Nilpotent Slodowy slices and W-algebras

(joint work with Tomoyuki Arakawa and Jethro van Ekeren)

Darboux seminar

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Vertex operator algebras

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It is also required that V is $\mathbb{Z}$-graded with $\dim V_n < \infty$ for each n.

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and R_V is the *Zhu's C_2 algebra.*

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- *The associated variety* $X_V = \mathrm{Specm}\, R_V$ *captures important properties of* V*.*

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$$\mathbb{V}\colon \{\text{4D } \mathcal{N}=2 \text{ SCFT's}\} \longrightarrow \{\text{2D chiral CFTs } (\textit{VOAs})\}$$

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For any 4D $\mathcal{N} = 2$ SCFT $\mathcal{T}$, we have

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It is *lisse* if $X_V = \{\text{point}\}$.

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The associated variety $X_{L_k(\mathfrak{g})}$ is difficult to compute in general !

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Example : $L_k(\mathfrak{g}) \cong L(k\Lambda_0)$ is admissible if and only if

$$k = -h^\vee + p/q, \text{ with } (p,q) = 1 \text{ and } \begin{cases} p \geqslant h^\vee & \text{if } (q, r^\vee) = 1, \\ p \geqslant h & \text{if } (q, r^\vee) \neq 1. \end{cases}.$$

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- $L_k(\mathfrak{g})$ is admissible $\Rightarrow X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k}$, for some nilpotent orbit $\mathbb{O}_k$ of $\mathfrak{g}$ [Arakawa '15].

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$L_k(\mathfrak{g})$ is quasi-lisse $\iff X_{L_k(\mathfrak{g})} \subset \mathcal{N}$, the nilpotent cone of $\mathfrak{g}$.

$$\{\text{integrable } \widehat{\mathfrak{g}}\text{-repr.}\} \quad \subsetneqq \quad \{\text{admissible } \widehat{\mathfrak{g}}\text{-repr.}\} \quad \subseteq \quad \{\text{modular inv. repr.}\}.$$

Example : $L_k(\mathfrak{g}) \cong L(k\Lambda_0)$ is admissible if and only if

$$k = -h^\vee + p/q, \text{ with } (p,q) = 1 \text{ and } \begin{cases} p \geqslant h^\vee & \text{if } (q, r^\vee) = 1, \\ p \geqslant h & \text{if } (q, r^\vee) \neq 1. \end{cases}.$$

- $L_k(\mathfrak{g})$ is admissible $\Rightarrow X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k}$, for some nilpotent orbit $\mathbb{O}_k$ of $\mathfrak{g}$ [Arakawa '15].

In particular, $L_k(\mathfrak{g})$ is quasi-lisse if k is admissible.

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▶ The VOAs $L_{-2}(D_4), L_{-3}(E_6), L_{-4}(E_7), L_{-6}(E_8)$ are precisely the VOAs that appeared in [BL^{2}PRvR] as the main examples of $\mathbb{V}(\mathcal{T})$!

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- More generally, the generic singularities has been determined ([Kraft and Procesi '81-82] in the classical types, [Fu-Juteau-Levy-Sommers ' 17] in the exceptional types).

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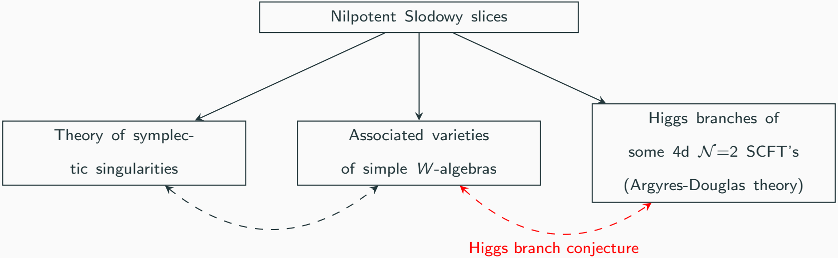

W-algebras and Argyres-Douglas theory

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  Conversely, from the coincidence of the singularities of different nilpotent Slodowy slices, we can guess many isomorphisms.

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For example, if $\mathcal{W}_k(\mathfrak{g}, f) \cong \mathbb{C}$, then k is collapsing.

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Motivations

- Collapsing levels are important in the Argyres-Douglas theory.

- If k is collapsing, the vertex algebra homomorphism $\mathcal{W}^k(\mathfrak{g}, f) \twoheadrightarrow \mathcal{W}_k(\mathfrak{g}, f) \cong L_{k^\natural}(\mathfrak{g}^\natural)$ induces an algebra homomorphism,

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When k and $k^\natural$ are admissible, such coincidences can be understood by considering singularities of nilpotent Slodowy slices...

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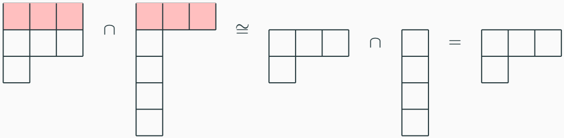

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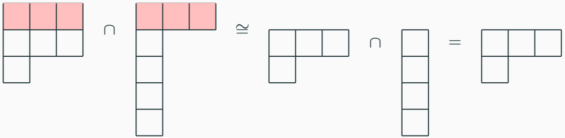

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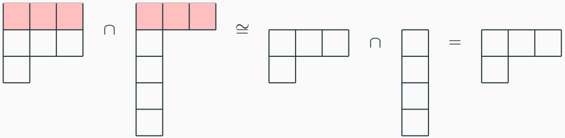

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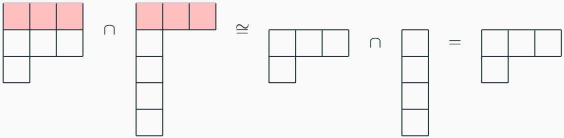

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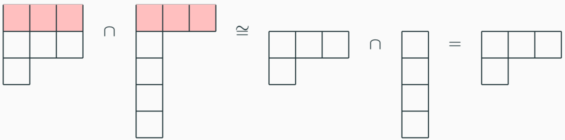

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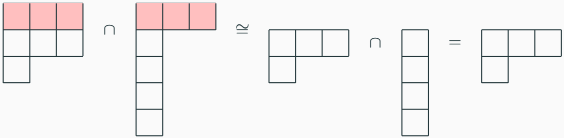

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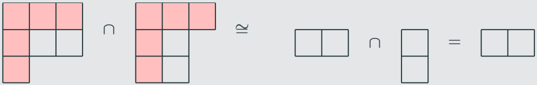

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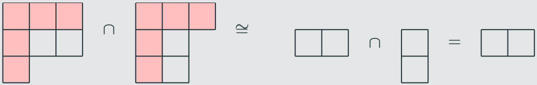

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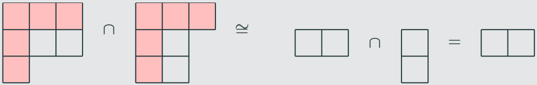

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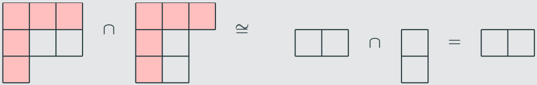

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Thank you !