Hydrodynamic diffusion and its breakdown near AdS_2 fixed points

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References and acknowledgments

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- All references and text in magenta contain hyperlinks.

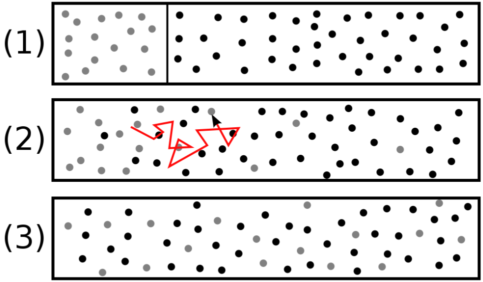
Hydrodynamics

- Universal description of interacting systems in the long wavelength $x \gtrsim \ell_{\rm eq}$, late time regime $t \gtrsim \tau_{\rm eq}$ ('small gradients').

- Based on a truncation of the dynamics to the relaxation of a few conserved densities, following from the symmetries of the system. Eg in a fluid: conservation of translations, rotations, particle number and possibly boosts.

- Provides an effective description of many interesting systems that cannot be described perturbatively: liquid phase of water, electrons in ultra-pure Graphene, Quark-Gluon-Plasma, superfluids, etc.

Classical diffusive hydrodynamics



CREDIT: [WIKIPEDIA]

- In this talk, I will focus on diffusive hydrodynamics: relaxation of the gradient of a conserved density ρ.

- Examples: particle number (chemical potential), energy (temperature gradient), shear momentum (transverse velocity), etc.

Classical diffusive hydrodynamics

(see eg [Kovtun'12] for more details)

- Conservation equation (Fick's law)

$$\partial_t \rho + \nabla_i j^i = 0$$

- Constitutive relation

$$j^i = -\frac{D}{\chi} \nabla^i \rho + O(\nabla^2, \partial_t \nabla)\,, \quad \chi = \frac{\partial \rho}{\partial \mu}$$

- Decompose linear perturbations in plane waves

$$\rho(t, x) = \rho_0 + \delta\rho e^{-i\omega t + ikx}$$

- Retarded Green's function

$$G^R_{\rho\rho}(\omega, k) = \frac{i\chi D k^2}{\omega + iDk^2}$$

- In this low frequency, low k approximation, single, diffusive pole

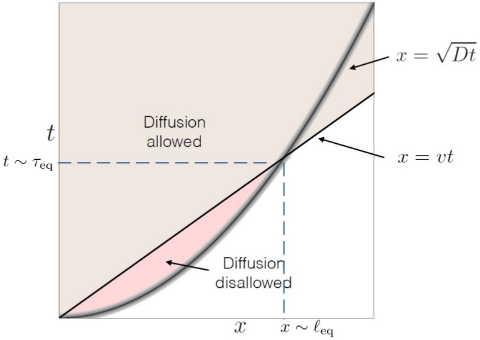
$$\omega = -iDk^2 + O(k^4) \quad \Rightarrow \quad G_{\rho\rho}(t, x) \sim \nabla^2 \frac{e^{-x^2/(4Dt)}}{t^{d/2}}$$

Limits of applicability: causality

Causality: upper bound on the diffusivity [Hartman et al'17]

$$D \lesssim v^2 \tau_{\mathrm{eq}}$$

- v: 'effective lightcone velocity'. General argument from linear in t growth of operators.
- Emergent infrared Lorentz invariance: expect $v = c_{ir}$.
- More generally:
  - Lieb-Robinson velocity?
  - Butterfly velocity?
  - Fermi velocity?
- More precise definition of local equilibration scales τ_{eq}, ℓ_{eq}?



Adapted from [Hartman et al'17]

*There are other limitations on the applicability of hydrodynamics (eg unstable frames, long time tails) but I won't discuss them in this talk.

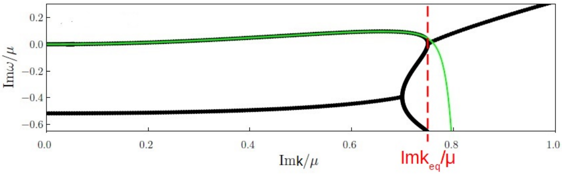
Limits of applicability: convergence

- In principle, the (linear) hydro series can be pushed to any order in k

$$\omega = -i \sum_{n=1}^{+\infty} \omega_{2n} k^{2n}$$

Does this series converge?

- Hard question to answer in general. Use holography to compute the series: in real space (constitutive relation) [Heller&al'13], [Heller&al'20] and in Fourier space (dispersion relation) [Withers'18].
- In Fourier space, [Withers'18] showed that the radius of convergence $k_{\rm eq}$ of the series of the shear diffusive mode of the RN-AdS$_4$ black brane matches a singularity in the complex k plane.



Adapted from [Withers'18]

Limits of applicability: convergence

- Further recent confirmation that the convergence radius is set by the collision of the hydro mode with the nearest non-hydro mode by [Jansen&Pantelidou'20].
- Further related studies and arguments for the above in [Grozdanov&al'19], [Grozdanov&al'19], [Abbasi&Tahery'20].
- Are there cases where the convergence radius can be determined without referring to a specific microscopic theory?
- Yes, if hierarchy of scales: examples using holography.
    1. In the presence of a weakly-broken symmetry (irrelevant deformation in the IR);
    2. In the absence of any weakly-broken symmetry near AdS_2 horizons.

Neutral translation-breaking black brane: model

- Classical solution to the action, [Bardoux&al'12], [Andrade&Withers'13]

$$S = \int d^4x \sqrt{-g} \left(R + 6 - \frac{1}{2} \sum_{i=1}^{2} (\partial \varphi_i)^2 \right)$$

- Metric and matter fields (breaks translations homogeneously)

$$ds^2 = -r^2 f(r) dt^2 + r^2 dx_i^2 + \frac{dr^2}{r^2 f(r)}, \quad \varphi_i = m x^i \,,$$

$$f(r) = 1 - \frac{m^2}{2r^2} - \left(1 - \frac{m^2}{2r_0^2}\right) \frac{r_0^3}{r^3} \quad \Rightarrow \quad T = \frac{3r_0}{4\pi} \left(1 - \frac{m^2}{6r_0^2}\right).$$

Slow relaxation in the presence of weak explicit breaking

- In this model, at high temperature $T \gg m$, translations are weakly broken at a rate Γ, [Andrade&Withers'13], [Davison'13]. This leads to the effective description at sufficiently large scales, [Davison&Goutéraux'14]:

$$\partial_t \pi^x + \partial_x \left(c_s^2 \delta\varepsilon \right) = -\Gamma \pi^x$$

- Together with energy conservation

$$\partial_t \delta\varepsilon + \partial_x \pi^x = 0$$

this leads to

$$\partial_t^2 \pi^x + \Gamma \partial_t \pi^x - c_s^2 \partial_x^2 \pi^x = 0$$

- The rate at which momentum relaxes can be computed $\Gamma \sim m^2/T \ll T$ when $T \gg m$, and so momentum relaxes slowly compared to typical excitations of the system at $\Lambda \sim T$.

Slow relaxation in the presence of weak explicit breaking

- Momentum couples to energy fluctuations: motion of poles governed by the equation

$$\omega^2 + i\Gamma\omega - c_s^2 k^2 + O(\lambda^3) = 0\,, \quad \Gamma \ll \Lambda_{uv}$$

in the scaling limit $\omega \sim k \sim \Gamma \sim \lambda$.

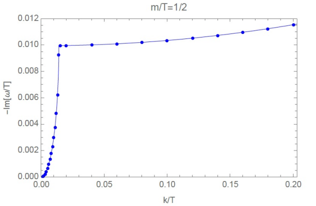
- Crossover between diffusion of energy + weak relaxation of momentum when $k \lesssim k_{\rm eq} \sim \Gamma$

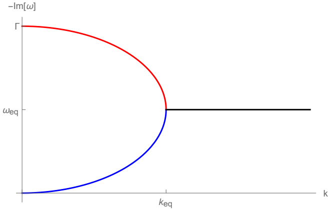
$$\omega = -i\frac{c_s^2}{\Gamma}k^2 + \ldots\,, \quad \omega = -i\Gamma + i\frac{c_s^2}{\Gamma}k^2 + \ldots$$

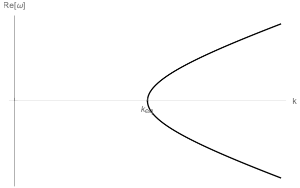
and propagating modes when $k \gtrsim k_{\rm eq} \sim \Gamma$

$$\omega = \pm c_s k + \ldots$$

Data from [Davison&Goutéraux'14]







Slow relaxation in the presence of weak explicit breaking

- We can be more precise about $k_{\rm eq}$. The solutions to

$$\omega^2 + i\Gamma\omega - c_s^2 k^2 = 0\,,$$

are

$$\omega_\pm = -i\frac{\Gamma}{2} \pm \sqrt{k^2 c_s^2 - \frac{\Gamma^2}{4}}$$

- The first non-analyticity is then at $(\omega_{\rm coll}, k_{\rm coll}) \simeq (-i\Gamma/2, \Gamma/(2c_s))$ $\Rightarrow (\omega_{\rm eq}, k_{\rm eq}) \simeq (\Gamma/2, \Gamma/(2c_s))$.

- Differently from [Withers'18], the dispersion relation is very well approximated by truncating to the first non-trivial terms in Γ and k $\Rightarrow$ analytical determination of the convergence radius: consequence of the hierarchy of scales $\Gamma \ll \Lambda_{uv}$.

- Different than the usual hydro expansion in k, which would diverge as $k \to k_{\rm eq}^-$

Slow relaxation in the presence of weak explicit breaking

- Even though energy diffusion $\omega = -i(c_s^2/\Gamma)k^2 + \ldots$ can be augmented to incorporate slowly-relaxing momentum $\omega = -i\Gamma + \ldots$ when $\Gamma \ll \Lambda_{uv}$, it formally breaks down at $(\omega_{\rm eq}, k_{\rm eq})$.

- The diffusivity is $D_\varepsilon \simeq c_s^2/\Gamma$ and is naturally expressed in terms of a velocity and a timescale, which are directly related to the motion of poles in the complex frequency plane.

- The diffusivity can also be written in terms of the local equilibration scales:

$$D_\varepsilon \simeq \frac{c_s^2}{\Gamma} \simeq \frac{1}{2}\frac{\omega_{\rm eq}}{k_{\rm eq}^2}$$

- Valid in the regime when $\Gamma \ll \Lambda_{uv}$.

$$D_\varepsilon \simeq \frac{1}{2}\frac{\omega_{\rm eq}}{k_{\rm eq}^2} = \frac{1}{2}v_{\rm eq}^2\tau_{\rm eq}\,, \qquad \tau_{\rm eq} = \frac{1}{\omega_{\rm eq}}\,, \qquad v_{\rm eq} = \frac{\omega_{\rm eq}}{k_{\rm eq}}$$

- Thanks to the hierarchy of scales $\Gamma \ll \Lambda_{uv}$, we could obtain a simple relation between a diffusivity, the local equilibration timescales and a characteristic velocity.

- What about more generally, when no symmetry is weakly broken, but still in the presence of a hierarchy of scales? Here, low temperature.

- Relation between diffusivity and 'chaos parameters' $D_\varepsilon \simeq v_B^2/\lambda_L$ [Blake'16], valid for low temperatures?

- We will investigate this question using solvable models of transport: holography, SYK, focusing on the diffusion of energy and transverse momentum.

Neutral translation-breaking black brane: low T spectrum

- At low temperatures $m \gg T$, translations are strongly broken and the only hydrodynamic mode is that of diffusion of energy:

$$\omega = -iD_\varepsilon k^2 + O(k^4)\,, \quad D_\varepsilon(T \to 0) = \frac{\sqrt{3}}{\sqrt{2}}\frac{1}{m}$$

- The horizon geometry becomes $AdS_2 \times R^2$ at $T = 0$ ($r_e = \sqrt{6}m$).

- At small temperatures $T \ll m$, the emergent $SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$ symmetry fixes the form of the IR retarded Green's function

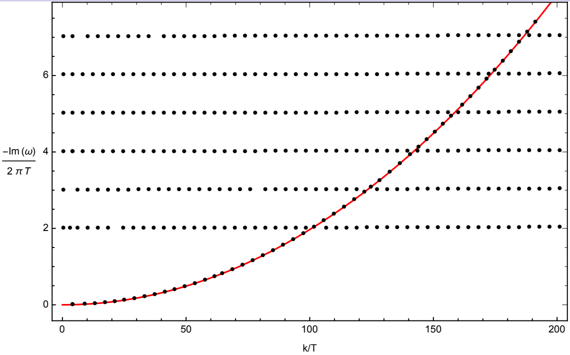
$$\mathcal{G}_{IR} \propto T^{2\Delta(k)-1}\frac{\Gamma\left(\frac{1}{2}-\Delta(k)\right)\Gamma\left(\Delta(k)-\frac{i\omega}{2\pi T}\right)}{\Gamma\left(\frac{1}{2}+\Delta(k)\right)\Gamma\left(1-\Delta(k)-\frac{i\omega}{2\pi T}\right)}\,,$$

$$\Delta(k) = \frac{1}{2} + \sqrt{\frac{9}{4} + 2\frac{k^2}{m^2}}\,,$$

and generates an infinite tower of gapped, IR modes

$$\omega_n = -i2\pi T(n + \Delta(0)) + O(k^2), \qquad n = 0, 1, 2, \ldots$$

Neutral translation-breaking black brane: low T spectrum

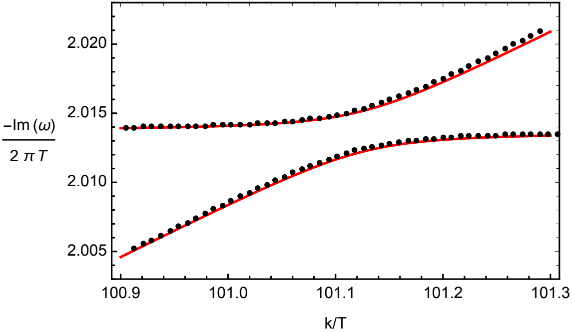


- Red line: analytical approximation to the location of the hydro pole in the scaling limit $\omega \sim T \sim k^2 \sim \lambda$, for $T/m = 1/1000$:

$$\omega(k) = -i\lambda\sqrt{\frac{3}{2}}\frac{k^2}{m}\left(1 + \lambda\frac{k^2}{m^2} + \lambda^2\left(\frac{4\pi T^2}{3m^2} + \frac{k^4}{m^4}\right) + O(\lambda^3)\right).$$

- Crosses IR poles at $\omega \simeq \omega_n = -i2\pi T(n+2)\,, k^2 \simeq k_n^2 = i\omega_n/D_\varepsilon$
- $T \ll m \Rightarrow$ agreement way beyond $k \ll T$ (usual hydro expansion).

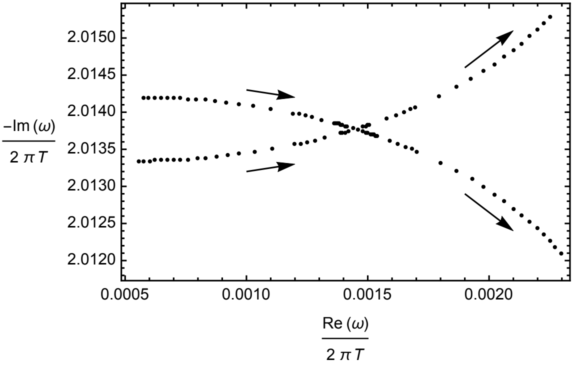
Neutral translation-breaking black brane: avoided crossings



Avoided crossings (with vanishingly small gaps as $T \to 0$ rather than pole collisions) as a function of real k. The red line is the analytical approximation around $\omega = \omega_n + \delta\omega$, $k^2 = k_n^2 + \delta(k^2)$

$$\left(\mathcal{D}_n \delta(k^2) - i\delta\omega\right)\left(1 - i\tau_n \delta\omega\right) - i\gamma_n \delta\omega = 0\,,$$

$$\mathcal{D}_n \to D_\varepsilon\,, \quad \tau_n \to \frac{9m}{16\sqrt{6}(2+n)\pi^2 T^2}, \quad \gamma_n \to \sqrt{\frac{3}{2}}(n(n+4)+3)\frac{\pi T}{m},$$

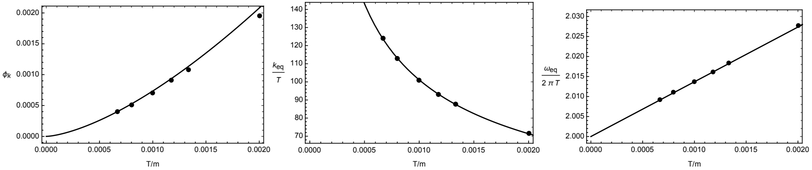


The collision occurs for complex values of k (see also [Withers'18], [Grozdanov&al'19], [Grozdanov&al'19], [Abbasi&Tahery'20], [Jansen&Pantelidou'20])

$$\phi_k \to \frac{2^4}{6^{3/4}}\left(\frac{\pi T}{m}\right)^{3/2}, \quad k_{eq}^2 \equiv |k|^2 \to \frac{\omega_{eq}}{D_\varepsilon}\left(1 - \frac{4\sqrt{6}\pi T}{3m} + \ldots\right),$$

$$\omega_{eq} \equiv |\omega| \to 4\pi T\left(1 + \frac{8\sqrt{6}\pi T}{9m} + \ldots\right).$$

Neutral translation-breaking black brane: complex collision

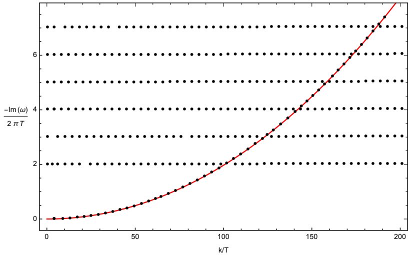


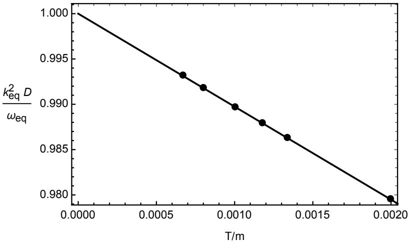
The analytical approximations work very well

$$\phi_k \to \frac{2^4}{6^{3/4}} \left(\frac{\pi T}{m}\right)^{3/2}, \quad k_{eq}^2 \equiv |k|^2 \to \frac{\omega_{eq}}{D_\varepsilon} \left(1 - \frac{4\sqrt{6}\pi T}{3m} + \dots\right),$$

$$\omega_{eq} \equiv |\omega| \to 4\pi T \left(1 + \frac{8\sqrt{6}\pi T}{9m} + \dots\right).$$

Neutral translation-breaking black brane: diffusivity





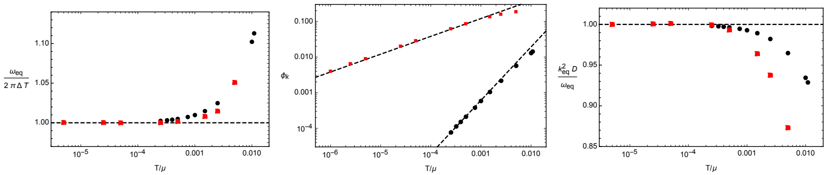
$$\omega(k) = -i\epsilon\sqrt{\frac{3}{2}}\frac{k^2}{m}\left(1 + \epsilon\frac{k^2}{m^2} + \epsilon^2\left(\frac{4\pi T^2}{3m^2} + \frac{k^4}{m^4}\right) + \dots\right).$$

Agreement between the approximation and the numerics implies

$$D_\varepsilon(T \to 0) = \frac{\omega_{\rm eq}}{k_{\rm eq}^2}$$

Relates hydrodynamic data at $(\omega\,, k \ll T)$ to data that mark the edge of validity of hydrodynamics $(|\omega| \simeq \omega_{\rm eq} \sim T,\ |k| \simeq k_{\rm eq} \sim \sqrt{Tm} \gg T)$

Reissner-Nordström black brane



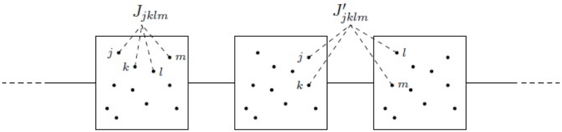
- We find exactly similar results for the Reissner-Nordström black brane, both for energy diffusion and shear momentum diffusion: not specific to energy diffusion. Instead, hierarchy of scales $T \ll \mu$.
- The irrelevant deformation is different for the two diffusive modes

$$\Delta_\varepsilon(k=0) = 2\,, \quad \Delta_\Pi(k=0) = 1\,.$$

- Temperature dependence of the collision phase consistent with

$$\phi_k \sim T^{\Delta - 1/2}$$

Sachdev-Ye-Kitaev chain



CREDIT: [GU&AL'16]

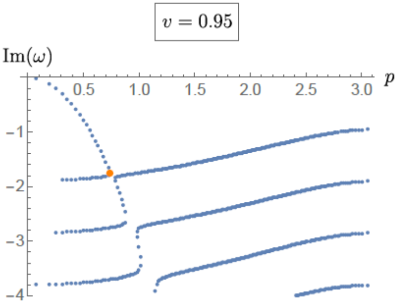
- The Sachdev-Ye-Kitaev model ([SACHDEV&YE'93], [KITAEV'15]) and its higher-dimensional generalizations [GU&AL'16] are another set of solvable models of strongly-coupled matter.

$$H = i^{q/2} \sum_{x=0}^{M-1} \left(\sum_{1 \le i_1 < \ldots < i_q \le N} J_{i_1 \ldots i_q, x} \chi_{i_1,x} \cdots \chi_{i_q,x} \right.$$

$$\left. + \sum_{\substack{1 \le i_1 < \ldots < i_{q/2} \le N \\ 1 \le j_1 < \ldots < j_{q/2} \le N}} J'_{i_1 \ldots i_{q/2} j_1 \ldots j_{q/2}, x} \chi_{i_1,x} \cdots \chi_{i_{q/2},x} \chi_{j_1,x+1} \cdots \chi_{j_{q/2},x+1} \right) .$$

- In the limit of infinite coupling $J, J' \to +\infty$, emergent reparameterization invariance suggests duality to near-AdS_2 gravity.

Exact large q solution

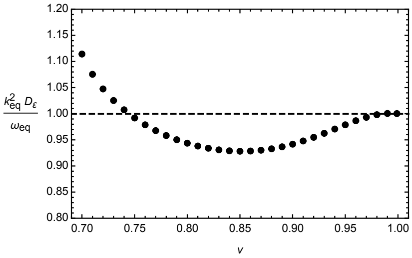


CREDIT: [CHOI&AL'20]

- By allowing $q \to +\infty$, the model can be solved analytically for all coupling strengths [CHOI&AL'20].

- The pole spectrum at strong coupling $v \to 1$ is very close to the holographic results. One difference is that collisions occur for real p at strong enough coupling $v \gtrsim 0.65$.

Diffusivity relation



- We have reproduced their results and extracted the local equilibration scales.

- In the limit of strong coupling $v \to 1$, the $q = +\infty$ SYK chain also verifies

$$D_\varepsilon = \frac{\omega_{\rm eq}}{k_{\rm eq}^2}$$

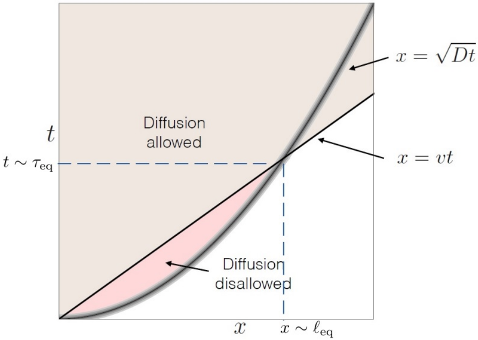
Diffusivity bound

- Our results are compatible with the upper bound formulated by [Hartman et al'17]

$$D \lesssim v_{\rm eq}^2 \tau_{\rm eq}$$

with $v_{\rm eq} \equiv \omega_{\rm eq}/k_{\rm eq}$.

- In particular, they are compatible with the emergence of an 'effective lightcone velocity' $v_{\rm eq}$ even for non-relativistic systems.



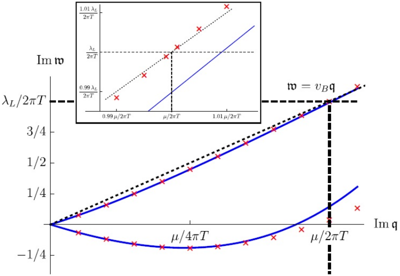
Adapted from [Hartman et al'17]

- Our results do not provide an absolute determination of $v_{\rm eq}$: even in cases where D cannot simply be determined in terms of the parameters charaterizing the breakdown of hydro, we still expect the bound to hold.

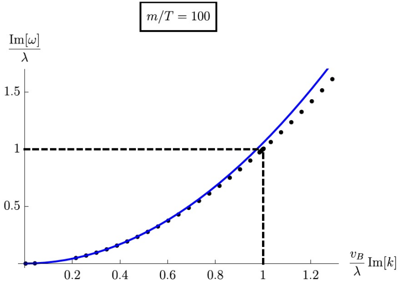
- In the examples of energy diffusion we studied, the diffusive approximation to the location of the pole is very good including at the energy pole skipping point [Grozdanov&al'17], [Blake&al'18], [Blake&al'18].

- This is the origin of the chaos relation

$$D_\varepsilon = v_B^2/\lambda_L$$

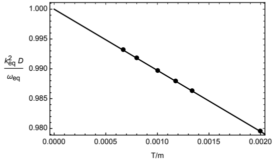


Credit: [Grozdanov&al'17]

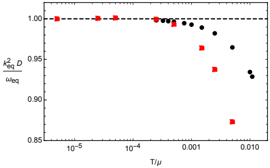


Credit: [Blake&al'18]

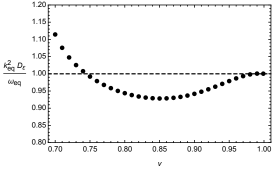
Summary and outlook



Translation-breaking black brane



RN-AdS black brane



SYK chain

- In states with a near-AdS_2 infrared fixed point, the excellent applicability of diffusive hydrodynamics across avoided crossings with an infinite tower of gapped infrared poles results in the relation

$$D = \frac{\omega_{\mathrm{eq}}}{k_{\mathrm{eq}}^2}$$

where ω_{eq} and k_{eq} are determined by infrared data, fixed by the symmetries of the state.

- As for the slow momentum-relaxing case, consequence of a hierarchy of scales.

Summary and outlook

- Extension to other near-AdS_2/SYK states with non-universal leading irrelevant deformation [Blake&Donos'16], [Milekhin'21]?
- Addition of charge to the neutral, translation-breaking black brane: extra diffusive mode, governs the resistivity, of direct interest for strange metallic transport [Hartnoll'14].
- Other, non-AdS_2 fixed points (Lifshitz, hyperscaling violation)?
- Other hierarchy of scales (eg angular momentum, magnetic field)?