An Exact Integrated Correlator in $\mathcal{N}=4$ SU(N) SYM

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Intro: 4d $\mathcal{N}=4$ SYM SU(N)

1) → Superconformal invariance
2) → Olive-Montonen E/M
3) → Pestun susy loc partition fct
4) → holo relation w/ IIB on $AdS_5 \times S^5$]

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1) ϕ_I $I=1,..6$

$$O_2(\underline{x}, Y) = \mathrm{Tr}(\phi_I \phi_J)\, Y^I Y^J$$

Y ↑ null polariz. vector

$[0,2,0]$, $20'$

sym. traceless rep of $SU(4)_R$

$1/2$ BPS $\Delta = 2$

$$\langle O_2(\underline{x}_1, Y_1)\, O_2(\underline{x}_2, Y_2)\rangle = \frac{Y_1 \cdot Y_2}{|\underline{x}_{12}|^4} \checkmark$$

$$\underline{x}_{12} = \underline{x}_1 - \underline{x}_2$$

4pt: $\langle O_2(\underline{x}_1) \ldots O_2(\underline{x}_4)\rangle$

$$= \frac{1}{|\underline{x}_{12}|^4}\, \frac{1}{|\underline{x}_{34}|^4}\, \mathcal{T}_N(U, V) \Big]$$

N ↑ SU(N) — U, V ↑ cross ratios

- nontrivial in U, V
- " in coupling const.

O_2 supercnf. primary in Stress-energy mult.

$$\langle T_{\mu_1\nu_1}(\underline{x}_1) \ldots T_{\mu_4\nu_4}(\underline{x}_4)\rangle$$

⇝ 4pt graviton Amplitudes in IIB

EM duality:

$$\tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{YM}^2}$$

$$y = \tau_2 \pi$$

Weak coupling: $g_{YM} \to 0 \leftrightarrow \tau_2 \gg 1 \leftrightarrow y \gg 1$

strong " : $g_{YM} \gg 1 \leftrightarrow \tau_2 \to 0 \leftrightarrow y \to 0$

Invariance:

$$\tau \to \gamma\cdot\tau = \frac{a\tau+b}{c\tau+d} \quad \text{w/ } \gamma = \underset{\substack{\Uparrow \\ SL_2(\mathbb{Z})}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boxed{\tau \to -\frac{1}{\tau}}$$

when $\theta = 0$: $g_{YM} \to \frac{1}{g_{YM}}$

Result: From Susy loc. we consider [Pestun]

$\mathcal{N} = 2^*$ mass deform. of $\mathcal{N} = 4$ on S^4

$$G_N(\tau,\bar\tau) = \frac{1}{4}\underline{\Delta_\tau}\,\underline{\partial_m^2} \log Z_N(m;\tau,\bar\tau)\Big|_{m=0}$$

$$= \int \prod_{i=1}^{4} dx_i\, \mu(\{x_i\}) \underline{\langle O_2(x_1) \cdots O_2(x_4)\rangle}$$

$$\Delta_\tau = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$$

[Binder, Chester, Pufu, Wang] ←

$$\left(\partial_m^4 \log Z_N \Big|_{m=0} = \int \tilde\mu(\{x_i\}) \langle O_2 \cdots O_2 \rangle \right)$$

$\langle O_2 O_2 O_p O_p \rangle$ ✓ | $\langle O_{p_1} O_{p_2} O_{p_3} O_{p_4} \rangle$?

$\partial_{\tau_p} \partial_{\bar\tau_p} \partial_m^2$ ↗

$$G_N(\tau,\bar\tau) = \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-t\frac{|m+n\tau|^2}{\tau_2}} \underline{B_N(t)}\,dt$$

$\tau \in \mathbb{H}$ ←

$$B_N(t) = \frac{Q_N(t)}{(1+t)^{2N+1}}$$

- $Q_N(t)$ a degree $2N-1$ pol. w/ $\mathbb{Z}$ coeff.

$$Q_2(t) = 9t - 30t^2 + 9t^3$$

$$B_N(t) = \frac{1}{t} B_N(1/t)$$

- Contains ∞ numb. of I & $\bar{I}$
- $SL_2(\mathbb{Z})$ manifest

Laplace diff.:

$$(\Delta_\tau - 2)\, G_N = N^2 \left[G_{N+1} - 2G_N + G_{N-1} \right] - N \left[G_{N+1} - G_{N-1} \right].$$

$\to$ Large N

$$(\Delta_\tau - 2)\, G_N \sim \left[N^2 \partial_N^2 - 2N\partial_N \right] G_N$$

$$\boxed{G_N(\tau,\bar\tau) \,\text{''}{=}\text{''}\, \frac{N(N-1)}{8} + \sum_{s=\frac{3}{2}}^{\infty} c_{(s)}^{(N)}\, E_{(s)}(\tau,\bar\tau)} \leftarrow \Delta$$

$$\Uparrow$$

$$C_x$$

$$E_s(\tau,\bar\tau) = \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{\tau_2^s/\pi^s}{|m+n\tau|^{2s}} = \underset{\uparrow}{\zeta_{2s}\, y^s} + \underset{\uparrow}{\zeta_{2s-1}\, y^{1-s}}$$

$$[\Delta - s(s-1)]\, E_s = 0$$

@ Large N

$$G_N(\tau,\bar\tau) \sim \frac{N^2}{4} + \sum_{\ell=0}^{\infty} N^{1/2-\ell} \sum_{s=3/2}^{\ell+3/2} d_\ell^s \, E_s(\tau,\bar\tau)$$

not 't Hooft! τ fixed as $N\to\infty$

[IIB] $\tau = \chi + i/g_s$

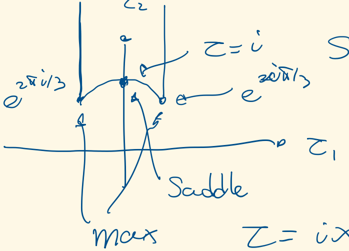
Convergence:

$$\sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-t\pi \frac{|m+n\tau|^2}{\tau_2}} \, B_N(t) \quad \Bigg|$$

$$\underset{O(t)}{\uparrow} \qquad O(t^{-2})$$

fixed m,n, $\tau\in\mathbb{H}$

$$\int_0^\infty \ldots \sim \frac{1}{|m+n\tau|^4}$$



$S\tau = \tau$

$\tau = i\infty$

$\tau = S\, i\infty = 0$ | min

$$g_N(\tau,\bar\tau) = \sum_{k\in\mathbb{Z}} g_{N,k}(\tau,\bar\tau)$$

$$= \sum_{k\in\mathbb{Z}} \underbrace{e^{2\pi i k \tau_1}}_{e^{i\theta k}} F_{N,k}(\tau_2)$$

$K>0$ I

$K<0$ $\bar{I}$

$K=0$ pert. th.

Checks & Predictions:

i) small g_{YM} & fixed N → SYM pert. Th.

ii) Large N fixed $\lambda = g_{YM}^2 N$ → 't Hooft limit

iii) Large N fixed τ → 4 graviton amplitudes.

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i) SYM pert. th. $\overset{\checkmark}{a} = g_{YM}^2 N/4\pi^2$ N fixed

$$g_{N,0}(\tau_2) = \underline{(N^2-1)}\Big[\frac{3\zeta_3}{2} a - \frac{75}{8}\zeta_5 a^2 + \frac{735}{16}\zeta_7 a^3 \neq \frac{6615}{32}\zeta_9 a^4\big(1 + \frac{2}{7}N^{-2}\big) + O(a^5)\Big]$$

4

$\langle \mathcal{O}_2\, \mathcal{O}_2\, \mathcal{O}_2\, \mathcal{O}_2 \rangle$

[[Drummond, Duhr, Eden, Heslop, Pennington, Smirnov]

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ii) $\lambda = g_{YM}^2 N$

$$\mathcal{G}_N(\tau, \bar\tau) = \mathcal{G}_{N,0}(\lambda) \sum_{g=1}^{\infty} N^{2-2g}\, \mathcal{G}^{(g)}(\lambda)$$

$$\mathcal{G}^{(0)}(\lambda) = \sum_{n=1}^{\infty} 4(-1)^{n+1} \frac{\zeta_{2n+1}\, \Gamma(n+3/2)^2}{\pi^{2n+1}\, \Gamma(n)\, \Gamma(n+3)} \lambda^n$$

$|\lambda| < \pi^2$

$$\mathcal{G}^{(0)}(\lambda) = \lambda \int_0^\infty dx\, x^3 \frac{J_1(x\sqrt{\lambda}/\pi)^2 - J_2(x\sqrt{\lambda}/\pi)^2}{4\pi^2 \sinh(x)^2} \nearrow$$

[Chester, Pufu]

$$\mathcal{G}^{(0)}(\lambda) = \frac{1}{4} + \sum_{n=1}^{\infty} \text{"}(2n+1)!\text{"} \frac{\zeta_{2n+1}}{(\sqrt{\lambda})^{2n+1}}$$

$$\pm\, i \left[\frac{127}{2^8} \mathrm{Li}_0(e^{-2\sqrt{\lambda}}) + \# \frac{\mathrm{Li}_1(e^{-2\sqrt{\lambda}})}{\lambda^{1/2}} \right.$$

$$\mathrm{Li}_0(e^{-2\sqrt{\lambda}}) \sim e^{-2\sqrt{\lambda}} + \dots + e^{-4\sqrt{\lambda}} + e^{-6\sqrt{\lambda}} + \dots$$

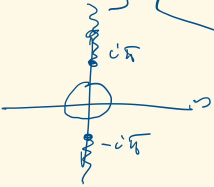
$$\exp(-2\sqrt{\lambda}) \longrightarrow \exp(-2L^2/\alpha')$$

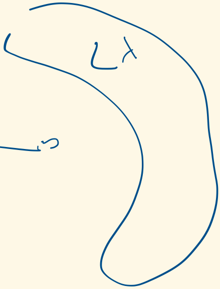
$\alpha' = (\text{string length})^2$

$L =$ AdS scale.

[world-sheet instantons]

$G^{(0)}(\lambda) \longrightarrow \infty$





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iii) Large N fixed τ

$$B_N(t) \sim N^{1/2}\left(\boxed{t^{1/2}} + \underline{t^{-3/2}}\right)$$
$$+ N^{-1/2}\left(t^{3/2} + t^{-5/2}\right)$$
$$+ N^{-3/2}\left(\#\left(t^{5/2} + t^{-7/2}\right) + \#\left(t^{1/2} + t^{-3/2}\right)\right)$$
$$+ \dots$$

$$E_s = \sum_{(m,n)\neq(0,0)} \int_0^\infty e^{-t\pi\frac{|m+n\tau|^2}{\tau_2}} \quad \boxed{\frac{t^{s-1}}{\Gamma(s)}}$$

$$E_s\,\Gamma(s) = E_{1-s}\,\Gamma(1-s)$$

we find

$$G_N(\tau,\bar\tau) \sim \boxed{\frac{N^2}{4}}\ \boxed{-\frac{3N^{1/2}}{2^4}E_{3/2}} + \frac{45}{2^8 N^{1/2}}E_{5/2} + \frac{1}{N^{3/2}}\underline{\left[\# E_{\frac{7}{2}} + \# E_{\frac{3}{2}}\right]}^{\downarrow}$$

[Chester, Green, Pufu, Wang, Wen] $+ \dots$

$\mathrm{II}\,B \qquad \tau = \chi + \dfrac{i}{g_s}$

4 graviton amplitude

$$\begin{tikzcd}
R & \boxed{R^4} & \boxed{D^4R^4} & \boxed{D^6R^4} \\
(\alpha')^{-4} \arrow[d] & (\alpha')^{-1} & \alpha' \arrow[d] & (\alpha')^{2} \arrow[d] \\
N^2 & \sqrt{N} & 1/\sqrt{N} & 1/N
\end{tikzcd}$$

[Chester, Green, Pufu, Wang, Wen]

$$\partial_m^4 Z_N \big| \sim \frac{1}{N}\, \boxed{E_{3/2,3/2}}$$

Open Questions

- $\partial_m^4 Z_N$: (II) $\qquad$ Eisenstein $\times$
- $N^{2-2g}\, G^{(g)}(\lambda) \qquad \exp(-N^{\#} f(\lambda))$

  $\wr$

  $N^{1/2-\ell}\, f(\tau,\bar\tau) \quad \rightsquigarrow \quad \exp(-N^{\#} g(\tau,\bar\tau))$
- geometric orig. behind $\quad Q_N(t) = \mathbb{Z}\, t^{i}$

  Lap. diff. eq?