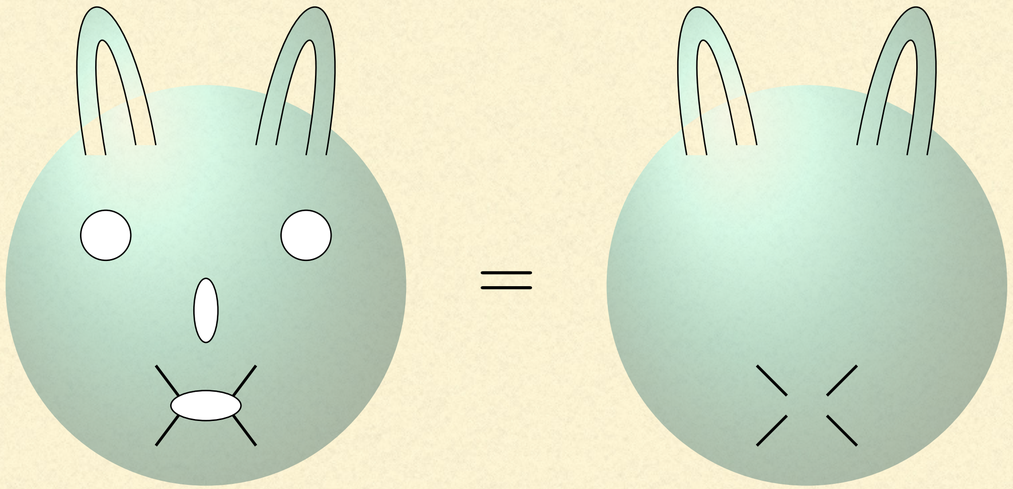
- MOTIVATIONS (From Strings to Fields and From Fields to Strings)
- BACKGROUND (On symmetric product orbifolds and covering maps)
- TECHNICALITIES (A large N limit, a matrix model and its solution)
- MEANING (How CFT Feynman diagrams cover stringy moduli space…)
- ANSWERS (…and what that concretely gives for the dual string theory)
- OUTLOOK (Onwards to 4d Yang-Mills!)

MOTIVATION

DERIVING AdS/CFT

- How exactly do large N QFTs reorganise themselves into theories of strings?
- D-brane physics indicates open-closed string duality as the underlying reason [Maldacena-'97].
- Holes close up and backreaction alters the background.
- But difficult to see this explicitly happen at large $g_s N = \lambda$.
- Therefore, cannot delineate scope of gauge-string duality beyond examples.

$\sum_{\text{holes}}$



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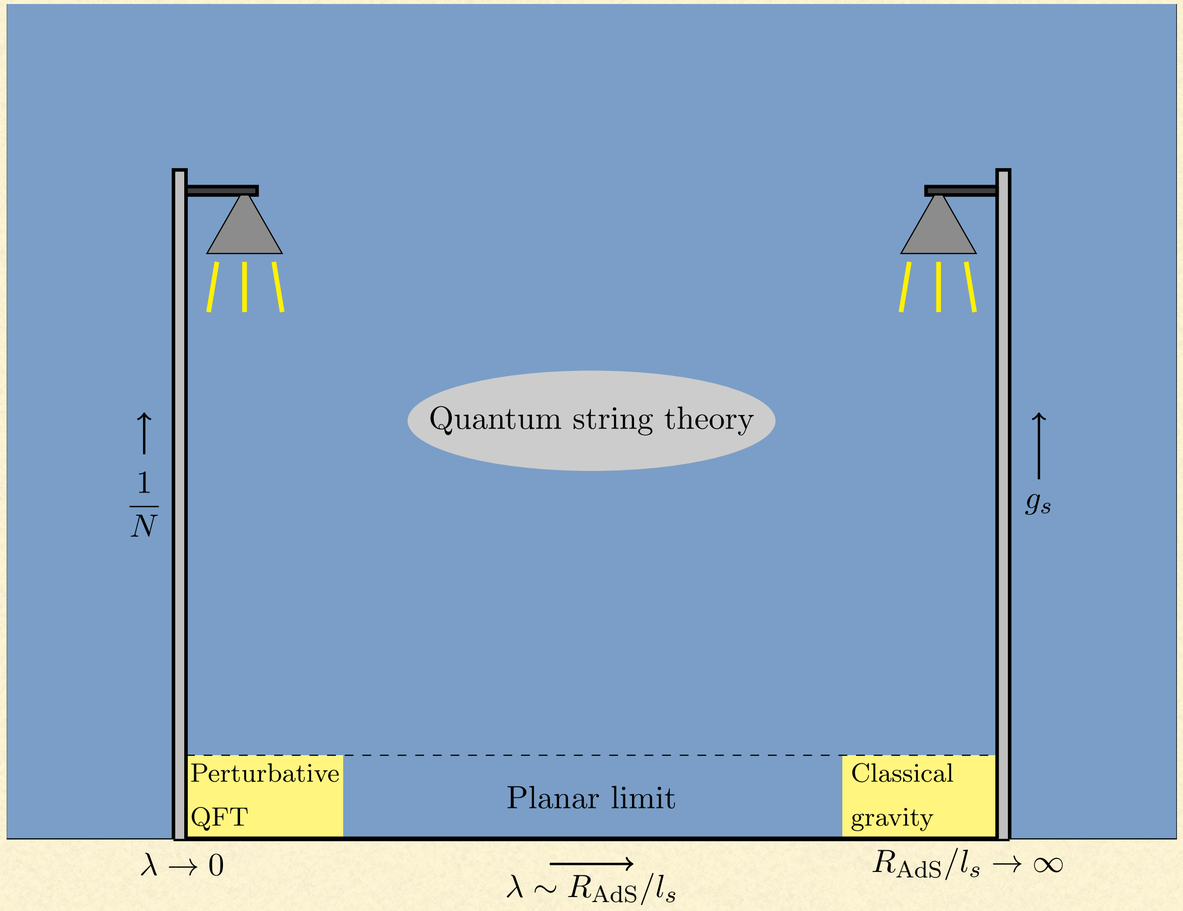
A DIFFERENT LAMP POST



- Shift focus in this talk to the corner where we understand the field theory but not the bulk.
- Look at $\lambda \to 0$ i.e. highly curved AdS or tensionless limit. Very stringy regime.
- Dictionary: $R_{AdS} \propto \lambda^{\alpha}$; $g_s^2 \propto \frac{\lambda^2}{N^2}$.
- Finite number of holes to sum over at zero coupling. Well defined genus expansion.
- Interactions treated perturbatively as correlators in a free QFT.

WHAT CONSTITUTES A DERIVATION?

$$\left\langle \mathcal{O}_{h_1}^{(w_1)}(x_1)\mathcal{O}_{h_2}^{(w_2)}(x_2)\ldots\mathcal{O}_{h_n}^{(w_n)}(x_n)\right\rangle_{S^d}\Big|_g = \int_{\mathcal{M}_{g,n}} \left\langle \mathcal{V}_{h_1}^{w_1}(x_1;z_1)\mathcal{V}_{h_2}^{w_2}(x_2;z_2)\ldots\mathcal{V}_{h_n}^{w_n}(x_n;z_n)\right\rangle_{\Sigma_{g,n}}$$

- Operational definition: Relate (single trace) gauge invariant (euclidean) correlators to perturbative string amplitudes - $\forall(g,n)$.
- Based on the dictionary between states: $\mathcal{O}_h^{(w)}(x) \leftrightarrow \mathcal{V}_h^w(x;z)$. ($h$ = conformal dimension).
- Both sides have autonomous definitions: as a fixed point for QFT on LHS and in terms of a perturbative sigma model 2d CFT for the RHS. Mathematically well posed question.
- Can we make the equality manifest? Can we `tautologise' the correspondence?

PROOF OF CONCEPT

- Recent work on a tensionless limit of the AdS_3/CFT_2 correspondence makes much of this discussion very concrete and explicit - testing ground for a derivation.

$$\left\langle \mathcal{O}_{h_1}^{(w_1)}(x_1)\mathcal{O}_{h_2}^{(w_2)}(x_2)\ldots\mathcal{O}_{h_n}^{(w_n)}(x_n)\right\rangle_{S^2}\Big|_g = \int_{\mathcal{M}_{g,n}} \left\langle \mathcal{V}_{h_1}^{w_1}(x_1;z_1)\mathcal{V}_{h_2}^{w_2}(x_2;z_2)\ldots\mathcal{V}_{h_n}^{w_n}(x_n;z_n)\right\rangle_{\Sigma_{g,n}}$$

- CLAIM: String Theory on $AdS_3 \times S^3 \times T^4$ and $k = 1$ unit of NS-NS flux $\equiv Sym^N(T^4)$ free Symmetric Orbifold CFT as $N \to \infty$; $(g_s^2 \propto 1/N)$. [Eberhardt, Gaberdiel, R.G. - '18-'19].

[RG@Strings20]

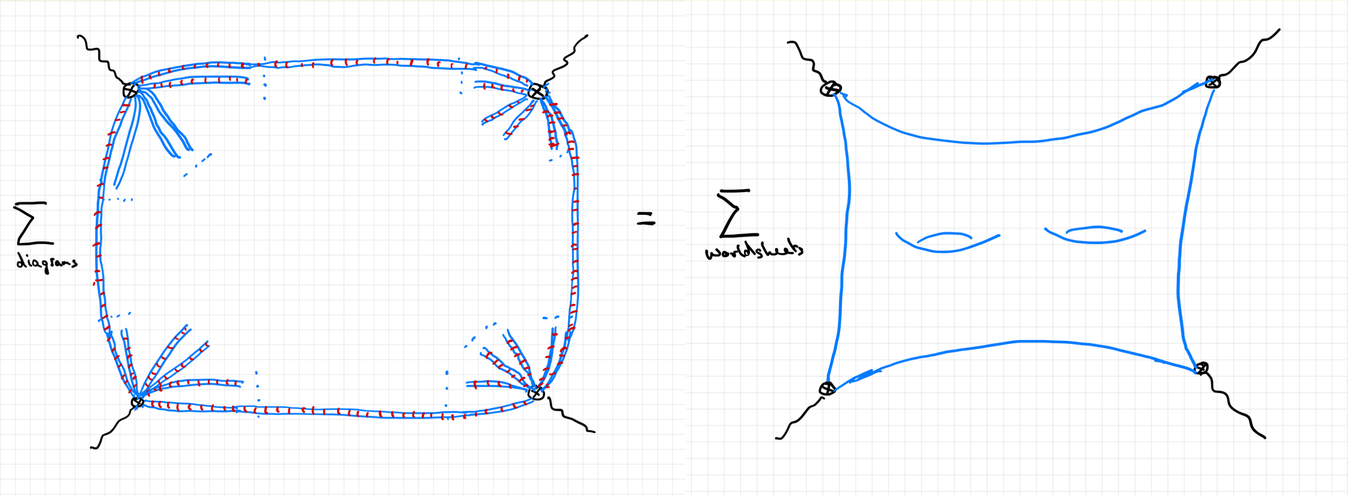
- Worldsheet correlators on the RHS $\propto \prod_{i=4}^{n} \delta^{(2)}(x_i - \Gamma(z_i))$ i.e. localise to points on $\mathcal{M}_{0,n}$ corresponding to specific branched covers $x = \Gamma(z)$ with branching w_i at insertions z_i : $x \sim x_i + a_i^{\Gamma}(z - z_i)^{w_i}$.

- Follows from a twistorial incidence relation $\langle(\xi^- + \Gamma(z)\xi^+)\rangle_{phys} = 0$. [Dei, Gaberdiel, R.G., Knighton '20].

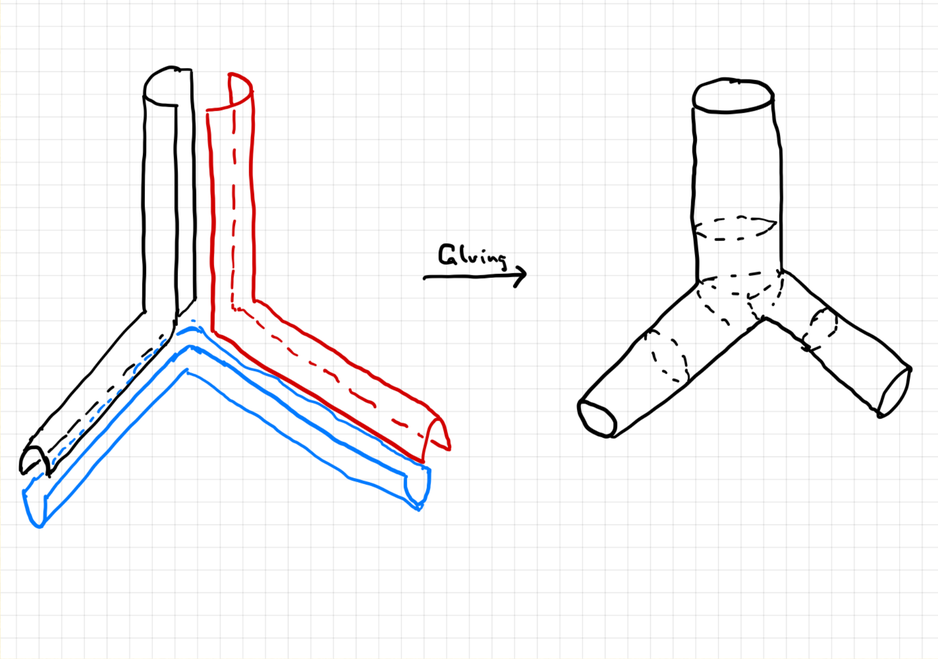
BACK AND FORTH

- $$\left\langle \mathcal{O}_{h_1}^{(w_1)}(x_1)\mathcal{O}_{h_2}^{(w_2)}(x_2)\ldots\mathcal{O}_{h_n}^{(w_n)}(x_n)\right\rangle_{S^d}\Big|_g = \int_{\mathcal{M}_{g,n}} \left\langle \mathcal{V}_{h_1}^{w_1}(x_1;z_1)\mathcal{V}_{h_2}^{w_2}(x_2;z_2)\ldots\mathcal{V}_{h_n}^{w_n}(x_n;z_n)\right\rangle_{\Sigma_{g,n}}$$
- An apparent asymmetry in this equality. Easier to go from RHS to LHS - Strings to Fields.
- To go from Fields to Strings (LHS to RHS), need to reconstruct a worldsheet integrand - not unique.
- Nevertheless can have a canonical or natural form for the correlator on the RHS.
- "From Free Fields to AdS" program to recast QFT correlators into stringy correlators. [R.G. - '03-'05].
- BASIC IDEA: Sum over distinct worldline topologies in Feynman diagrams for a large N theory = Sum over distinct worldsheets (moduli space) after gluing up double lines.

FROM WORLDLINES TO WORLDSHEETS



Implementation of
open-closed string duality.



SLOGAN: EACH FEYNMAN GRAPH
$\leftrightarrow$ A CLOSED WORLDSHEET.

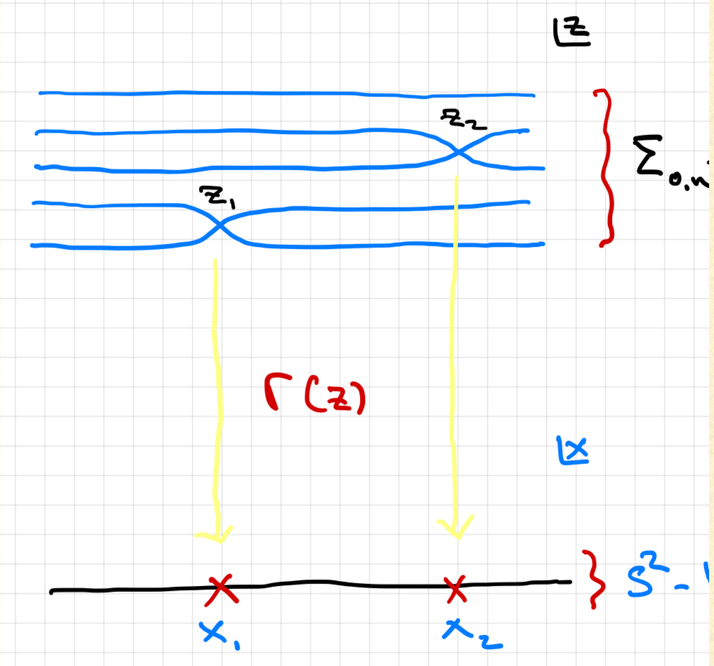
Exploits the Strebel parametrisation of $\mathcal{M}_{g,n}$ [R.G.'05].

- A refinement of 't Hooft's idea of associating a genus to double line Feynman graphs [R.G. '04].

BACKGROUND

ORBIFOLD CORRELATORS AND COVERINGS

- Implement the Fields to Strings program in our test case. $CFT_2 = (T^4)^K/S_K$; $(K \to \infty)$.
- Consider $\left\langle \sigma^{(w_1)}(x_1)\sigma^{(w_2)}(x_2)\ldots\sigma^{(w_n)}(x_n)\right\rangle_{S^2}$ - ground states of w-cycle twisted sector.
- Insight of Lunin-Mathur['00] : can compute these by going to a covering space. [cf. replica trick]
- Vacuum path integral of single copy of T^4 CFT but branching behaviour at insertions of operators.
- Locally, $x = \Gamma(z)$ with branching w_i at insertions z_i : $x \sim x_i + a_i^{\Gamma}(z - z_i)^{w_i}$.

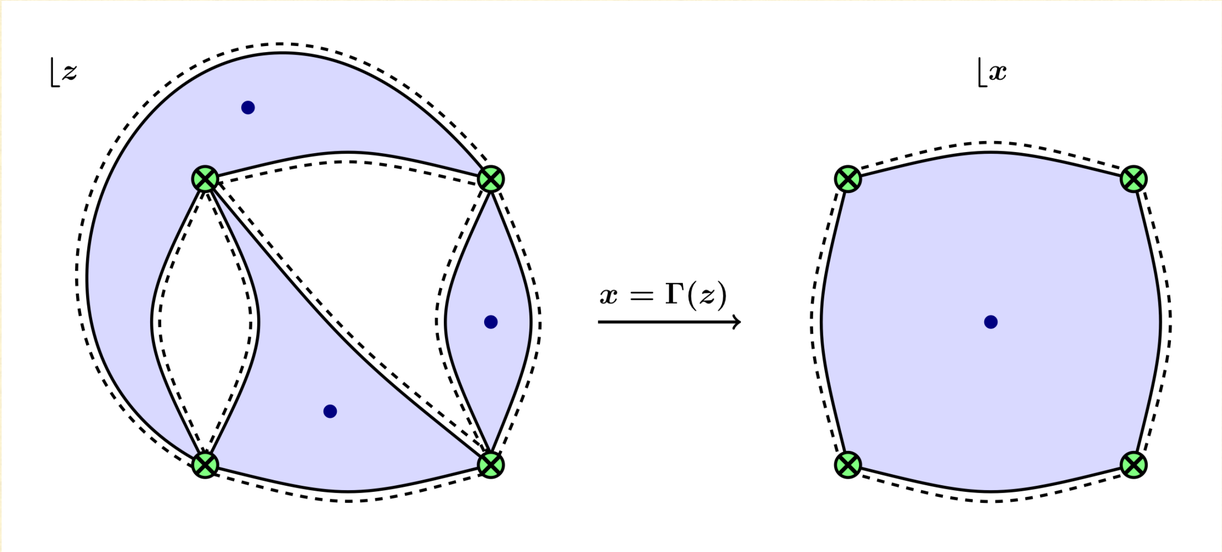


CALCULATING WITH COVERINGS

- Original correlator $\left\langle \sigma^{(w_1)}(x_1)\sigma^{(w_2)}(x_2)\ldots\sigma^{(w_n)}(x_n)\right\rangle_{S^2}$ given by a sum over contributions from all the allowed covering maps $x = \Gamma(z)$ with specified branching behaviour $x \sim x_i + a_i^{\Gamma}(z - z_i)^{w_i}$.
- Covering map specified by the data $\{z_i, w_i\}$ and three $x's$. Remaining $(n-3)$ of x_i determined.
- [Equivalently, specifying $\forall x_i, w_i$ determines $(n-3)$ of z_i - discrete set of points on $\mathscr{M}_{0,n}$.].
- Coordinate dependence comes from pullback $\partial\Gamma(z)$ - induced metric on covering space - and Liouville action of this conformal factor.
- Weight $\propto e^{-\mathcal{S}_L[\Phi=\ln|\partial\Gamma|^2]}$. Here $\mathcal{S}_L[\Phi] = \dfrac{c}{48\pi}\displaystyle\int d^2z[2\partial\Phi\bar{\partial}\Phi + R\Phi]$.
- With appropriate regularisation and normalisation, gives the ground state correlators.

FEYNMAN COVERINGS

- Can associate free field like Feynman diagram with each covering map contribution.
- Bifundamental like double line graph - pullback of Jordan curve on spacetime S^2.
- $2w_i$ edges coming out of vertices $z_i = \Gamma^{-1}(x_i)$.
- N preimages of $x = \infty$ (poles of $\Gamma(z)$) in the coloured loops. N=degree of map.
- Graph triangulates the covering space (= worldsheet).



[Pakman-Rastelli-Razamat-'09]

COMPUTING COVERINGS

- Covering maps are hard to explicitly write down. Even for a four point function on sphere.
- Stick to genus zero covering space, but n-branch points and degree N maps ($\Gamma(z) : \Sigma_{0,n} \to S^2$).
- $$\Gamma(z) = \frac{p_N(z)}{q_N(z)} = \frac{p_N(z)}{\prod_{a=1}^{N}(z-\lambda_a)} \Rightarrow \quad \partial\Gamma(z) = M_\Gamma \frac{\prod_{i=1}^{n-1}(z-z_i)^{w_i-1}}{\prod_{a=1}^{N}(z-\lambda_a)^2} \qquad (z_n = \infty)$$
- Requiring no simple pole at $z = \lambda_a$ $\Rightarrow \sum_{i=1}^{n-1} \frac{w_i - 1}{\lambda_a - z_i} = \sum_{b\neq a}^{N} \frac{2}{\lambda_a - \lambda_b}$, $\quad (a = 1,\ldots,N)$.

`Scattering Equations' - to be solved for the N poles: $\lambda_a' s$ [Roumpedakis -'18]

TECHNICALITIES

A GROSS-MENDE LIMIT

- Study $\left\langle \sigma^{(w_1)}(x_1)\sigma^{(w_2)}(x_2)\ldots\sigma^{(w_n)}(x_n)\right\rangle_{S^2}\Big|_{g=0}$ in a special limit where dimensions/energies are large.

  [Gaberdiel-R.G.-Knighton-Maity-'20]

- Recall correlator gets contributions from a finite number of covering maps $\sim N^{2n-6}$ where $N = 1 + \sum_{i=1}^{n} \frac{w_i - 1}{2}$ = degree of the map (Riemann-Hurwitz).

- Thus Lunin-Mathur covering maps at a finite number of points on moduli space $\mathscr{M}_{0,n}$.

- How do we see the full stringy moduli space $\mathscr{M}_{0,n}$ if we start with $\left\langle \sigma^{(w_1)}(x_1)\sigma^{(w_2)}(x_2)\ldots\sigma^{(w_n)}(x_n)\right\rangle_{S^2}$?

- Take $w_i \to \infty$; $\frac{w_i}{N} = \alpha_i$ fixed. Gross-Mende like limit $h_i = \frac{w_i^2 - 1}{4w_i} \to \frac{\alpha_i N}{4}$.

COVERING MAPS & A MATRIX MODEL

- Find the covering maps in this limit. Recall $\partial\Gamma(z) = M_\Gamma \dfrac{\prod_{i=1}^{n-1}(z-z_i)^{w_i-1}}{\prod_{a=1}^{N}(z-\lambda_a)^2}$

- Scattering equations become $\Rightarrow \displaystyle\sum_{i=1}^{n-1} \frac{\alpha_i}{\lambda_a - z_i} = \frac{1}{N}\sum_{b\neq a}^{N} \frac{2}{\lambda_a - \lambda_b}\,, \qquad (a = 1,\ldots,N).$

- Saddle point of a large N Penner-like matrix model with potential $W(z) = \displaystyle\sum_{i=1}^{n-1} \alpha_i \log(z - z_i)\,;$

  [cf. Dijkgraaf-Vafa-'09]

  Solve for the resolvent $u(z) = \displaystyle\int_C \frac{\rho(\lambda)}{z-\lambda}$ where $\rho(\lambda) = \dfrac{1}{N}\displaystyle\sum_{a=1}^{N}\delta(\lambda - \lambda_a).$

  [Gaberdiel-R.G.-Knighton-Maity - '20]

LOOP EQUATIONS

- Resolvent obeys the loop equation (in terms of $y(z) = W'(z) - 2u(z)$) [Wadia -'80]

$$y^2(z) - \frac{2}{N} y'(z) = \left(W'(z)\right)^2 - \frac{2}{N} W''(z) - 4R(z)$$

Defines the spectral curve of the matrix model (with $R(z) = \frac{1}{N} \sum_{a=1}^{N} \frac{W'(\lambda_a) - W'(z)}{(\lambda_a - z)}$).

- The spectral curve captures the original covering map $\Gamma(z)$. By definition:

Liouville Field of Lunin-Mathur

$$y(z) = \sum_{i=1}^{n-1} \frac{\alpha_i}{(z - z_i)} - \frac{2}{N} \sum_{a=1}^{N} \frac{1}{(z - \lambda_a)} = \frac{1}{N} \frac{\partial^2 \Gamma(z)}{\partial \Gamma(z)} = \frac{1}{N} \partial \ln \partial \Gamma = \frac{1}{N} \partial \Phi$$

TO LEADING ORDER…

- At large N, the spectral curve $y_0(z)$ obeys $y_0^2(z) = \left(W'(z)\right)^2 - 4R_0(z)$.

Polynomials

- $$y_0^2(z) = \frac{\tilde{W}_{n-2}^2(z) - \prod_{i=1}^{n-1}(z - z_i)\tilde{R}_{n-3}(z)}{\prod_{i=1}^{n-1}(z - z_i)^2} \equiv \frac{Q_{2n-4}(z)}{\prod_{i=1}^{n-1}(z - z_i)^2}\,.$$

- Has n double poles at $z = z_i$. And $(2n-4)$ zeroes $(z = a_k)$ - of the polynomial $Q_{2n-4}(z)$.

- Take as unknowns, $(n-3)$ parameters in $\tilde{R}_{n-3}(z)$ and $(n-3)$ cross ratios z_i.

- Fixed by $(2n-6)$ periods: $\frac{1}{2\pi i}\oint_{A_l} y_0(z)dz \equiv \nu_l\,,\quad \frac{1}{2\pi i}\oint_{B_l} y_0(z)dz \equiv \mu_l$. Parametrises the different covering map solutions. $y_0(z) = \partial \ln \partial\Gamma$ to leading order. Will see the interpretation of these parameters soon.

…AND A BIT BEYOND

- Going back to the $\frac{1}{N}$ corrected loop equation, we have $y^2(z) - \frac{2}{N}y'(z) = -\frac{2}{N^2}\left[\frac{\Gamma'''}{\Gamma'} - \frac{3}{2}\left(\frac{\Gamma''}{\Gamma'}\right)^2\right]$

Schwarzian of the covering map

$$\Rightarrow -\frac{2}{N^2}S[\Gamma] = \left(W'(z)\right)^2 - \frac{2}{N}W''(z) - 4R(z) = \frac{\tilde{Q}_{2n-4}(z)}{\prod_{i=1}^{n-1}(z-z_i)^2}$$

- Note that the left hand side transforms as a quadratic differential.

- It's behaviour near the double poles are $\sim \frac{w_i^2 - 1}{N^2}\frac{1}{(z-z_i)^2}$. (Residue $\sim \alpha_i^2$ to leading order)

MEANING

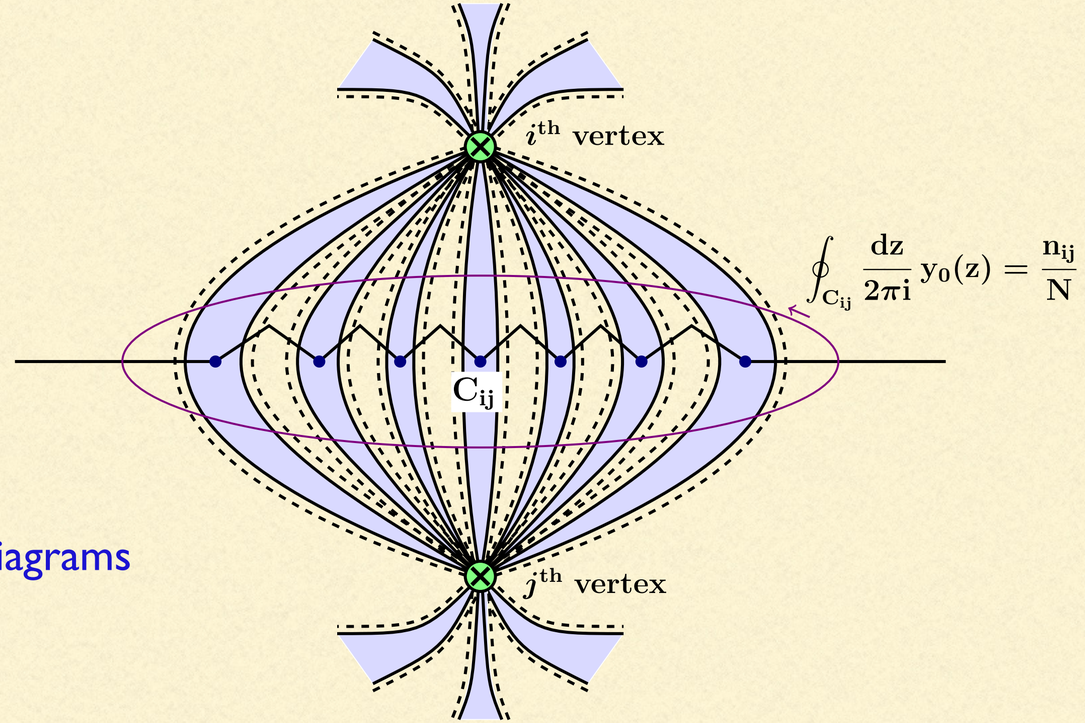
DIAGRAMMATIC INTERPRETATION

- The spectral curve $y_0(z)$ - determines 'eigenvalue density' of poles λ_a in coloured loops of Feynman diagrams - coalesces into cuts, transverse to the edges of the graph.

- $$y_0^2(z) = \frac{Q_{2n-4}(z)}{\prod_{i=1}^{n-1}(z-z_i)^2} = \frac{\alpha_n^2\, dz^2}{\prod_{i=1}^{n}(z-z_i)^2}\prod_{k=1}^{2n-4}(z-a_k);$$
  $(n-3)$ branch cuts between the zeroes $\{a_k\}$.

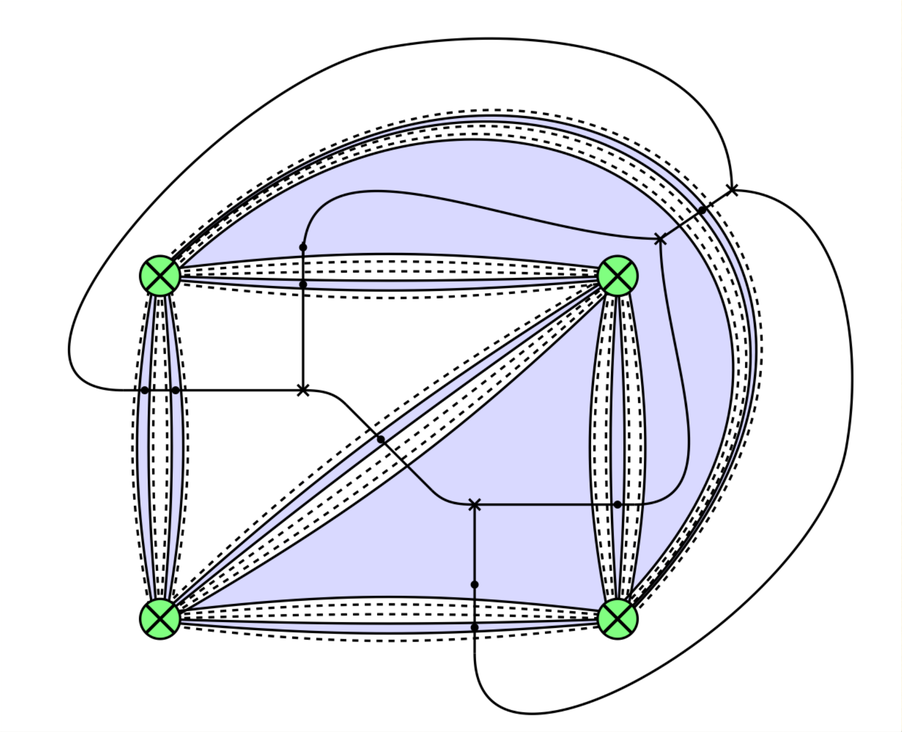
- The periods count number of wick contractions n_{ij}.
  The constraint $\sum_{j\neq i} n_{ij} = 2w_i$ follows from residues at the poles z_i.

- $(2n-6)$ independent $\{n^{(l)}, \tilde{n}^{(l)}\}$ which parametrise different Feynman diagrams and therefore inequivalent coverings $\Gamma(z)$.
  $$\frac{1}{2\pi i}\oint_{A_l} y_0(z)dz \equiv \nu_l = \frac{n^{(l)}}{N}\ , \quad \frac{1}{2\pi i}\oint_{B_l} y_0(z)dz \equiv \mu_l = \frac{\tilde{n}^{(l)}}{N}$$

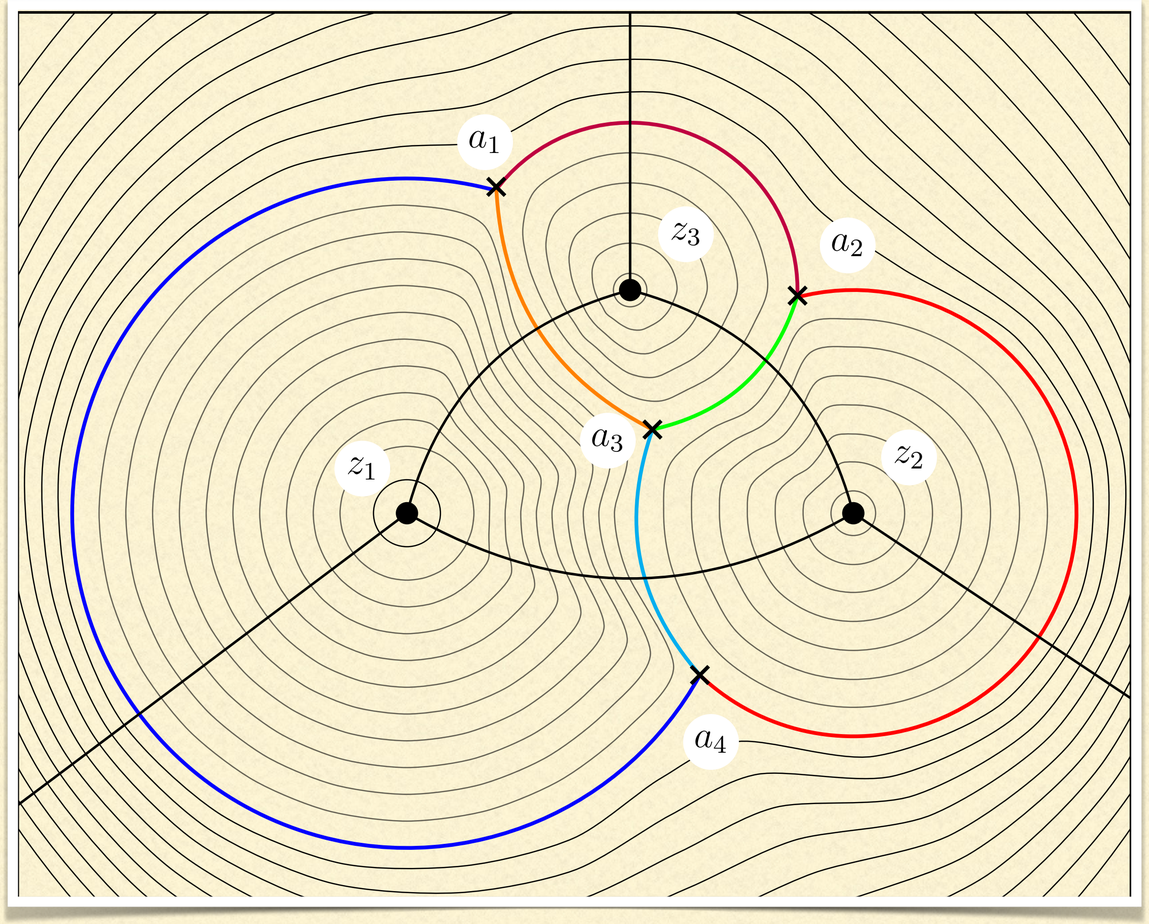


THE SKELETON GRAPH & IT'S DUAL

- Global picture therefore of a cut system dual to the edges of the (skeleton) Feynman diagram. Dual graph has $(3n-6)$ edges, n faces (each with a pole z_i) and $(2n-4)$ vertices $\{a_k\}$.
- Geometric significance of this graph follows from a remarkable property of the spectral curve.
- $-y_0^2(z)dz^2 \equiv \phi_S(z)dz^2$ is a Strebel differential.
- Unique meromorphic quadratic differential on $\Sigma_{0,n}$ with double poles at z_i (with real residues) and real Strebel lengths (periods) i.e. $\int_{a_k}^{a_m} \sqrt{\phi_S(z)}dz = l_{km} \in \mathbb{R}_+$.



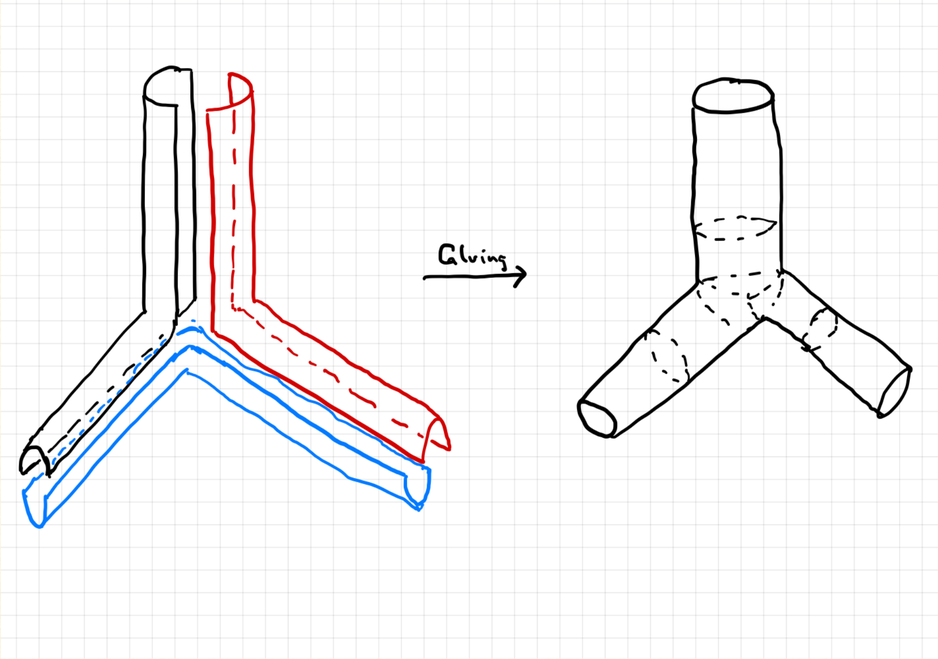
THE STREBEL GRAPH



- The Strebel differential foliates the Riemann surface into closed `horizontal trajectories': $\phi_S(z(t))\left(\frac{dz(t)}{dt}\right)^2 > 0.$
- Disk domains (faces) each containing one of the n double poles.
- Separated by a critical graph connecting the zeroes.
- Cuts that form the graph dual to the (skeleton) graph for the Feynman diagram.

IMPLEMENTING OPEN-CLOSED DUALITY

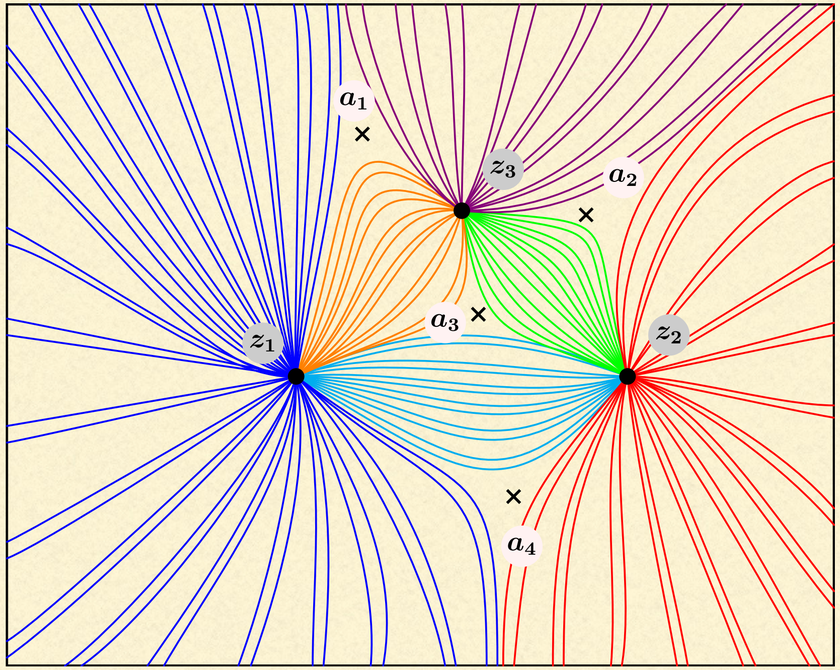
- Strebel differentials give a one to one parametrisation of $\mathscr{M}_{0,n}$ (for fixed residues α_i) through the $(2n-6)$ real lengths l_{km}.
- Associate a closed string surface to the Strebel graph.
- An implementation of gluing: for each Feynman graph, a dual Strebel graph which specifies how the ribbon graphs close up to the dual string worldsheet.
- Precise realisation of the prescription for open-closed duality. [R.G. '04-'05].



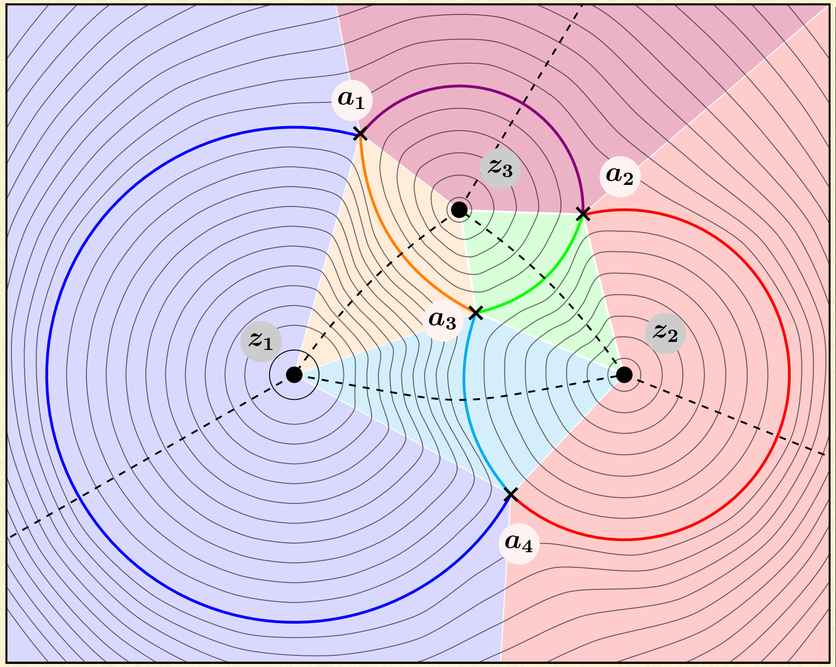
- In the present case, at large N, $l_{km} \propto n_{km}/N$ - similar to Razamat-'08. Takes continuous values.

OPEN-CLOSED TRIPTYCH

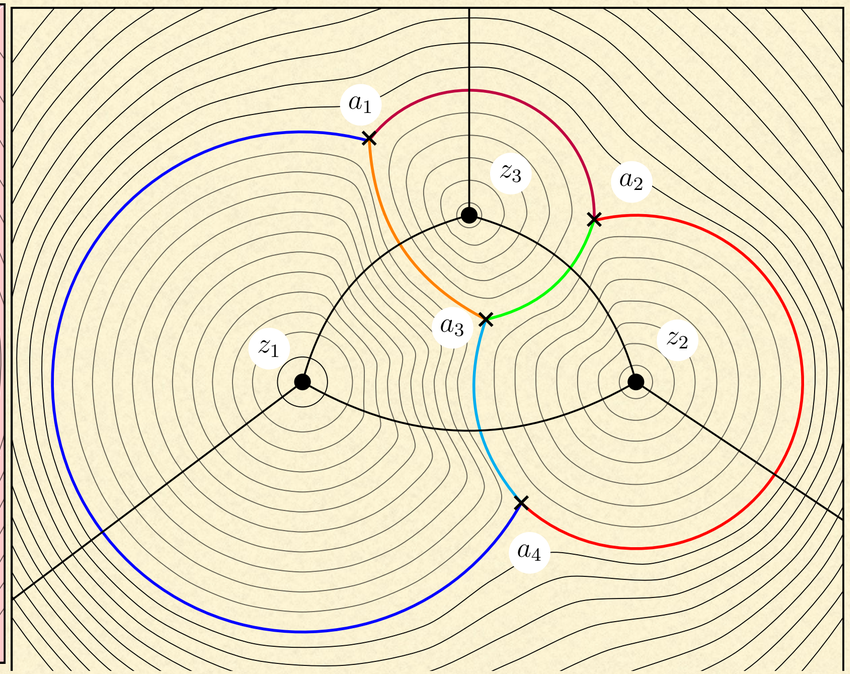
1) Ribbon Graphs



2) Glued up Strips



3) Strebel Surface



ANSWERS

THREE AVATARS

- Sum over all the inequivalent covering maps (i.e. Feynman diagrams) goes over to an integral over moduli space $\mathscr{M}_{0,n}$: $$\sum_{\{n_{ij}\}} \longrightarrow \int \prod_{l=1}^{n-3} [d\nu_l d\mu_l] = \int_{\mathscr{M}_{0,n}} |\omega^{(n-3)}(z_i)|^2.$$

- What about the integrand? Lunin-Mathur $\propto e^{-\mathcal{S}_L[\Phi]}$ with the Liouville action $$\mathcal{S}_L[\Phi] = \frac{c}{48\pi}\int d^2z[2\partial\Phi\bar{\partial}\Phi + R\Phi].$$ Recall $\partial\Phi \sim N y_0(z) \sim N\sqrt{-\phi_S(z)} \sim \sqrt{S[\Gamma]}$.

- To leading order, a Schwarzian action $\mathcal{S}_L[\Phi] \sim \int d^2z \left| S[\Gamma(z)] \right|$.

- Compare with AdS_2 case, for softly broken conformal symmetry [Maldacena-Stanford-Yang -'16].

FROM FIELDS TO STRINGS

- The weight on moduli space can also be viewed in more worldsheet terms - the Strebel differential $\phi_S(z)dz^2$ defines an almost flat worldsheet metric $ds^2 = |\phi_S(z)|dzd\bar{z}$.

- The Liouville action $\mathcal{S}_L[\Phi] \sim \int d^2z\sqrt{\det(g_S)}$ - Nambu-Goto action with induced Strebel metric.

- Alternatively, $\mathcal{S}_L[\Phi] \sim \int d^2z \frac{1}{\partial X \bar{\partial} \bar{X}} \partial^2 X(z)\, \bar{\partial}^2 \bar{X}(\bar{z})$ (where $X(z) \equiv \Gamma(z)$ parametrises boundary S^2). Similar to the action of the pre-holographic "rigid string". [Polyakov, Kleinert-'86]

- All these forms match with the on-shell AdS_3 sigma model action of the correlator in the tensionless limit - with $\Phi(z, \bar{z})$ being the radial direction. [Eberhardt-Gaberdiel-R. G. -'19].

OUTLOOK

- Exhibited a test case where the general program of reassembling large N QFTs into string theories can be explicitly carried out.
- Could also close the circle from strings to fields due to a tractable worldsheet theory.
- Holds out the hope that we can use this program to generalise to large N QFTs in general.
- General lessons from the worldsheet theory - underlying topological string,…

  [Gaberdiel-R.G.]
- General lessons for field theories - underlying geometric picture of Feynman diagrams….

  [Bharathkumar-Bhat-R.G.-Maity]
- Laundry list of problems [Gaberdiel, R.G., Knighton-Maity,'20].

THANK YOU