Back to the future with the ε expansion

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February 11 2021

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Phase transitions and CFTs

Quantum field theory post-1970 has close ties to Landau paradigm. Rough idea: continuous phase transition between symmetry-preserving and -breaking phases = fixed point of effective action for order parameter.

Such an order parameter can be a scalar ($\mathbb{Z}_2$ magnets = Ising), a vector, or even a matrix (liquid ^{3}He) — it transforms as some irrep of a global symmetry group G.

Playbook to studying phase transition:

1. determine global symm. group G + write down order parameter(s)
2. write down all relevant G- and Lorentz-singlets
3. study RG diagram, pick out the fixed points, determine critical exponents.

Step 3 is delicate — often requires Monte Carlo or similar approaches.

Bootstrap?

This point of view is partially outdated. A continuous phase transition has conformal symmetry — and we should use this $SO(d+1,1)$ judiciously.
Reflection positivity (when applicable) is also extremely constraining.

Consistent with universality, there are only finitely many fixed points with given dimension d, symmetry G and say central charge $\leqslant c_{\text{max}}$.
As matter of principle, can study these "exactly". Success for 3d Ising and $O(N)$ (+ many other) universality classes.

Becomes more and more tedious if G is "small" and/or there are many fields. But long-term vision is clear: should explore the atlas of conformal field theories.

Epsilon expansion

We want a large class of non-supersymmetric toy models. Most RG flows are long: they change scaling dimensions and central charges by $O(1)$ quantities — can't be studied analytically.

To get a short RG flow, need a small parameter. Wilson's idea is to work in $d - \varepsilon$ dimensions, where there are couplings g^i of mass dimension ε, and beta functions of the form

$$\beta^i(g^j) = -\varepsilon g^i + C^i_{\ jk}\, g^j g^k + \dots$$

for some $C^i_{\ jk}$.

There can be fixed points $\beta^i(g^j_*) = 0$ due to cancellations:

$$g^i_* = \#\varepsilon + \#\varepsilon^2 + \#\varepsilon^3 + \dots$$

so ε is good small parameter for perturbation theory. Need to set $\varepsilon \to 1$ to get real physics.

Multiscalar fixed points

Simplest setup that can give 3d CFTs: $d = 4 - \varepsilon$ + quartic interactions.
Most general theory with N scalars:

$$\mathcal{L} = (\partial_\mu \phi^i)^2 + \mu^\varepsilon V(\phi^i), \quad V(\phi^i) = \lambda_{ijkl}\, \phi^i \phi^j \phi^k \phi^l.$$

Impose $\mathbb{Z}_2$ to rule out cubic terms. One-loop β functions

$$\beta(\lambda)_{ijkl} = -\lambda_{ijkl} + \lambda_{ijmn}\lambda_{klmn} + \lambda_{ikmn}\lambda_{jlmn} + \lambda_{ilmn}\lambda_{jkmn}$$

setting $\varepsilon = 1$.

Scheme dependence feeds into higher loops. Can all be ignored today.
(Higher loops only make sense after fixing one-loop.)

Only focus: examine solutions to $\beta(\lambda) = 0$.

Multiscalar CFTs (2)

Well-explored since birth of RG [Wilson 1972] + many, many others.

On the menu today:

- Can we classify all solutions for a given N?
- Can we put rigorous bounds on theory space, *à la* bootstrap?
- Can we prove theorems about observables?

Theories with N scalars + quartic interaction are not the end of the story. Much can be recycled in other contexts (ϕ^n interactions, other matter content). Not covered today.

Philosophy goes back to fundamental paper [Brézin–Le Guillou–Zinn-Justin 1973] for isotropic systems, follow-up work by Michel and many others. Recent revival of this strategy by [Osborn-Stergiou 2017], [Rychkov-Stergiou 2018], [Codello et al. 2019, 2020].

Important solutions

Trivial theory $V(\phi) = 0$. Ising = Wilson-Fisher theory

$$V(\phi) = \frac{1}{3}\phi^4$$

or generalization to $O(N)$, $V(\phi^i) \propto (\phi^i\phi^i)^2$. Other known solutions:

- cubic: $O(N)$ deformed by $\sum_i \phi_i^4$ w/ discrete symmetry group
- biconical-type solutions with symmetry $O(m) \times O(m')$.
- ...

No solutions known with $G = \mathbb{Z}_2$ beyond Ising. Exploration so far based on group theory (model building with large G).

Discussion of (an)isotropy

From Landau point of view, need to impose additional group-theoretical inputs = "isotropy":

- ϕ^i irrep of G
- unique quadratic invariant δ_{ij}

Useful to make connection with experiments (reduce # of relevant operators), but not for writing CFT atlas.

Classification of isotropic CFTs undertaken in [Wallace-Zia 1975] for $N = 2, 3$, [Brézin-Michel-Tolédano-Tolédano 1985] for $N = 4$.

State of the art

Beyond model building, classification?
This is hard: # of couplings $\sim \frac{1}{24} N^4$.

- $N = 1$: textbook.
- $N = 2$: solved in by [Osborn-Stergiou 2017]
- $N = 3$: 15 equations, too much for analytics;
  numerical hints that there is nothing more (conjecture).

New as of 2020: numerics [Codello et al., Osborn-Stergiou].

Can put an upper bound on number of CFTs using algebraic geometry:

$$\text{number of fixed points} < \exp(0.0458 N^4)$$

Seems to overcount badly (but how to prove this?).

Invariants

Given a potential $V(\phi^i)$, can perform $O(N)$ field redefinition

$$\phi^i \mapsto R^i{}_j \, \phi^j$$

or equivalently, change couplings. This is *not* a symmetry, just redundancy in description.

Take-away: individual couplings like λ_{1111} don't have physical meaning. Invariants like

$$\lambda_{iijj}, \quad \|\lambda\|^2 = \lambda_{ijkl}^2, \quad \text{eigenvalues of } \Lambda_{ij} := \lambda_{ijkk}, \quad \ldots$$

do.

Known results

Most basic invariant: norm

$$\|\lambda\|^2 = \lambda_{ijkl}^2 \geqslant 0.$$

Rychkov-Stergiou recently showed that

$$\beta(\lambda) = 0 \;\Rightarrow\; \|\lambda\|^2 \leqslant \begin{cases} \frac{1}{36}(3+4\sqrt{2}) \approx 0.240468 & N=2 \\ \frac{1}{12}(1+2\sqrt{3}) \approx 0.372008 & N=3 \\ \frac{1}{8}N & N \geqslant 4 \end{cases}.$$

Interpretation: fixed points can't live in the whole space of dimension $\sim N^4$, they live inside a sphere of radius $\sim \sqrt{N}$.
Proof: bound individual elements of λ_{ijkl} using $\beta = 0$.

Other invariants

There is one linear invariant:

$$a_0 := \lambda_{iijj}.$$

In addition to $\|\lambda\|^2$, one extra quadratic invariant:

$$a_2 = \frac{6}{N+4}\left[\lambda_{ijkk}^2 - \frac{1}{N}\, a_0^2\right] \geqslant 0$$

defined such that $a_2 = 0$ for isotropic theories.

Explorations

Can write simple Python code and hunt for fixed points for given N.
Can use this to draw points in theory space.
Some heuristics:

- all fixed points have $a_0 \geqslant 0$, even though this invariant is not sign-definite
- there's no theory other than Gaussian with $\|\lambda\| < 0.33$, but 0.33 does occur.

Let's prove this.

Lower bound on $\|\lambda\|$

We'll argue that any fixed point must satisfy:

$$\lambda_{ijkl} = 0 \quad \text{or} \quad \|\lambda\| \geqslant \frac{1}{3}.$$

Or: can't have arbitrarily weak CFTs.

Proof: fixed point obeys

$$\lambda_{ijkl} = \lambda_{ijmn}\lambda_{klmn} + 2 \text{ terms.}$$

Now bound first term on RHS using Cauchy-Schwartz:

$$\sum_{mn}(\lambda_{ijmn}\lambda_{klmn})^2 \leqslant \sum_{mn}\lambda_{ijmn}^2 \sum_{pq}\lambda_{klpq}^2 \;\Rightarrow\; \|\text{RHS}\| \leqslant 3\|\lambda\|^2.$$

But then

$$\|\lambda\| \leqslant 3\|\lambda\|^2 \;\Rightarrow\; 3\|\lambda\|(\|\lambda\| - 1/3) \geqslant 0. \quad \square$$

Lower bound (2)

C-S argument gives info about limiting cases.
Here: learn that the bound is saturated if there exist matrices R, S such that

$$\lambda_{ijkl} = R_{ij} S_{kl}.$$

By permutation symmetry of λ_{ijkl} can argue that $R = S$ and

$$(S^2)_{ij} = \mathrm{tr}(S) S_{ij}.$$

But then every eigenvalue ν of S must obey

$$\nu = 0 \quad \text{or} \quad \nu = \mathrm{tr}(S).$$

Only possible if $\leqslant 1$ non-zero eigenvalue. So $\exists\, u_i$ s.t. $S_{ij} \propto u_i u_j$ which implies

$$\text{bound saturated} \quad \Leftrightarrow \quad \|\lambda\| = 1/3 \quad \Leftrightarrow \quad V(\phi) = \frac{1}{3}(u_i \phi^i)^4.$$

Conclusion: Ising model is the most weakly-coupled CFT, for any N!

More bounds

To proceed: notice that any fixed point obeys

$$\lambda_{iijj} = \lambda_{iimn}\lambda_{jjmn} + 2 \text{ terms}$$

and decompose this into invariants. Messy but doable:

$$\frac{1}{2N}\, a_0(N - a_0) = \|\lambda\|^2 + \frac{N+4}{12}\, a_2.$$

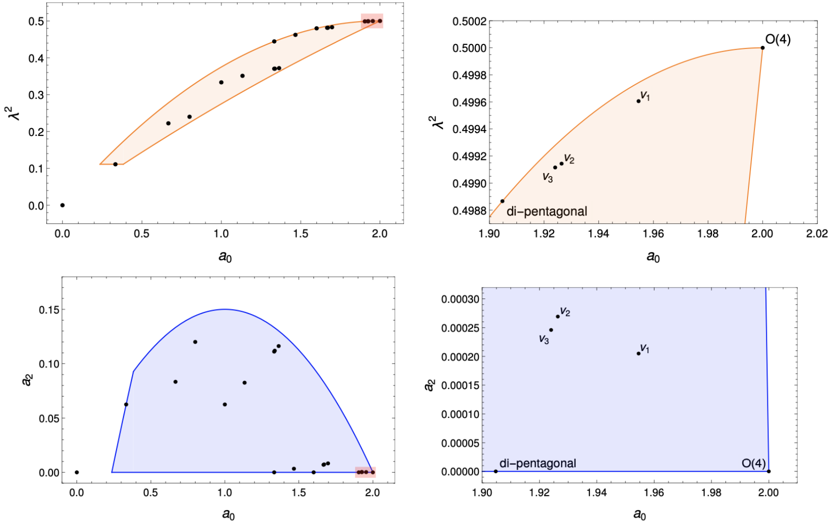
Constraining relation between the different invariants (only valid at fixed points).
Finally, can show that

$$\|\lambda\|^2 - a_2 - \frac{3}{N(N+2)}(a_0)^2 \geqslant 0.$$

This comes from decomposing λ_{ijkl} into 3 irreps → 3 positivity conditions.

Combined results

Bringing together the different bounds we've collected, get tight allowed regions (here $N = 4$):



Comments on spectra

Composite operators $\mathcal{O}_a$ have an anomalous dimension at one loop:

$$\Delta[\mathcal{O}_a] = \Delta_{\text{classical}}[\mathcal{O}_a] + \gamma_a \, \varepsilon + O(\varepsilon^2).$$

For simple theories, a finite number of them determine the critical exponents $\nu, \gamma, \ldots$.

Operators mix under RG. Determining γ_a means diagonalizing a mixing matrix. There are some selection rules, especially at order ε.
Focus on operators of form

$$[\mathcal{O}_a] = T_{a|i_1 \cdots i_r} \, \phi^{i_1} \cdots \phi^{i_r}$$

which have large degeneracy and only mix amongst themselves.
Coefficient tensors must obey equation of the form

$$\lambda_{ijkl} \, T_{a|kl} \propto \gamma_a \, T_{a|ij}$$

likewise for higher r.

Anomalous dimensions (2)

Since there is a huge degeneracy, what is the strongest statement we can make? First, get bound: for operators with r copies of ϕ, we find

$$[\mathcal{O}_a] \sim \phi^r : \quad |\gamma_a| \leqslant \frac{r(r-1)}{2} \|\lambda\|$$

saturated by Ising model — similar proof as before.

More interesting, can say something about averages i.e. sum rules

$$\langle \gamma^n \rangle_r = \frac{1}{(\#\text{ of operators})} \sum_a (\gamma_a)^n.$$

This tells you something about the statistics: what is the average anomalous dimension, what is the standard deviation?
Obey convexity condition.

Anomalous dimensions (3)

These averages are closely related to the invariants we discussed before. Can easily show that the sum $\langle\gamma\rangle_r$ is maximal at the $O(N)$ fixed point. This is an invariant way to characterize the $O(N)$ fixed point!

For the sum of squares $\langle\gamma^2\rangle_r$, the analysis is more involved. Can prove that as long as $N \leqslant 4\,(r - 9/8)^2$, we have

$$\langle\gamma^2\rangle_r^{\lambda} \leqslant \langle\gamma^2\rangle_r^{O(N)} < \frac{r(r-1)(r^2-r+1)}{(N+5)^2}.$$

Bootstrap philosophy: completely universal bounds that hold for all CFTs, only making use of unitarity.

$U(1)$ gauge theory

Can instead look at N complex scalars, imposing overall $U(1)$ + reality:

$$\mathcal{L} = |\partial_\mu \phi^i|^2 + g_{ijkl}\phi^i\phi^j\bar{\phi}^k\bar{\phi}^l.$$

(Sufficient for unitarity.) Note: charge conjugation not used as an input. Can be embedded in action with $2N$ real fields. No new interesting bounds, except Ising is replaced by $O(2)$ = XY model and $O(N)$ by $U(N)$.

Now gauge the $U(1)$ that rotates $\phi^i \mapsto \exp(i\alpha)\phi^i$ = bosonic QED:

$$\mathcal{L}' = \tfrac{1}{4}F_{\mu\nu}^2 + |D_\mu \phi^i|^2 + g_{ijkl}\phi^i\phi^j\bar{\phi}^k\bar{\phi}^l.$$

Beta function for coupling e reads

$$\beta(e) = -\frac{\varepsilon}{2}e + \frac{N}{48\pi^2}e^3 + \dots$$

so at any fixed point $e_*^2 = 0$ (previous situation) or $e_*^2 \sim \varepsilon/N > 0$.

Known cases

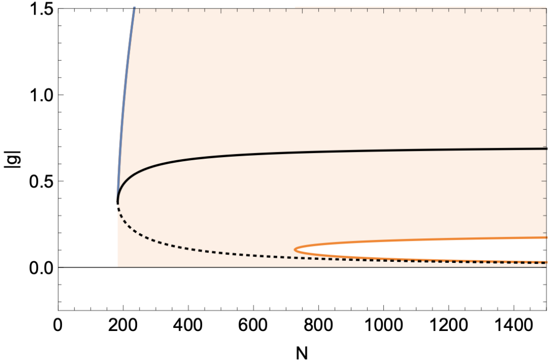
Well-known solution: $PSU(N) = SU(N)/\mathbb{Z}_N$ global symmetry, having an interaction

$$V(\phi) \propto (\sum_i |\phi^i|^2)^2$$

which famously exists only for $N \geqslant 183$.
Similar families of theories that exist only for $N \geqslant 198$ [Benvenuti et al. 2019].

Results

Gauge coupling e^2 interacts with the scalars through D_μ, and this feeds into the beta functions for the g_{ijkl}. The new terms are of order $1/N$ and $1/N^2$. Imhomogeneous equations.
To proceed, decompose g_{ijkl} into irreps of $PSU(N)$ instead of $O(N)$, but same logic applies.



No solutions at all for $N < N_* := 90 + 24\sqrt{15} \approx 182.9516$.

Discussion

- Should take the task of describing the CFT landscape seriously, and multiscalar theories are a logical place to start. Perhaps using some fresh ideas (machine learning?).
- Using simple group theory, can already show that fixed points live in a small sliver of theory space. Should be possible to do better.
- Machinery applies to many different frameworks (Yukawas).
- Many conjectures to prove (e.g. absence of theories with trivial symmetries). Almost nothing known about gauged CFTs.
- Possible to prove theorems about other elements of spectrum (e.g. spinning operators $\phi^i \partial^\ell \phi^j$), OPE coefficients? Primaries versus descendants.