Crossing symmetric dispersion relations in QFT

Aninda Sinha
Centre for High Energy Physics,
Indian Institute of Science,
Bangalore, India.

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Outline

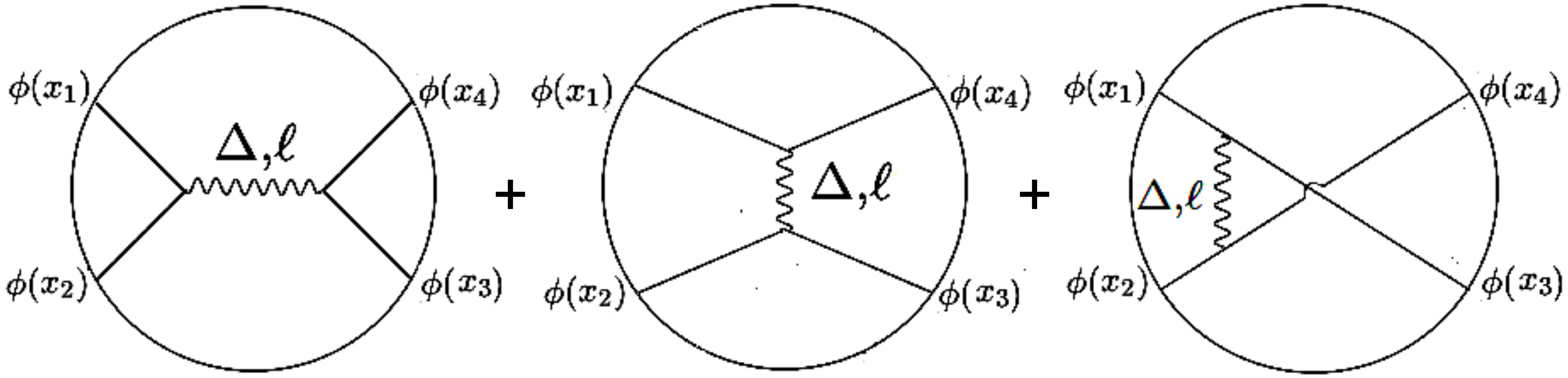
- A brief history of crossing symmetric dispersion relation
- Some quick checks that the new approach works
- Outline of the crossing-symmetric derivation
- Applications in QFT, Froissart bound, generalisations, Bieberbach conjecture
- Brief comments on Polyakov bootstrap in CFT

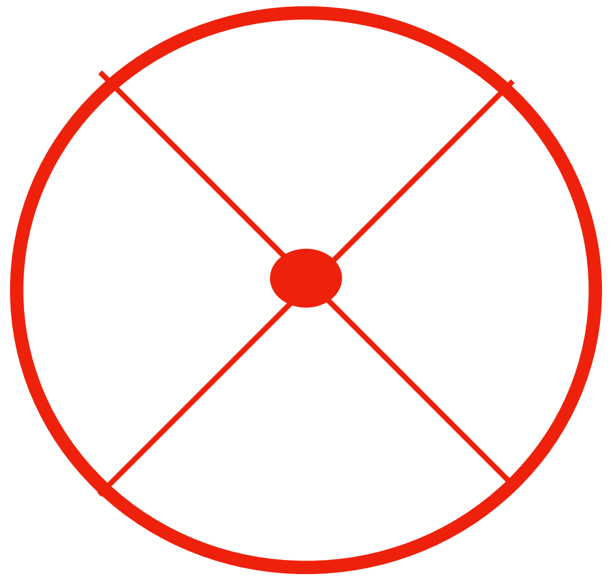
A brief history

- Consider 2-2 scattering of identical massive scalars.
- In perturbative QFT we use Feynman diagrams. These manifest crossing symmetry.
- A non-perturbative representation of the amplitude follows from fixed-t dispersion relation. This loses crossing symmetry, which needs to be imposed as a constraint.

- Where are the Feynman diagrams then from the dispersion relation point of view?
- There is an analogous question one can ask in conformal field theory—this was our motivation to look into this question: Polyakov in 1974 proposed a crossing symmetric bootstrap. This looked like……

$$\mathcal{M}(s,t) \overset{??}{=} \sum_{\Delta,\ell} c_{\Delta,\ell}$$



+  ??

Gopakumar, Kaviraj, Sen, AS '16; Gopakumar, AS '18,…………

- Based on 2012.04877 with Ahmadullah Zahed and 2101.09017 with Rajesh Gopakumar and Ahmadullah Zahed, 2103.12108 with Parthiv Haldar and Ahmadullah Zahed and work in progress with Prashanth Raman.

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Rigorous Parametric Dispersion Representation with Three-Channel Symmetry*

G. Auberson and N. N. Khuri
Rockefeller University, New York, New York 10021
(Received 30 June 1972)

Starting with an analyticity domain in the two Mandelstam variables which is contained in the domain obtained by Martin, we derive a parametric dispersion representation for scattering amplitudes in the equal-mass case. For pion-pion scattering this representation is a rigorous consequence of the axioms of local field theory; it displays in a symmetric and explicit way the contributions of all three channels, and it has only "physical" absorptive parts. This representation is useful for deriving sum rules involving only absorptive parts and relating all three channels. Some of these sum rules are given in this paper, the most important of which form a set of independent physical relations that lead to necessary and sufficient conditions ensuring full crossing symmetry.

A relatively unknown paper from 1972!

Note added in proof. One should note that Eq. (4.13) is tremendously simplified in the fully symmetric case, $\pi^0\pi^0 \to \pi^0\pi^0$. In that case one obtains

$$F_0(\bar{s}, \bar{t}, \bar{u}) = \alpha_0 + \frac{1}{\pi}\int_{8/3}^{\infty} \frac{d\bar{s}'}{\bar{s}'} A(\bar{s}'; \bar{t}_+(\bar{s}'; \bar{s}, \bar{t}, \bar{u}))\, H(\bar{s}'; \bar{s}, \bar{t}, \bar{u}),$$

where $H(\bar{s}'; \bar{s}, \bar{t}, \bar{u}) = [\bar{s}(\bar{s}' - \bar{s})^{-1} + \bar{t}(\bar{s}' - \bar{t})^{-1} + \bar{u}(\bar{s}' - \bar{u})^{-1}]$, and $\bar{t}_+(\bar{s}'; \bar{s}, \bar{t}, \bar{u}) = t_+(\bar{s}'; \bar{a})$ with $\bar{a} = \bar{s}\,\bar{t}\,\bar{u}(\bar{s}\,\bar{t} + \bar{t}\bar{u} + \bar{s}\,\bar{u})^{-1}$ and $\bar{t}_+(\bar{s}', \bar{a})$ given in Eq. (5.20). This representation holds for *any* point (s, t) for which $\tau \equiv \bar{t}_+(\bar{s}'; \bar{s}, \bar{t}, \bar{u}) + \frac{4}{3}$ lies in the Martin-Lehmann ellipses $E(s')$ for $A(s', \tau)$ given in Eq. (A2). The similarity of this representation to the Cini-Fubini approximation[4] is striking. This representation follows most directly from Eq. (5.2) by transforming from the (z, a) variables to the s, t, u variables.

The most useful/ encouraging formula was in a NOTE ADDED!

ACKNOWLEDGMENTS

We are indebted to F. J. Dyson and G. Wanders for useful comments.

See also Mahoux, Roy, Wanders 1974.

Crossing symmetric dispersion relation

$$\boxed{\mathcal{M}_0(s_1,s_2)=\alpha_0+\frac{1}{\pi}\int_{\frac{2\mu}{3}}^{\infty}\frac{ds_1'}{s_1'}\mathcal{A}\left(s_1';s_2^{(+)}\left(s_1',a\right)\right)\times H\left(s_1';s_1,s_2,s_3\right)}$$

$$H\left(s_1';s_1,s_2,s_3\right)=\left[\frac{s_1}{\left(s_1'-s_1\right)}+\frac{s_2}{\left(s_1'-s_2\right)}+\frac{s_3}{\left(s_1'-s_3\right)}\right]$$

$$s_2^{(+)}\left(s_1',a\right)=-\frac{s_1'}{2}\left[1-\left(\frac{s_1'+3a}{s_1'-a}\right)^{1/2}\right],$$

$$s_1+s_2+s_3=0 \quad \mu=4m^2 \qquad a=\frac{s_1s_2s_3}{s_1s_2+s_1s_3+s_2s_3} \qquad \alpha_0\equiv\mathcal{M}(s_1=0,s_2=0)$$

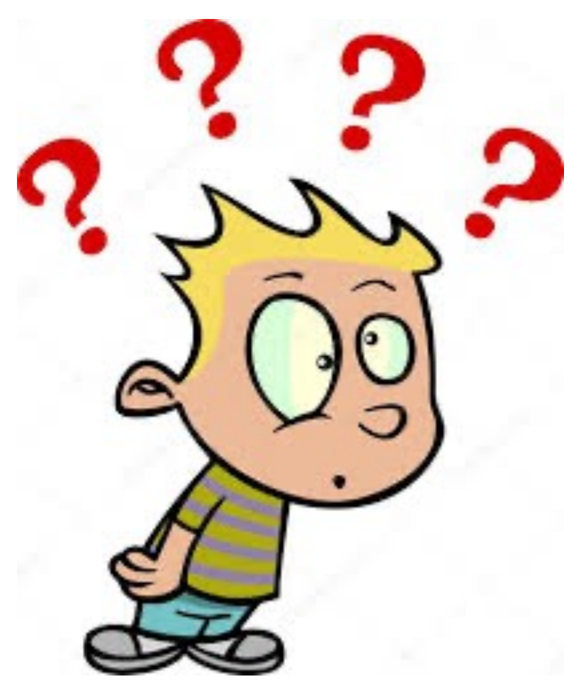
NB: $s_1=s-\frac{4}{3}m^2, s_2=t-\frac{4}{3}m^2, s_3=u-\frac{4}{3}m^2$

An important question

Expand*

$$-s_1 s_2 (s_1 + s_2)\mathscr{M}(s_1, s_2) = \frac{\Gamma\left(1 - s_1\right)\Gamma\left(1 - s_2\right)\Gamma\left(s_1 + s_2 + 1\right)}{\Gamma\left(s_1 + 1\right)\Gamma\left(-s_1 - s_2 + 1\right)\Gamma\left(s_2 + 1\right)}$$

In terms of poles in
$s_1, s_2, s_3 = -s_1 - s_2$



***Removing a kinematic pre-factor of** $(s_1 s_2 + s_1 s_3 + s_2 s_3)^2$

- A very naive 1st try would be to just sum over the residue x pole in each channel. This is doomed to fail.
- On the other hand if you just sum over s-u channel poles, this works.
- The nontrivial question is (x,y are crossing symmetric combinations):

$$\mathscr{M}(s_1, s_2, s_3) \overset{??}{=} \sum_k f_k(x, y)\left(\frac{1}{s_1 - s_k} + \frac{1}{s_2 - s_k} + \frac{1}{s_3 - s_k} + c_k\right)$$

Answer

Expand

$$-s_1 s_2(s_1+s_2)\mathcal{M}(s_1,s_2)=\frac{\Gamma\left(1-s_1\right)\Gamma\left(1-s_2\right)\Gamma\left(s_1+s_2+1\right)}{\Gamma\left(s_1+1\right)\Gamma\left(-s_1-s_2+1\right)\Gamma\left(s_2+1\right)}$$

In terms of poles in s_1, s_2 & $s_3=-s_1-s_2$

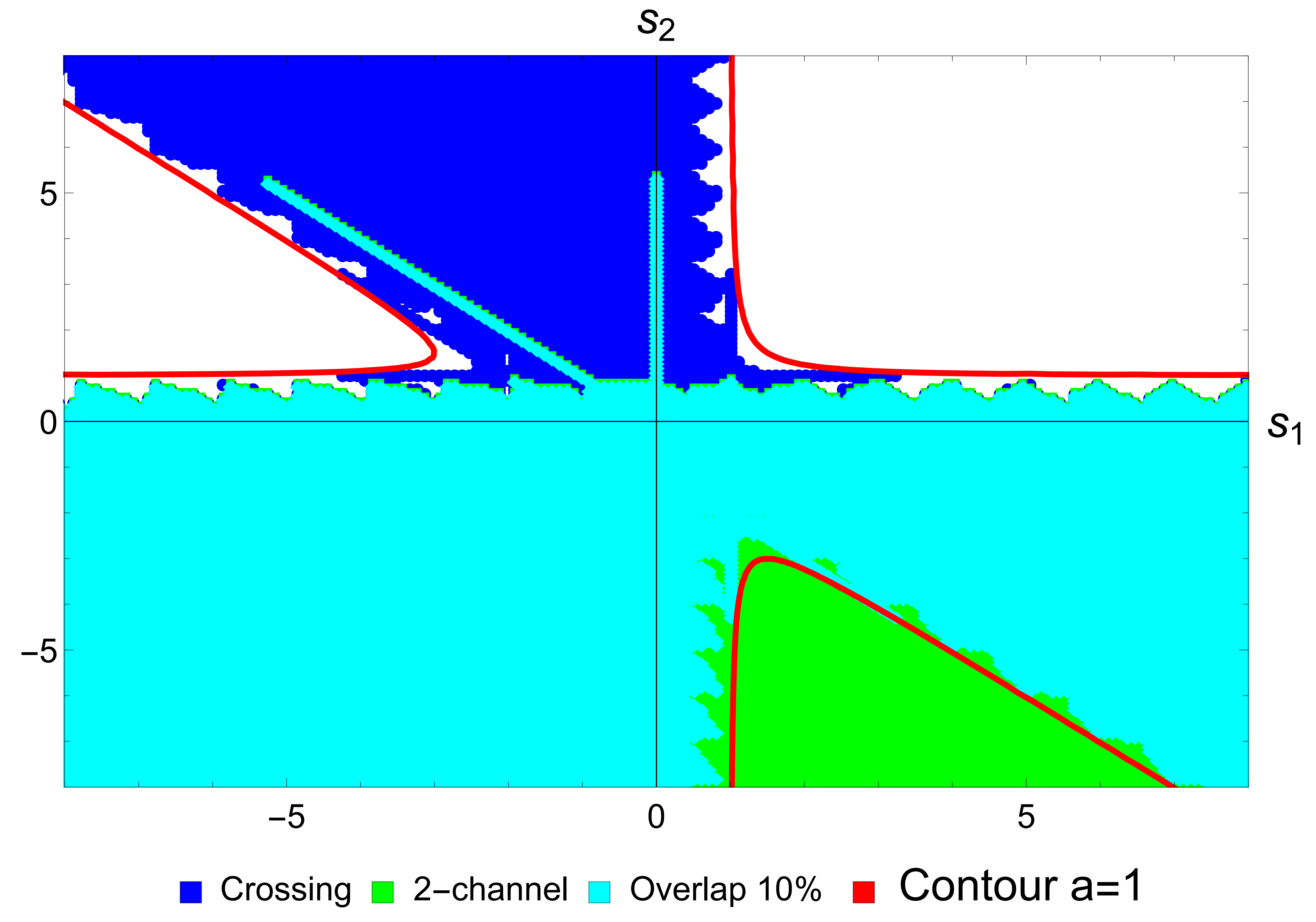
Answer

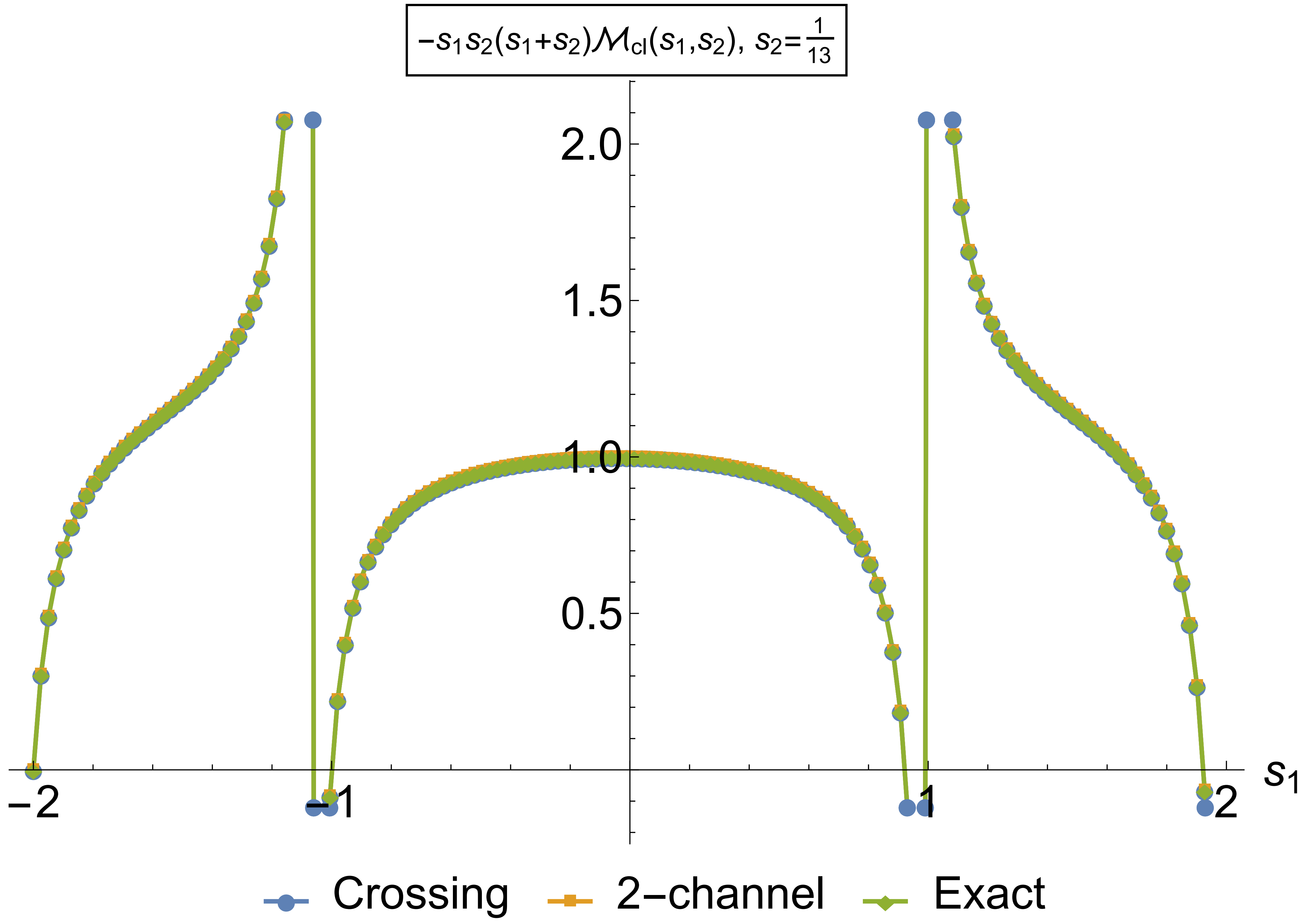
$$-s_1 s_2(s_1+s_2)\mathcal{M}(s_1,s_2)^{(crossing)}=1+\sum_{k=0}^{\infty}\left[\frac{(-1)^k}{k!(k+1)!}\left(\frac{1}{k-s_1+1}+\frac{1}{k-s_2+1}+\frac{1}{k-s_3+1}-\frac{3}{k+1}\right)\right.$$
$$\left.\times\frac{\Gamma\left(\frac{1}{2}\left(-k\sqrt{\frac{4a}{-a+k+1}+1}+k-\sqrt{\frac{4a}{-a+k+1}+1}+3\right)\right)\Gamma\left(\frac{1}{2}\left(k\sqrt{\frac{4a}{-a+k+1}+1}+k+\sqrt{\frac{4a}{-a+k+1}+1}+3\right)\right)}{\Gamma\left(\frac{1}{2}\left(k\left(\sqrt{\frac{4a}{-a+k+1}+1}-1\right)+\sqrt{\frac{4a}{-a+k+1}+1}+1\right)\right)\Gamma\left(\frac{1}{2}\left(-\sqrt{\frac{4a}{-a+k+1}+1}-k\left(\sqrt{\frac{4a}{-a+k+1}+1}+1\right)+1\right)\right)}\right],$$

$$\sim\frac{\Gamma(\frac{3+k}{2}-\lambda)\Gamma(\frac{3+k}{2}+\lambda)}{\Gamma(\frac{1-k}{2}-\lambda)\Gamma(\frac{1-k}{2}+\lambda)}$$

$$\boxed{a=\frac{s_1 s_2 s_3}{s_1 s_2+s_1 s_3+s_2 s_3}\equiv\frac{y}{x}}$$

Numerical checks





Similar expansion exists for the 2d Ising Mellin Amplitude. Also similar expansion exists for the open string amplitude with only s,t symmetry (w P. Raman).

s	t	Exact	k_{max}=100	k_{max}=400
$\frac{46}{13}$	$\frac{1}{10}$	1.32322	1.32361	1.32325
$\frac{83}{10}$	$-\frac{8}{5}$	-0.000619309	-0.00061931	-0.000619309
$\frac{41}{5}+\frac{21i}{10}$	$-\frac{8}{5}-\frac{43i}{10}$	$0.200577-0.0884721i$	$0.200577-0.088472i$	$0.200577-0.0884721i$
$\frac{3}{5}+\frac{31i}{10}$	$\frac{1}{5}+\frac{i}{2}$	$-0.242057+2.28081i$	$-0.247887+2.28194i$	$-0.242315+2.28022i$
$\frac{31i}{10}$	$\frac{i}{5}$	$0.769274+0.638919i$	$0.769435+0.63905i$	$0.769279+0.638931i$

- Good match for complex values

- Notice something nontrivial. If we consider just a fixed k.
- Expand around a=0.
- You will get negative powers of x.
- But LHS has no such powers. This means that once we sum over k, these negative powers will cancel. Keep this in mind.
- Lesson: To have a crossing symmetric expansion, it seems we have to introduce "spurious", "non-local" singularities to the basis elements. [[nothing to do with Polyakov double poles]]

$$\boxed{\text{A brief derivation of crossing symmetric dispersion relation}}$$

- Auberson, Khuri 1972; AS, Zahed '20.

The key idea

- If we have a completely crossing symmetric amplitude, then if there are no massless poles, we can expect

$$\boxed{\mathcal{M}(s_1, s_2, s_3) = \sum_{p,q} \mathcal{W}_{pq} x^p y^q}$$

$\mathcal{W}_{pq}$: **"Wilson" coefficients**

- The idea is to work with a different set of variables. Consider the cubic equation

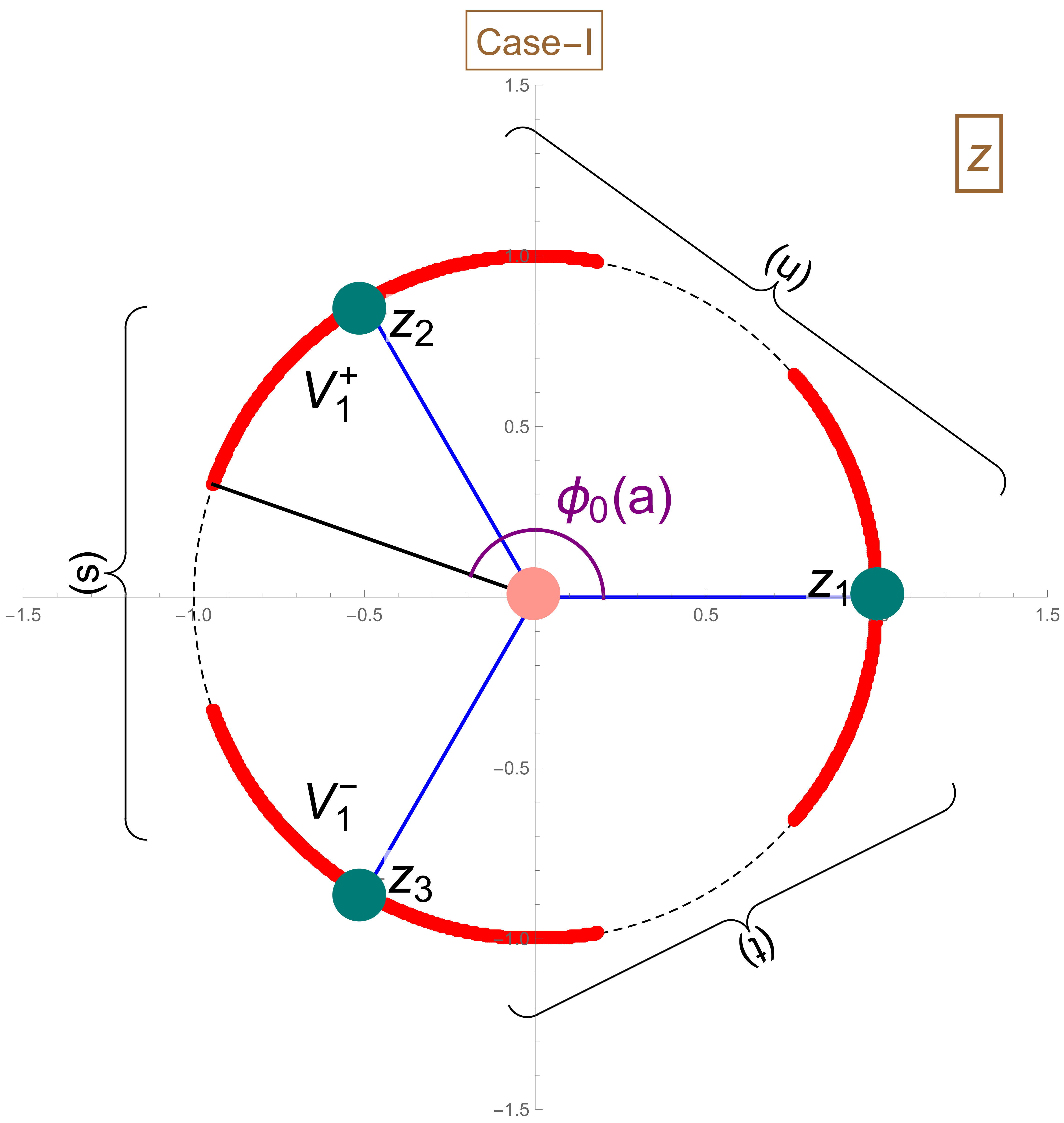
$$a = \frac{s_1 s_2 s_3}{s_1 s_2 + s_1 s_3 + s_2 s_3}$$

The key steps

- We can parametrize the solution using a new variable z

$$s_k = a - a\frac{(z - z_k)^3}{z^3 - 1}$$

- Here z_k are the cube roots of unity. Satisfying $z_1 + z_2 + z_3 = 0$



$$-\frac{8}{9}m^2 < a \leq 0$$

$$s_k = a - a\frac{(z-z_k)^3}{z^3-1}$$

$$z_1 + z_2 + z_3 = 0$$

$$a = \frac{s_1 s_2 s_3}{s_1 s_2 + s_1 s_3 + s_2 s_3} = \frac{s_1 s_2 (s_1 + s_2)}{s_1 s_2 + s_1^2 + s_2^2}$$

Low energy

High energy

$s_1 \to a, |s_2|, |s_3| \to \infty$ when $z \to z_1$ etc

The key steps

- Imposing $s_1 + s_2 + s_3 = 0$ the equation

$$a = \frac{s_1 s_2 s_3}{s_1 s_2 + s_1 s_3 + s_2 s_3}$$

- Becomes a quadratic equation giving s_2 in terms of s_1.

  Call these $s_2^{\pm}(s_1, a)$

$$s_2^{(\pm)}(s_1, a) = -\frac{s_1}{2}\left[1 \mp \left(\frac{s_1 + 3a}{s_1 - a}\right)^{1/2}\right]$$

The key idea

- Idea now is to write a dispersion relation in z for fixed a.
- Conveniently one can show

$$-y \equiv s_1 s_2 s_3 = \frac{27a^3z^3}{(z^3-1)^2}$$

$$-x \equiv s_1 s_2 + s_2 s_3 + s_1 s_3 = \frac{27a^2z^3}{(z^3-1)^2}$$

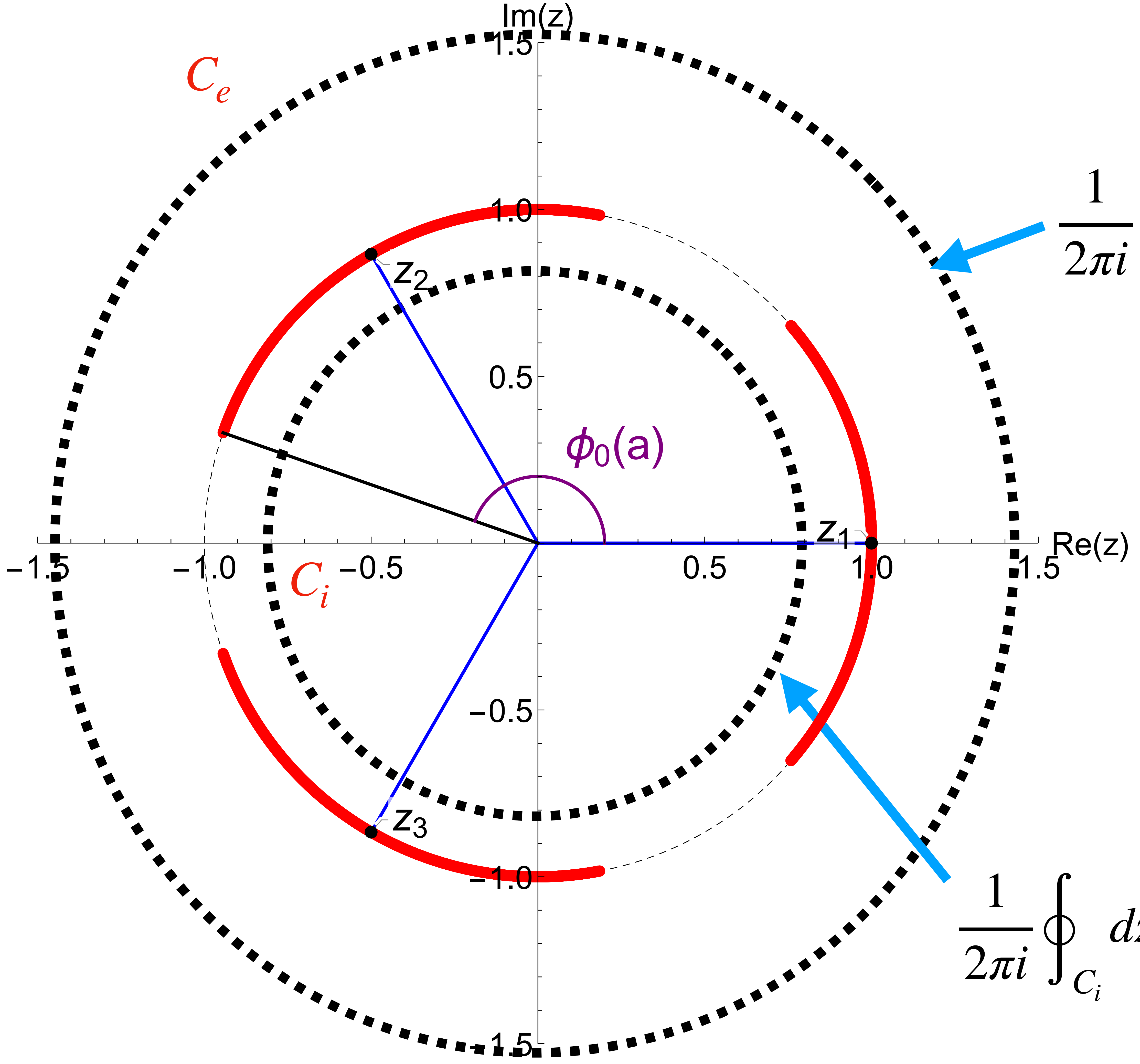
Make a mental note of the forms of these. These are what are called Koebe functions in the context of univalent functions.

- So that an expansion in powers of x, y is an expansion in powers of a, z^3.

The key idea

- If we assume $\mathscr{M}(s_1, s_2) = o(s_1^2)$ for $|s_1| \to \infty$ for fixed s_2 and similarly for the other channels, this translates to
- $\mathscr{M}(a, z) = o(1/(z - z_k)^2)$ as $z \to z_k$ with fixed a.
- We write dispersion integrals for $(z^3 - 1)\mathscr{M}(a, z)$.

The key idea



$$\frac{1}{2\pi i}\oint_{C_e} dz' \frac{z'^3-1}{z'^3(z'-z)}\mathscr{M}(z',a) = \mathscr{M}(s_1=0,s_2=0)$$

$$\frac{1}{2\pi i}\oint_{C_i} dz' \frac{z'^3-1}{z'^3(z'-z)}\mathscr{M}(z',a) = \frac{z^3-1}{z^3}\mathscr{M}(z,a) + \frac{\mathscr{M}(s_1=0,s_2=0)}{z^3}$$

- Subtracting and converting to s_1 variable leads to

$$\boxed{\mathscr{M}_0(s_1, s_2) = \alpha_0 + \frac{1}{\pi}\int_{\frac{2\mu}{3}}^{\infty} \frac{ds_1'}{s_1'} \mathscr{A}\left(s_1'; s_2^{(+)}\left(s_1', a\right)\right) \times H\left(s_1'; s_1, s_2, s_3\right)}$$

$$H\left(s_1'; s_1, s_2, s_3\right) = \left[\frac{s_1}{\left(s_1' - s_1\right)} + \frac{s_2}{\left(s_1' - s_2\right)} + \frac{s_3}{\left(s_1' - s_3\right)}\right]$$

$$s_2^{(+)}\left(s_1', a\right) = -\frac{s_1'}{2}\left[1 - \left(\frac{s_1' + 3a}{s_1' - a}\right)^{1/2}\right],$$

$$s_1 + s_2 + s_3 = 0 \qquad a = \frac{s_1 s_2 s_3}{s_1 s_2 + s_1 s_3 + s_2 s_3} \qquad \alpha_0 \equiv \mathscr{M}(s_1 = 0, s_2 = 0)$$

2012.04877 with A. Zahed

- An advantage of this formalism is that RHS is now manifestly crossing symmetric.
- This means we should be able to write down an immediate formula for the “Wilson” coefficients.

$$\mathscr{A}_1(s_1, s_2) = \sum_{\ell=0}^{\infty} (2\ell + 2\alpha)\, a_\ell(s_1) C_\ell^{(\alpha)}\left(\sqrt{\xi(s_1, a)}\right) ,$$

$$\xi(s_1, a) = \xi_0 + 4\xi_0 \left(\frac{a}{s_1 - a}\right), \ \xi_0 = \frac{s_1^2}{(s_1 - 2\mu/3)^2}$$

The partial wave expansion is convergent inside the Lehmann-Martin ellipse which translates to $-7.05\mu < a < \mu$

Locality constraints

$$\mathscr{W}_{n-m,m} = \int_{\frac{2\mu}{3}}^{\infty} \frac{ds_1}{s_1} \sum_{\ell=0}^{\infty} (2\ell + 2\alpha)\, a_\ell(s_1) \mathscr{B}^{(\ell)}_{n,m}(s_1)\,,$$

"Wilson coefficients"

$$\mathscr{B}^{(\ell)}_{n,m}(s_1) = \sum_{j=0}^{m} \frac{p^{(j)}_\ell(\xi_0)\,(4\xi_0)^j (3j-m-2n)(-n)_m}{\pi s_1^{2n+m}\, j!(m-j)!(-n)_{j+1}}\,,$$

- **Our formalism is such that $m \geq 0$. However, $n \leq m$ is not ruled out.**
- **This would lead to negative powers of x. We want to rule this out by demanding "locality."**
- **These are new constraints. $n < m, n \geq 1$. Only spins greater than $2n$ contribute.**

* $p^{(j)}_\ell(\xi_0) = \partial^j C^{(\alpha)}_\ell\left(\sqrt{\xi}\right)/\partial \xi^j|_{\xi=\xi_0}$

$$\boxed{\mathscr{M}(s_1, s_2, s_3) = \sum_{p,q} \mathscr{W}_{pq} x^p y^q}$$

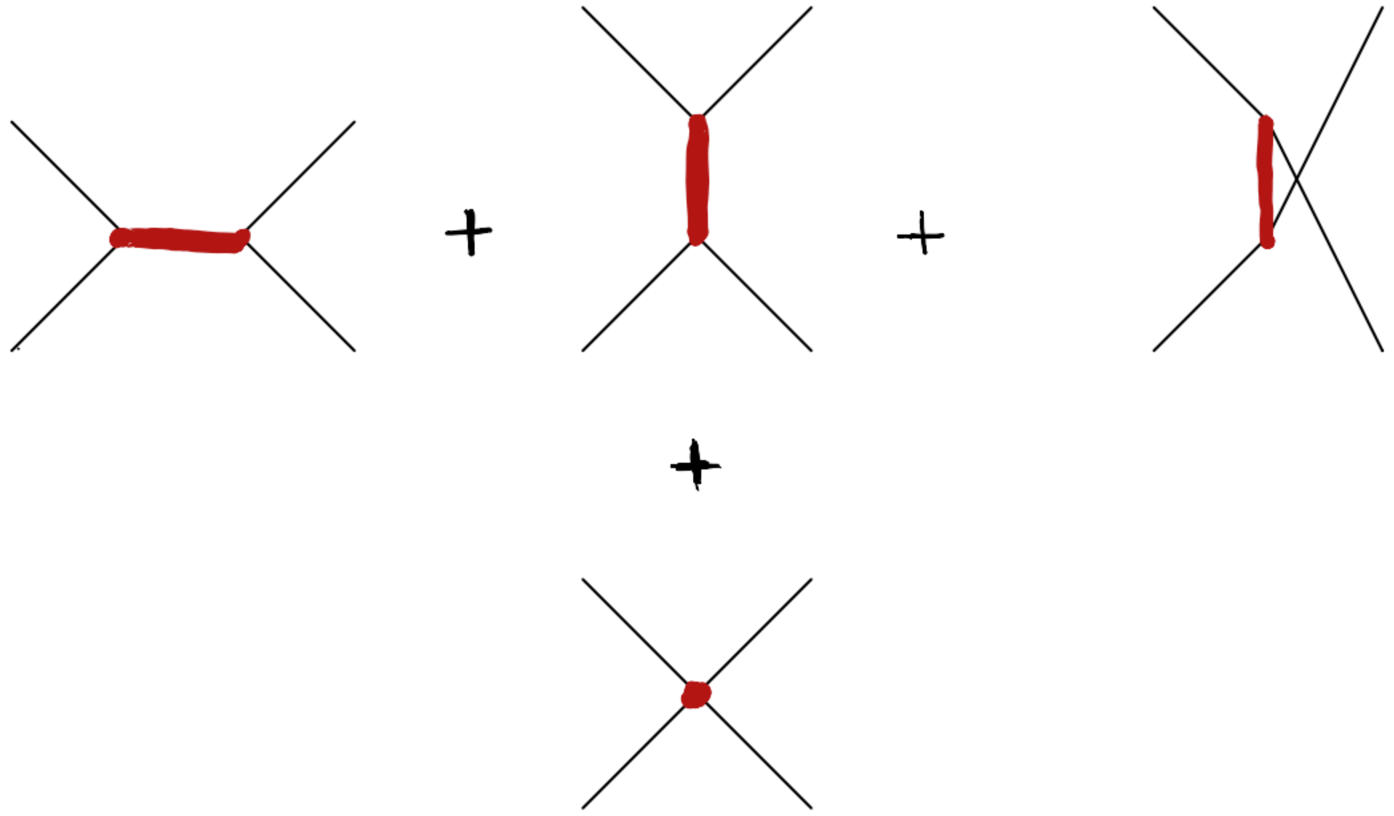
Structure of "Dyson" and "Feynman" blocks

$$\mathscr{M}(s_1, s_2) = \mathscr{M}(0,0) + \frac{1}{\pi}\sum_{\ell}(2\ell + 2\alpha)\int_{\frac{2\mu}{3}}^{\infty}\frac{d\sigma}{\sigma}H(\sigma; s_i)a_\ell(\sigma)C_\ell^{(\alpha)}(\sqrt{\xi(\sigma, a)})$$

Dyson block expansion
—"Regge bounded"

$$\mathscr{M}(s_1, s_2) = \mathscr{M}(0,0) + \frac{1}{\pi}\sum_{\ell}(2\ell + 2\alpha)\int_{\frac{2\mu}{3}}^{\infty}\frac{d\sigma}{\sigma(\sigma - \frac{2\mu}{3})^\ell}a_\ell(\sigma)\left[\frac{Q_\ell(s_1, s_2)}{\sigma - s_1} + \frac{Q_\ell(s_2, s_3)}{\sigma - s_2} + \frac{Q_\ell(s_3, s_1)}{\sigma - s_3} + poly_\ell\right]$$

Feynman
block expansion
—not "Regge bounded"

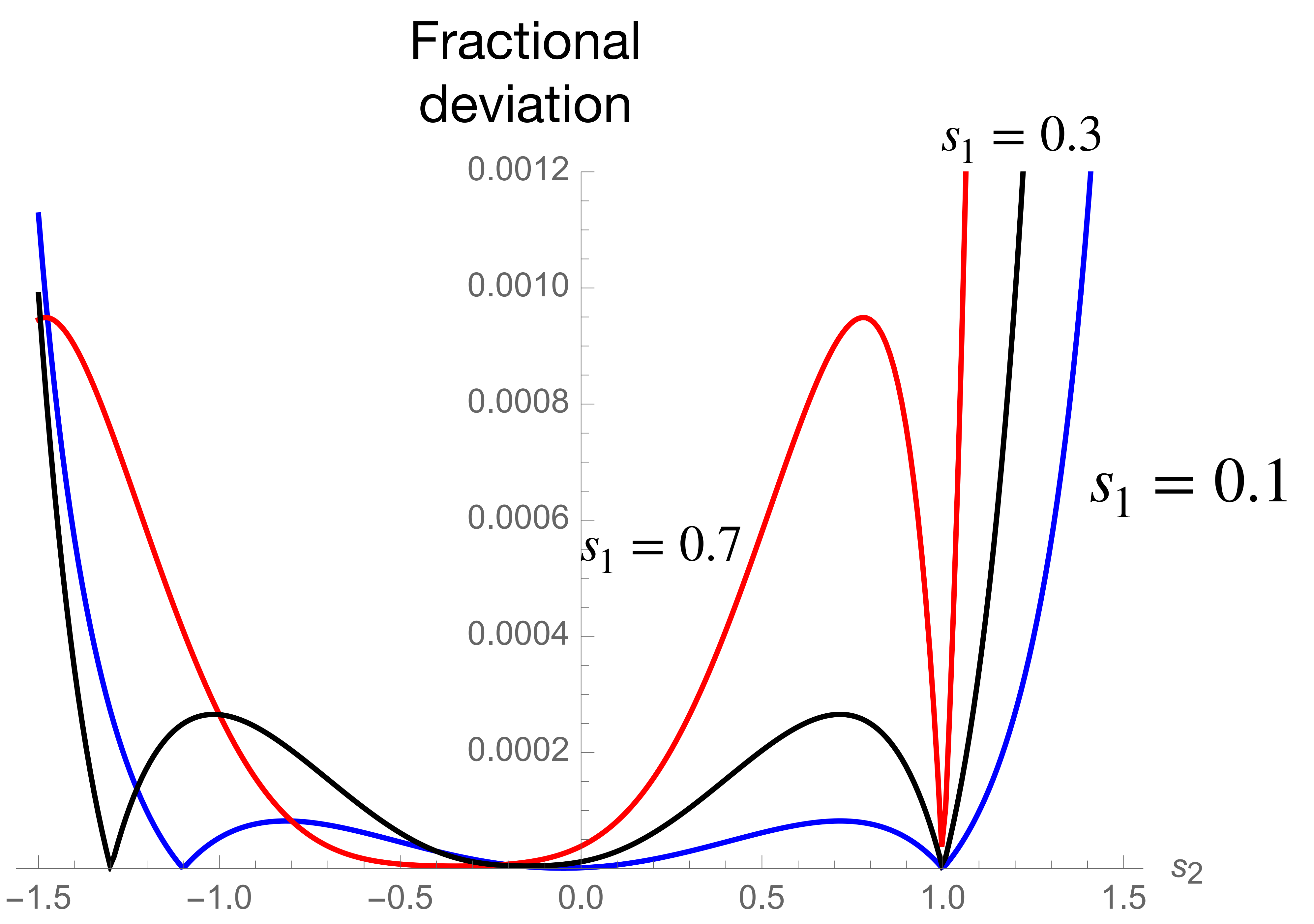


$$eg. \quad poly_2 = c_1 x + c_2 y$$

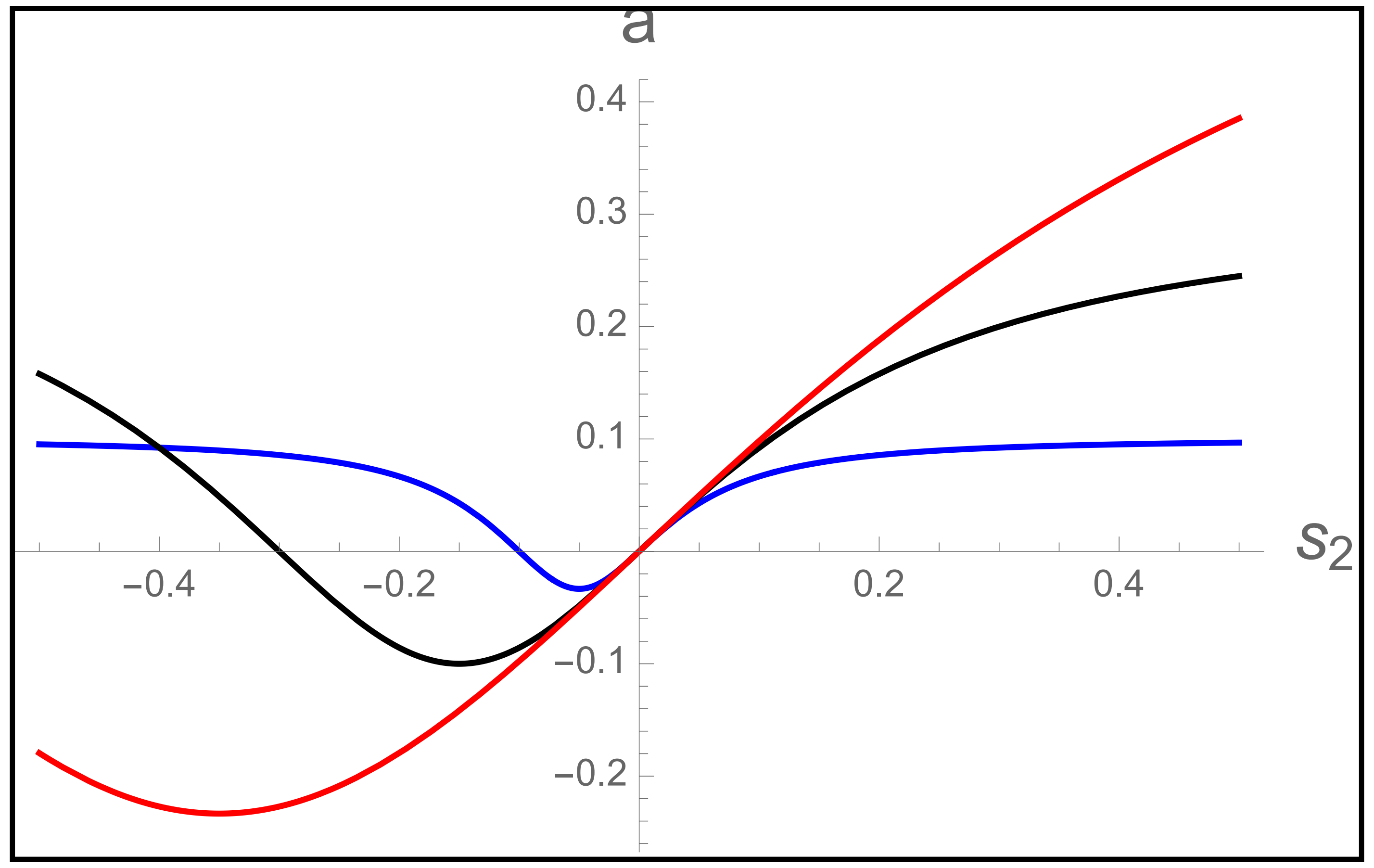
* $Q_\ell(s_1, s_2) = s_1(s_1 - \frac{2\mu}{3})^\ell C_\ell^{(\alpha)}(\cos\theta)$

“Dyson” block expansion

Expansion of the massless pole subtracted dilation amplitude in terms of crossing symmetric partial waves (locality constraints implicit).

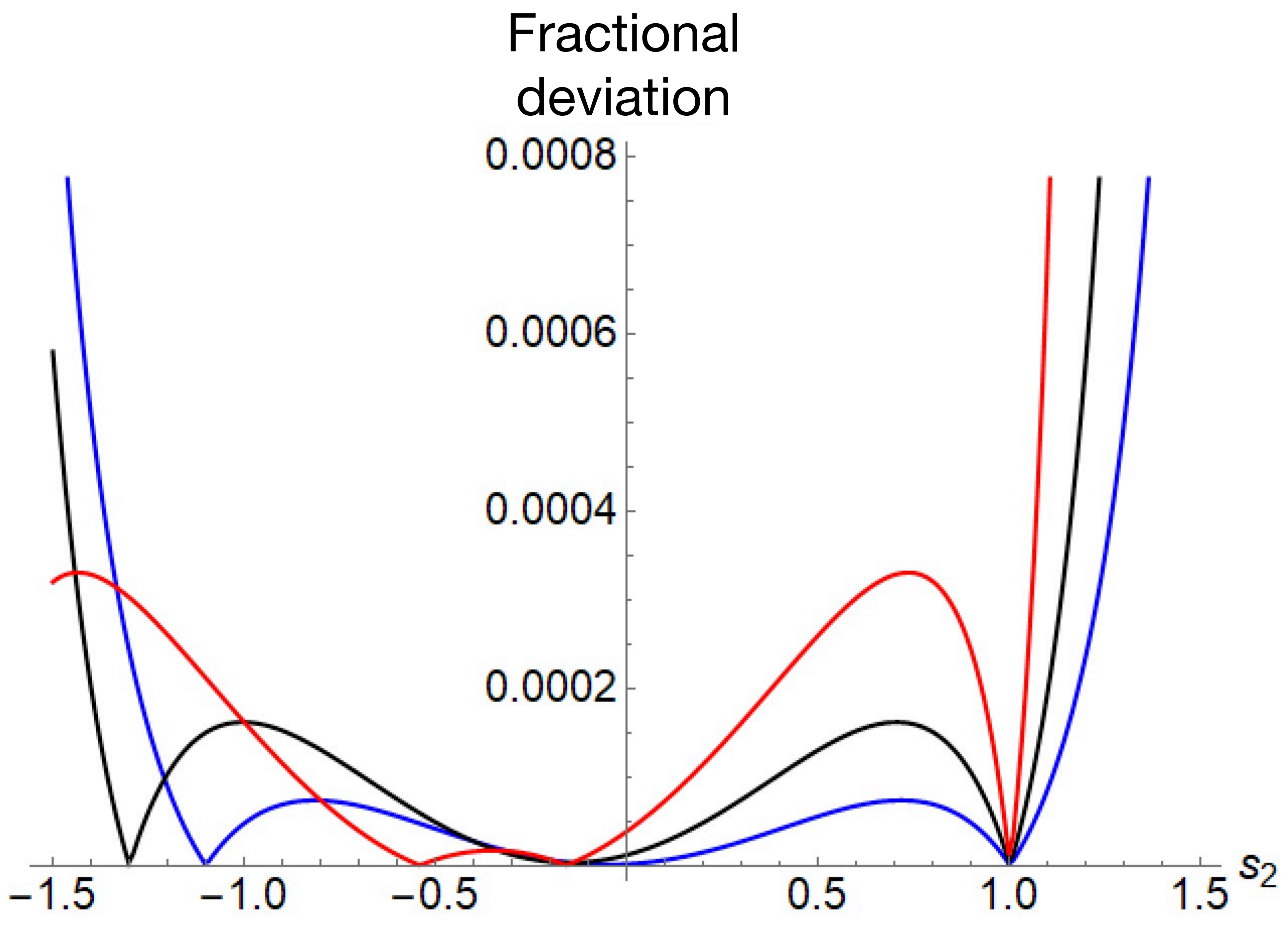


$\ell_{max} = 6, \quad k_{max} = 6$



“Feynman” block expansion

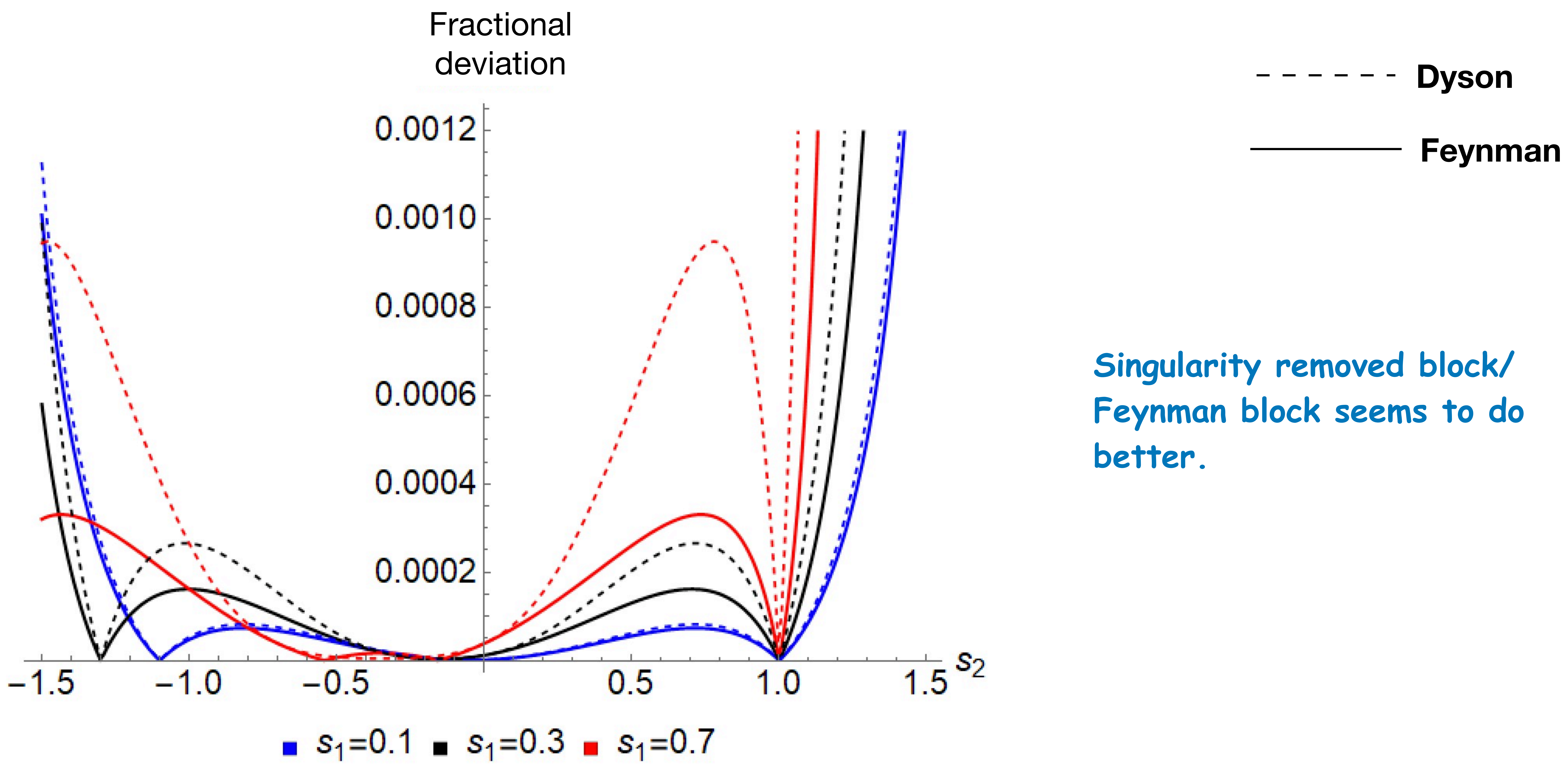
Expansion of the massless pole subtracted dilation amplitude in terms of crossing symmetric partial waves (locality constraints imposed—operationally throw away negative powers of x, partialwave*kernel wise).



$$\ell_{max} = 6, \quad k_{max} = 6$$

Importance of contact terms

s_1	s_2	Exact	Feynman	Without contact terms	Dyson
$\frac{1}{13}$	$\frac{1}{10}$	2.45013	2.45012	2.44898	2.45012
$\frac{7i}{27}$	$1+\frac{i}{10}$	$0.886092+10.2404\,i$	$0.886074+10.2397\,i$	$0.834015+10.2278\,i$	$0.886395+10.24\,i$
$\frac{3}{10}+\frac{31i}{10}$	$\frac{1}{11}+\frac{3i}{13}$	$0.327184+0.214067\,i$	$0.332949+0.217049\,i$	$0.790025+0.0663346\,i$	$0.32976+0.219688\,i$
$\frac{1}{13}+\frac{33i}{10}$	$\frac{5i}{13}$	$0.205184+0.179331\,i$	$0.211081+0.187269\,i$	$0.775999+0.0550316\,i$	$0.201598+0.183881\,i$
$\frac{31i}{10}$	$\frac{i}{5}$	$0.312277+0.112769\,i$	$0.315664+0.116603\,i$	$0.802506+0.0667358\,i$	$0.313287+0.116089\,i$



Singularity removed block/ Feynman block seems to do better.

Proposed nomenclature

	QFT	**CFT**
	Partial wave with singularity — Dyson block	Conformal partial wave with singularity — Polyakov block
	Partial wave with singularities removed — Feynman block	CPW with singularities removed — Witten block
	QFT	**CFT**

Applications in QFTs

- 2012.04877 with A. Zahed

Inequalities

$$\mathscr{W}_{n-m,m} = \int_{\frac{2\mu}{3}}^{\infty} \frac{ds_1}{s_1} \sum_{\ell=0}^{\infty} (2\ell + 2\alpha)\, a_\ell(s_1) \mathscr{B}^{(\ell)}_{n,m}(s_1)\,,$$

$$\mathscr{B}^{(\ell)}_{n,m}(s_1) = \sum_{j=0}^{m} \frac{p_\ell^{(j)}\left(\xi_0\right)\left(4\xi_0\right)^j (3j-m-2n)(-n)_m}{\pi s_1^{2n+m} j!(m-j)!(-n)_{j+1}}\,,$$

- **We can derive inequalities for $\mathscr{W}_{pq}$.**
- **These arise from properties of $\mathscr{B}^{(\ell)}_{n,m}$**
- **We consider $n \geq m, m \geq 0, n \geq 1$**

$* \, p_\ell^{(j)}\left(\xi_0\right) = \partial^j C_\ell^{(\alpha)}\left(\sqrt{\xi}\right)/\partial \xi^j |_{\xi=\xi_0}$

$$\boxed{\mathscr{M}(s_1, s_2, s_3) = \sum_{p,q} \mathscr{W}_{pq} x^p y^q}$$

Examples of inequalities

$$\mathscr{W}^{(\delta_0)}_{n-m,m} = \int_{\frac{2\mu}{3}+\delta_0}^{\infty} \frac{ds_1}{s_1} \sum_{\ell=0}^{\infty} \left(2\ell + 2\alpha\right) a_\ell(s_1)\mathscr{B}^{(\ell)}_{n,m}(s_1)\,,$$

$$\sum_{r=0}^{m} \chi_n^{(r,m)}(\mu,\delta_0)\mathscr{W}^{(\delta_0)}_{n-r,r} \geq 0$$

Recursion relation for $\chi's$ known 2012.04877 with A. Zahed

$$\sum_{r=0}^{m} \frac{n^{m-r}}{(m-r)!(\delta_0+\frac{2\mu}{3})^{m-r}} \frac{\mathscr{W}^{(\delta_0)}_{n-r,r}}{\mathscr{W}^{(\delta_0}_{n,0}} \geq 0\,, \quad n \gg m$$

A nontrivial limit

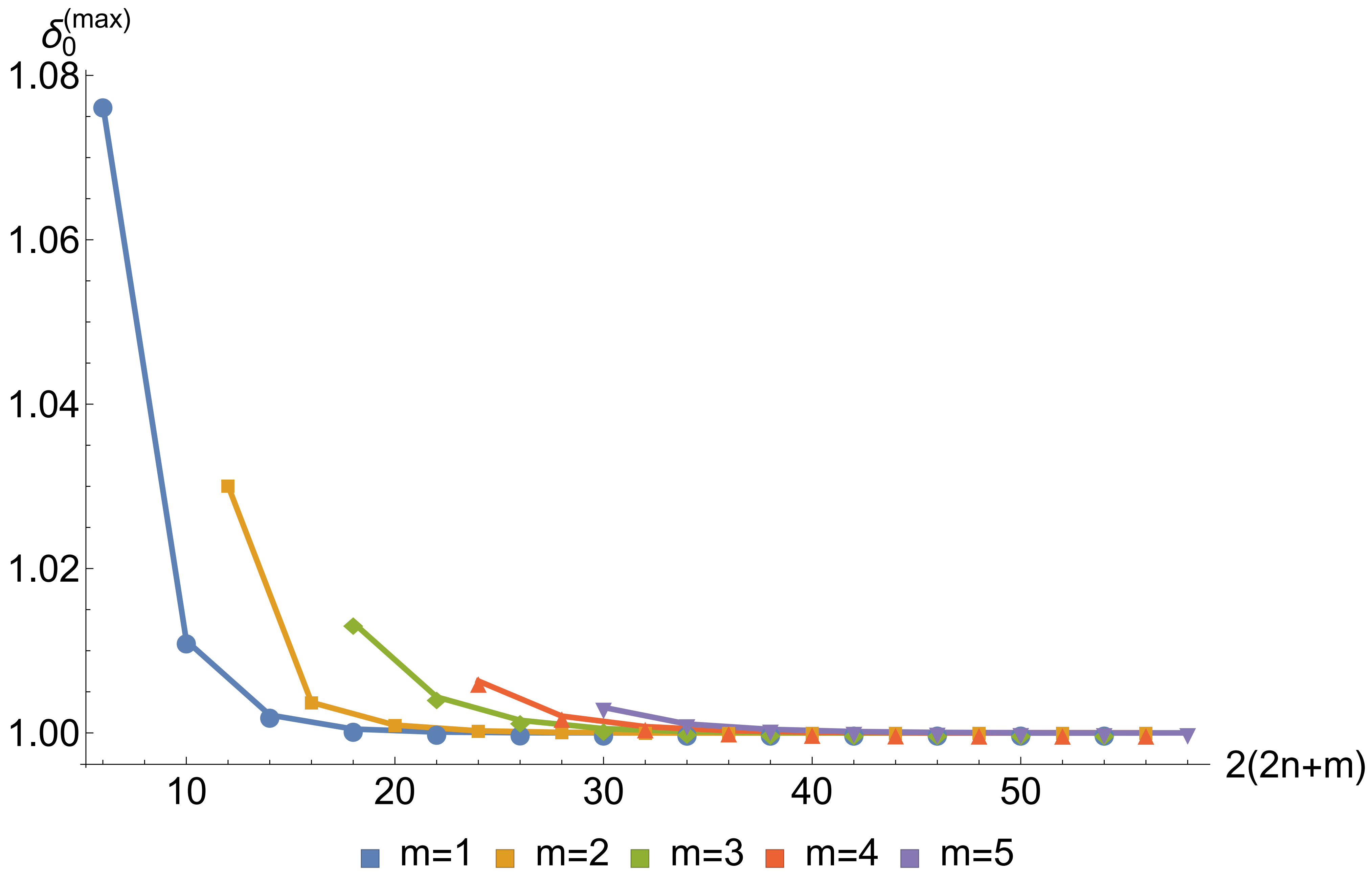
Testing ground: Dilaton scattering in type II string theory, Green & Wen '19

$$\mathscr{M}(s_1,s_2) = \frac{\Gamma(-s_1)\Gamma(-s_2)\Gamma(s_1+s_2)}{s_1 s_2(s_1+s_2)\Gamma(s_1)\Gamma(s_2)\Gamma(-s_1-s_2)} = \frac{1}{y} + 2\zeta(3) - 2\zeta(3)^2 y + 2\zeta(5)x + \frac{2}{3}(2\zeta(3)^3 + \zeta(9))y^2 - 4\zeta(3)\zeta(5)xy + \cdots$$

* **here $\mu = 0$, we can subtract the massless pole by choosing $\delta_0 > 0$**

Low energy constraints

- Say we were given the first few terms in the expansion.
- Can we say what maximum δ_0 we can use? This will be the location of the 1st massive string state.



Converges very rapidly to 1 **!!**

Comparison with recent work

- We can reproduce the limited number of inequalities available in the literature so far and generalise them.
- These include the "null-constraints" (which need crossing symmetry) of Tolley et al and Caron-huot and Duong which lead to two sided bounds in EFTs.
- These "null-constraints" or "locality constraints" appear to give a generalised Froissart bound...... AS, Zahed

2011.02957: Caron-huot, Duong; 2011.02400: Tolley et al

Extremal solution (string theory)

$$\sum_{r=0}^{m} \chi_n^{(r,m)}(\mu, \delta_0) \mathscr{W}^{(\delta_0)}_{n-r,r} = 0$$

- Extremal solution:

$\tilde{W}_{0,1} = -1.5$	$\tilde{W}^{(\mathrm{cl})}_{0,1} = -1.39348$

$\tilde{W}_{1,1} = -2.5$	$\tilde{W}_{0,2} = 1.5$
$\tilde{W}^{(\mathrm{cl})}_{1,1} = -2.47225$	$\tilde{W}^{(\mathrm{cl})}_{0,2} = 1.47958$

$\tilde{W}_{2,1} = -3.5$	$\tilde{W}_{1,2} = 4$	$\tilde{W}_{0,3} = -1.5$
$\tilde{W}^{(\mathrm{cl})}_{2,1} = -3.49239$	$\tilde{W}^{(\mathrm{cl})}_{1,2} = 3.98908$	$\tilde{W}^{(\mathrm{cl})}_{0,3} = -1.49593$

$$\tilde{W}_{p,q} = \frac{W_{p,q}}{W_{p+q,0}}$$

$\tilde{W}_{5,1} = -6.5$	$\tilde{W}_{4,2} = 17.5$	$\tilde{W}_{3,3} = -25$	$\tilde{W}_{2,4} = 20$	$\tilde{W}_{1,5} = -8.5$	$\tilde{W}_{0,6} = 1.5$
$\tilde{W}^{(\mathrm{cl})}_{5,1} = -6.49984$	$\tilde{W}^{(\mathrm{cl})}_{4,2} = 17.4994$	$\tilde{W}^{(\mathrm{cl})}_{3,3} = -24.9991$	$\tilde{W}^{(\mathrm{cl})}_{2,4} = 19.9993$	$\tilde{W}^{(\mathrm{cl})}_{1,5} = -8.49972$	$\tilde{W}^{(\mathrm{cl})}_{0,6} = 1.49995$

$\tilde{W}_{8,1} = -9.5$	$\tilde{W}_{7,2} = 40$	$\tilde{W}_{6,3} = -98$	$\tilde{W}_{5,4} = 154$	$\tilde{W}_{4,5} = -161$	$\tilde{W}_{3,6} = 112$	$\tilde{W}_{2,7} = -50$	$\tilde{W}_{1,8} = 13$	$\tilde{W}_{0,9} = -1.5$
$\tilde{W}^{(\mathrm{cl})}_{8,1} = -9.5$	$\tilde{W}^{(\mathrm{cl})}_{7,2} = 40$	$\tilde{W}^{(\mathrm{cl})}_{6,3} = -98$	$\tilde{W}^{(\mathrm{cl})}_{5,4} = 154$	$\tilde{W}^{(\mathrm{cl})}_{4,5} = -161$	$\tilde{W}^{(\mathrm{cl})}_{3,6} = 112$	$\tilde{W}^{(\mathrm{cl})}_{2,7} = -50$	$\tilde{W}^{(\mathrm{cl})}_{1,8} = 13$	$\tilde{W}^{(\mathrm{cl})}_{0,9} = -1.5$

$$\sum_{n,m=0}^{\infty} (-1)^n \frac{(2m+3n)\Gamma(m+n)}{m!n!} x^m y^n = \frac{3y-2x}{(x-y-1)} = \left(\frac{s_1}{1-s_1} + \frac{s_2}{1-s_2} + \frac{s_3}{1-s_3}\right) .$$

Bieberbach conjecture and QFT

de Branges's theorem

From Wikipedia, the free encyclopedia

In complex analysis, **de Branges's theorem**, or the **Bieberbach conjecture**, is a theorem that gives a necessary condition on a holomorphic function in order for it to map the open unit disk of the complex plane injectively to the complex plane. It was posed by Ludwig Bieberbach (1916) and finally proven by Louis de Branges (1985).

The statement concerns the Taylor coefficients a_n of a univalent function, i.e. a one-to-one holomorphic function that maps the unit disk into the complex plane, normalized as is always possible so that $a_0 = 0$ and $a_1 = 1$. That is, we consider a function defined on the open unit disk which is holomorphic and injective (*univalent*) with Taylor series of the form

$$f(z) = z + \sum_{n\geq 2} a_n z^n.$$

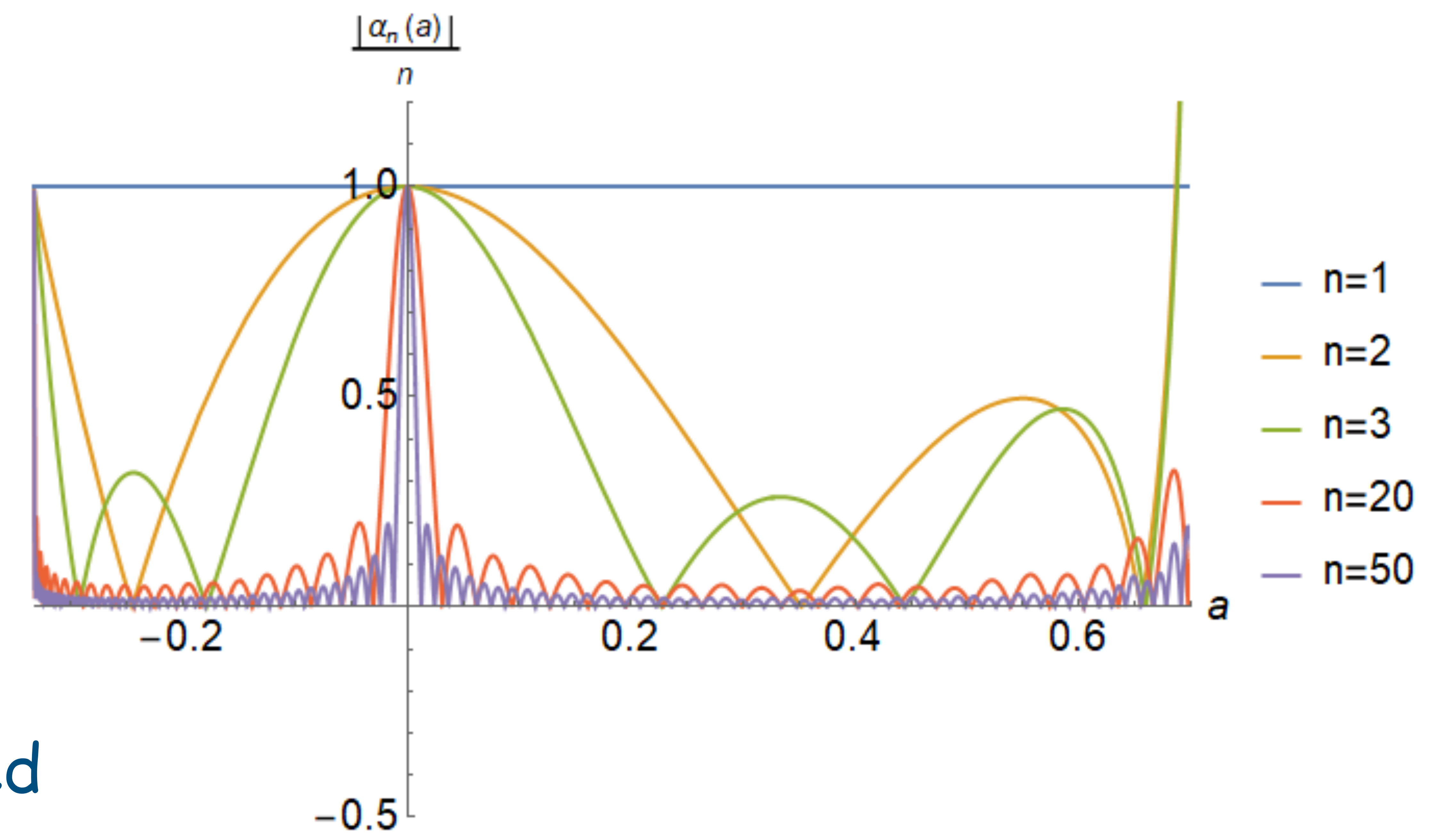
Such functions are called *schlicht*. The theorem then states that

$$|a_n| \leq n \quad \text{for all } n \geq 2.$$

$$z^3 \equiv \tilde{z}$$

$$\mathscr{M}(z,a) - \alpha_0 \to z^3 + \sum_{n\geq 2} \alpha_n z^{3n}$$

Crossing symmetric dispersion variables expansion



- 2103.12108 w P. Haldar and A. Zahed

Key point: Univalence

$$H = \frac{k(\tilde{z})(2s_1 - 3a)}{(s_1 - a)k(\tilde{z}) - s_1^3}$$

Kernel is a Mobius transformation of the Kobe function and is hence univalent provided no singularity inside unit disc. This gives
-8/9<a<16/9.

$$k(\tilde{z}) = 27a^2\frac{\tilde{z}}{(\tilde{z}-1)^2}$$

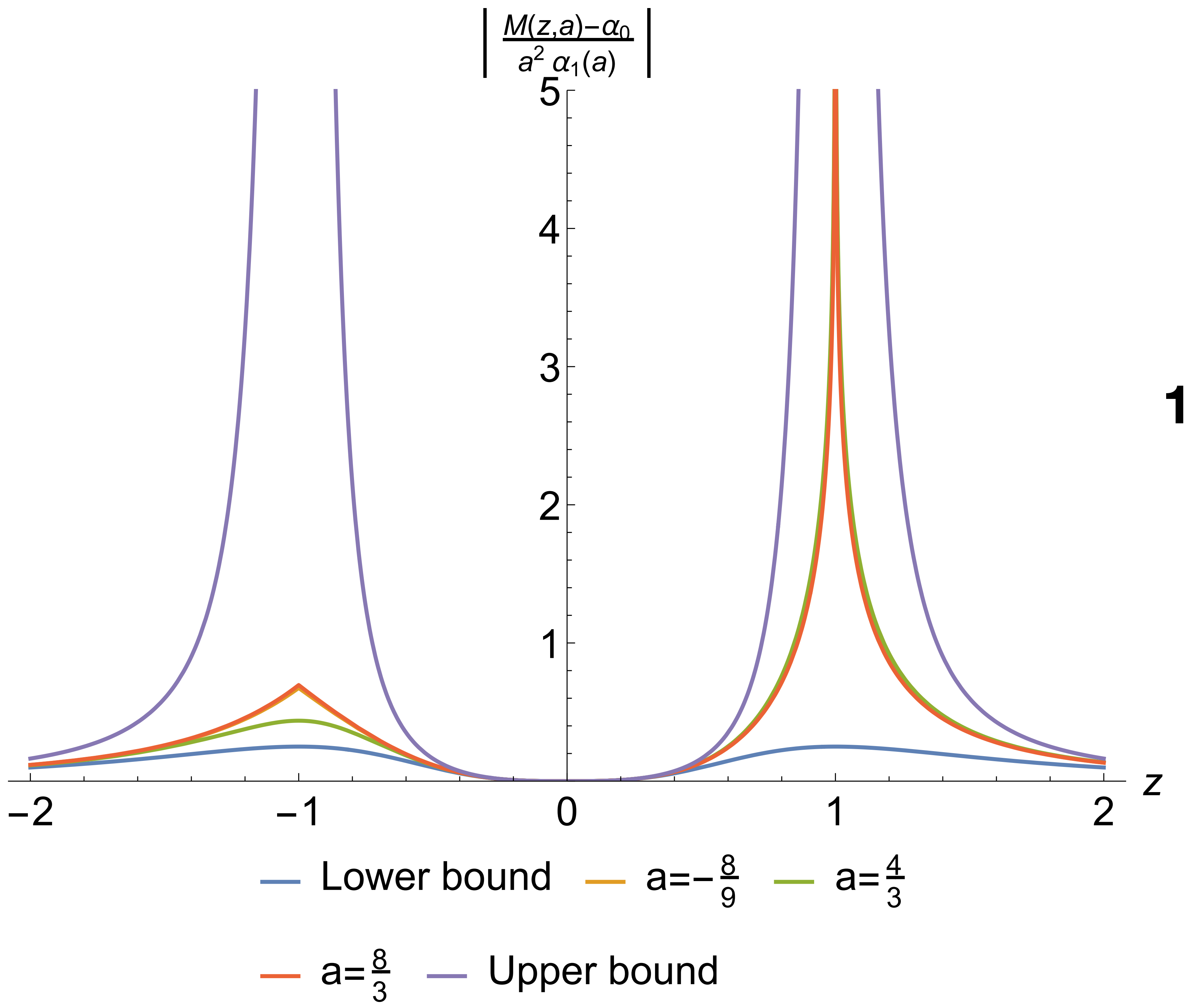
The amplitude for UNITARY theories then is a convex sum of univalent functions…..

Two sided bound on amplitude

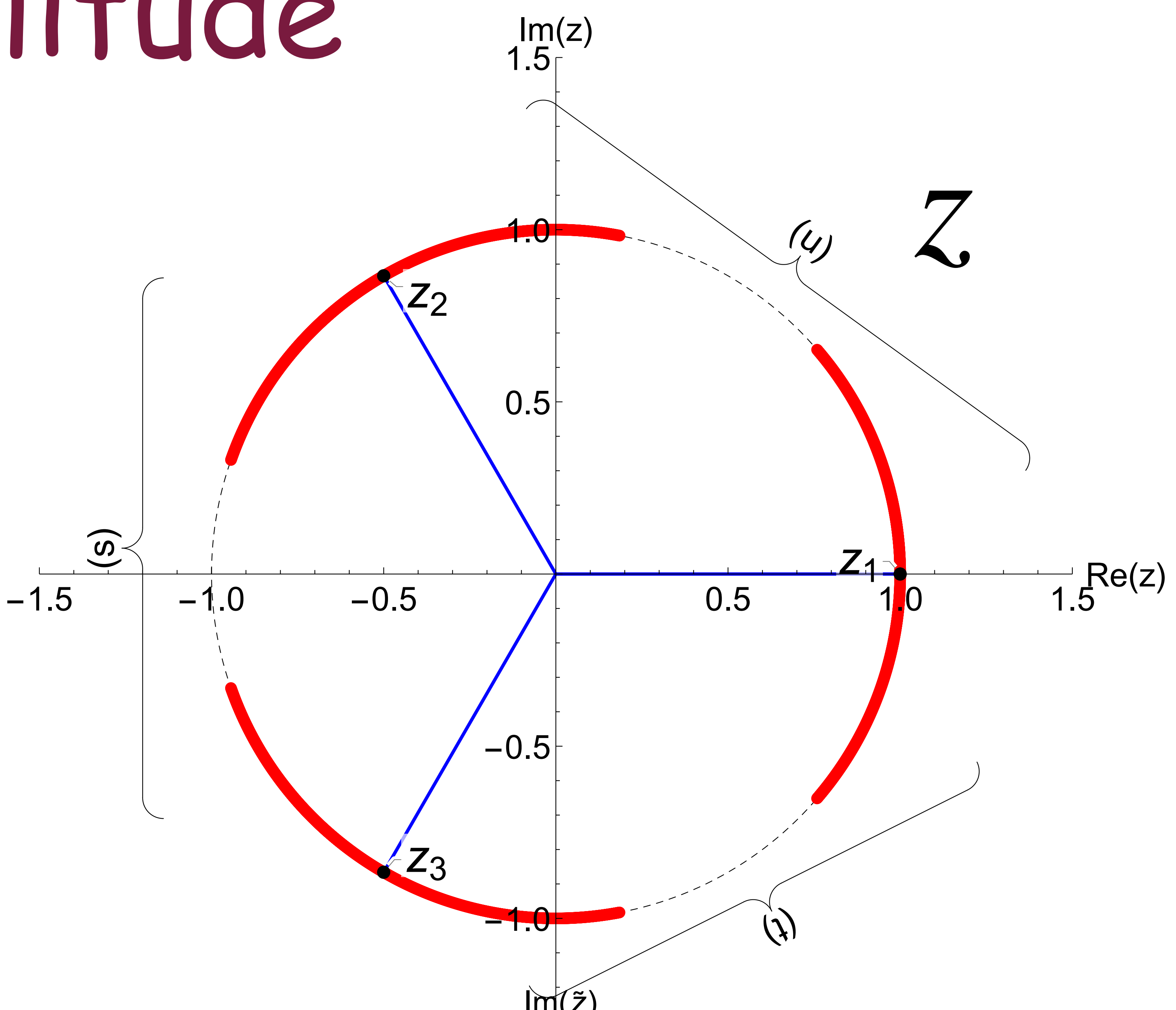
$$\left|\frac{z^3}{(1+z^3)^2}\right| \le \left|\frac{\mathcal{M}(z,a)-\alpha_0}{\alpha_1}\right| \le \left|\frac{z^3}{(-1+z^3)^2}\right|$$

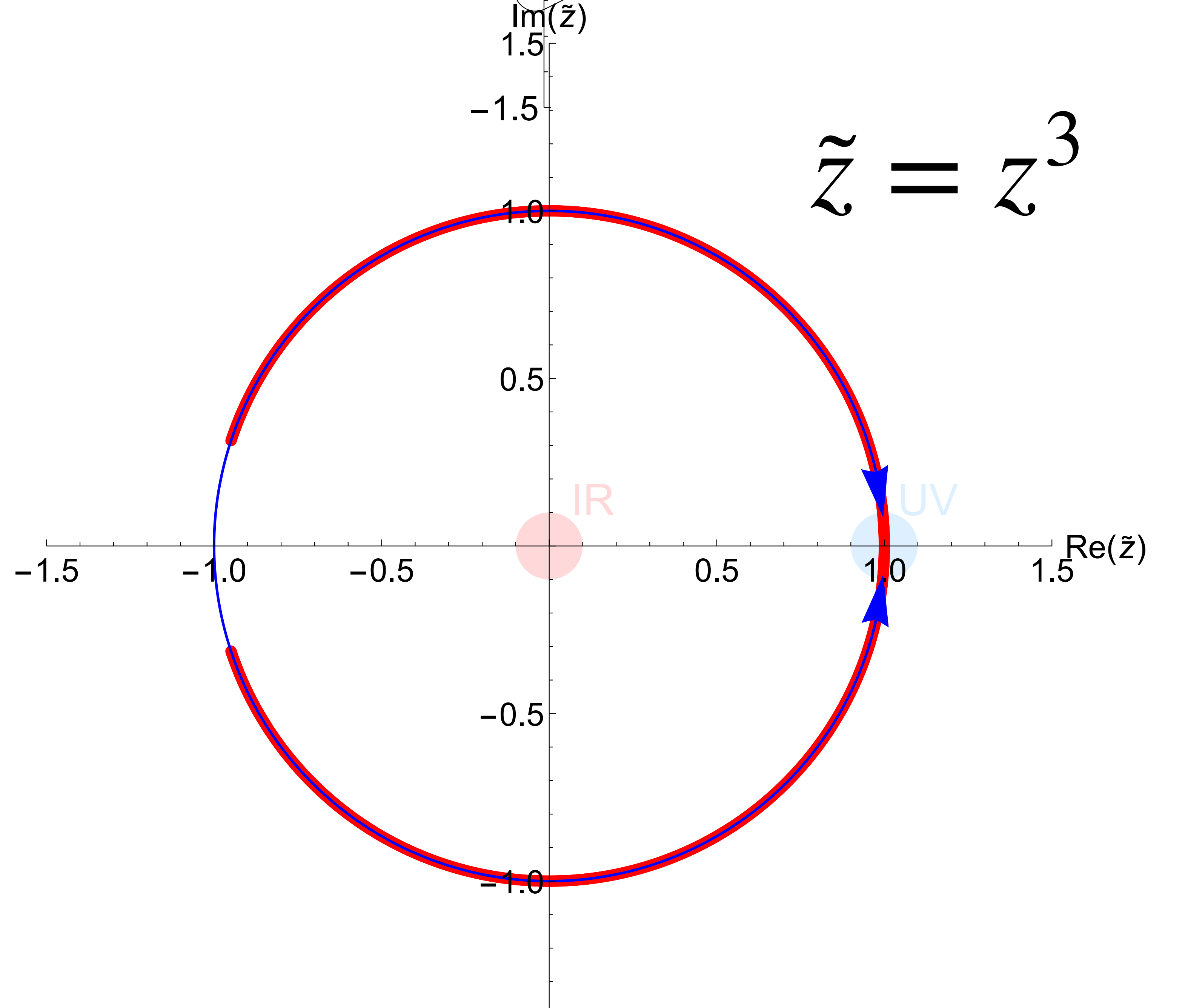
$$\frac{27a^2}{4}|a\mathcal{W}_{01}+\mathcal{W}_{10}| < |\mathcal{M}-\alpha_0| < \frac{|s_1|^2}{\sin^2\frac{\theta}{2}}|a\mathcal{W}_{01}+\mathcal{W}_{10}|$$

$$s_1 = |s_1|e^{i\frac{\theta}{2}}$$

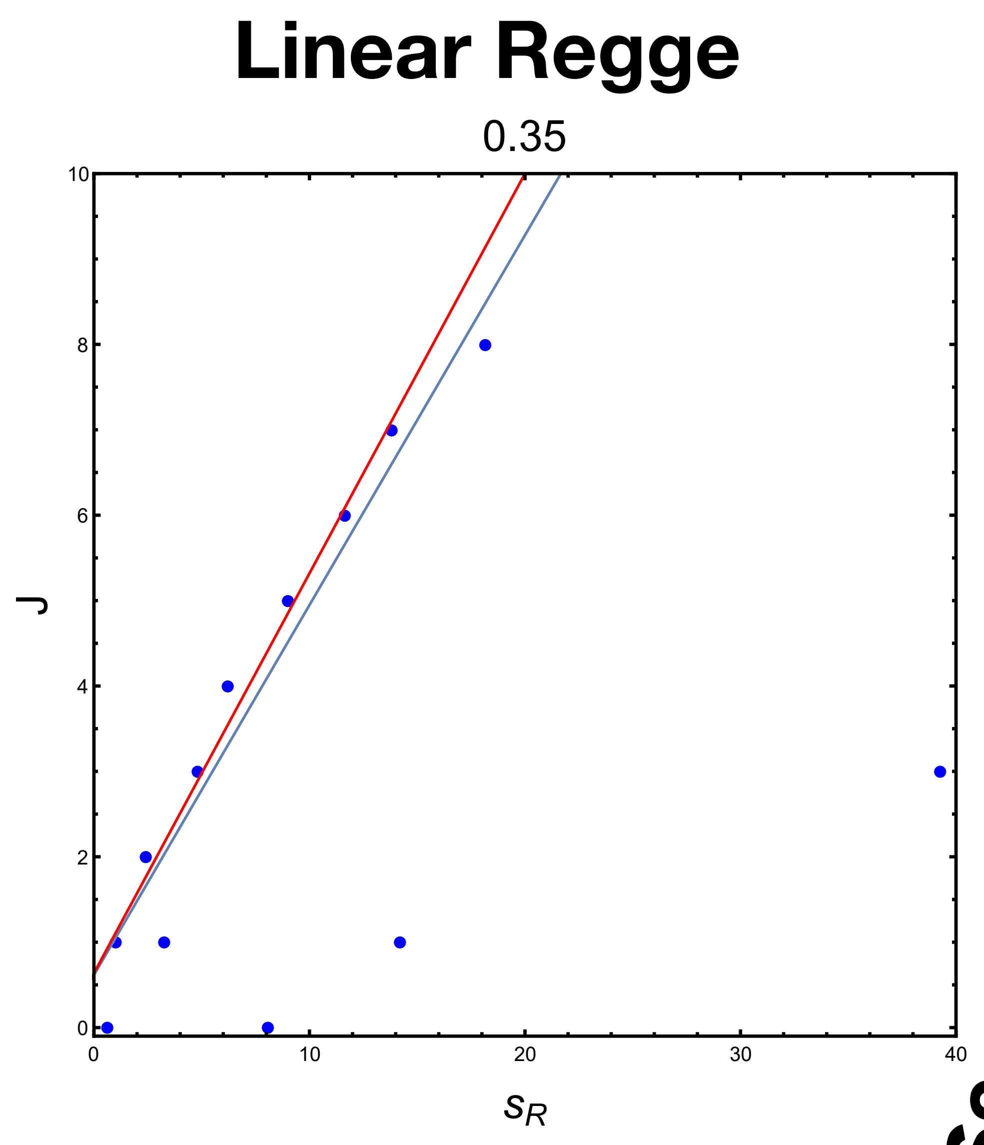


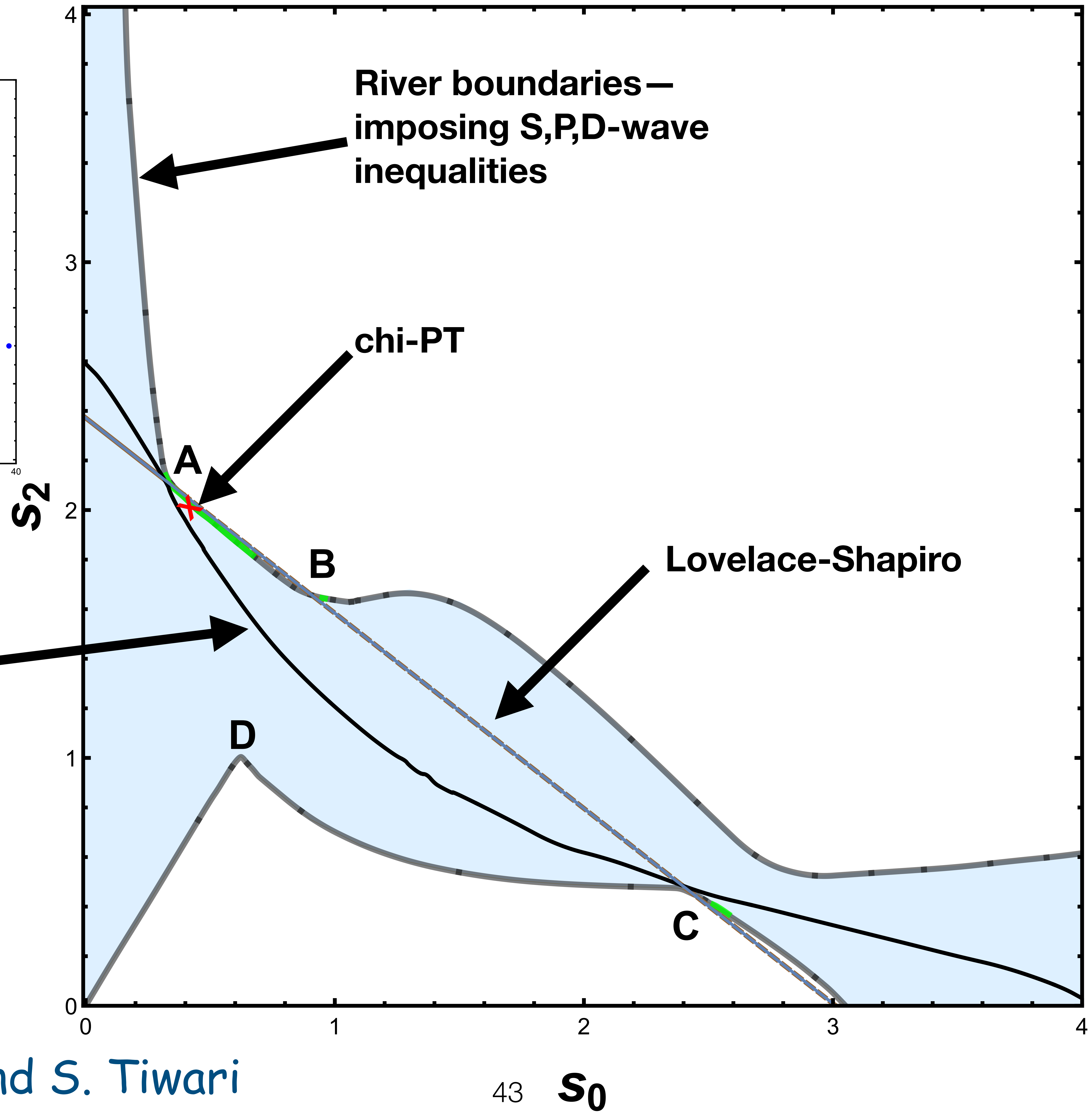
1-loop ϕ^4





Testing using S-matrix bootstrap

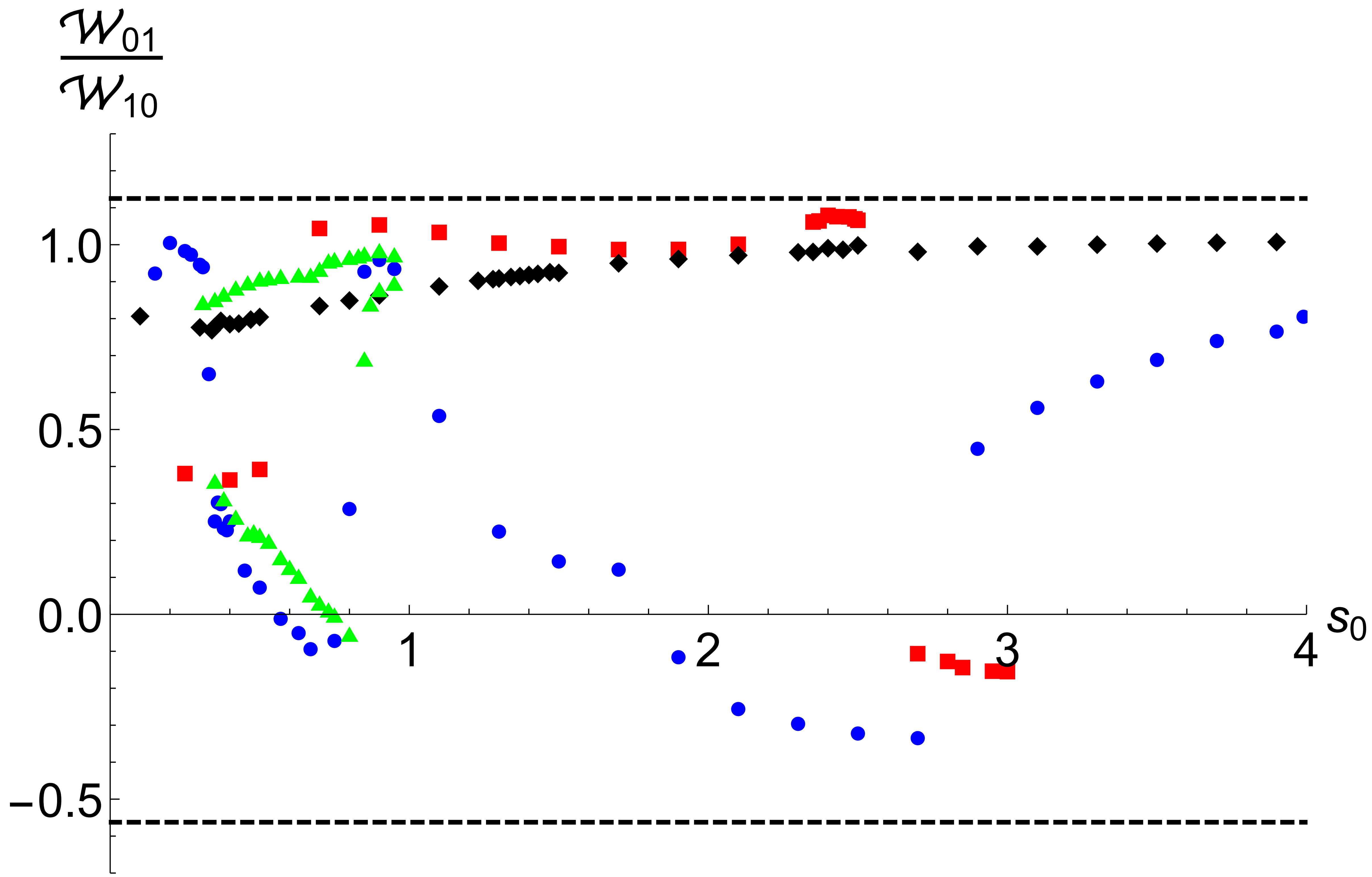




Builds on S-matrix bootstrap rebooted by Paulos, Penedones, van Rees, Vieira and adapted to pion bootstrap by Guerrieri, Penedones and Vieiera

- 2011.07944 w A. Bose and S. Tiwari

Two sided bound on Wilson coefficient



$$
{}^{*}\ -\frac{9}{16} < \frac{\mathscr{W}_{0,1}}{\mathscr{W}_{1,0}} < \frac{9}{8}\ {}^{**}
$$

Consequence of de Branges' theorem

***Same as Tolley et al; Caron-huot, van Duong**

****Stronger than any known result, dimension independent, coincides with Caron-huot, van Duong in the infinite dimension limit**

Froissart bound

A new look

- I will now briefly tell you how a Froissart like bound at high energy is derived from all this.
- The key element in deriving Froissart bound is to impose polynomial boundedness at the boundary of the Lehmann-Martin ellipse, namely at $t = 4$.
- We need this and some jiggery pokery black magic of the type

$$P_\ell(x) \geq c(x + \sqrt{x^2 - 1})^\ell\,, \quad x > 1$$

A new look

- Now since we have a dispersion relation (and we were using the twice subtracted form) we should recover the Froissart bound. The question is if the numerical coefficient in the bound can be made stronger, namely

$$\sigma < \frac{\pi}{m^2} \log^2 \frac{s}{s_0}$$

Numerical factor apparently 2 orders of magnitude bigger than experiments:

Froissart '59, Martin '65……, Martin-Roy, …..many many papers

A new look

- We will use the 1st locality constraint.
- Now recall that the locality constraints used an expansion around a=0—this is in the interior of the Lehmann-Martin ellipse, namely at $t = 4/3$.
- Therefore it should not come as a surprise that what we land up finding is in fact

$$\sigma < \frac{9\pi}{m^2} \log^2 \frac{s}{s_0}$$

A new look

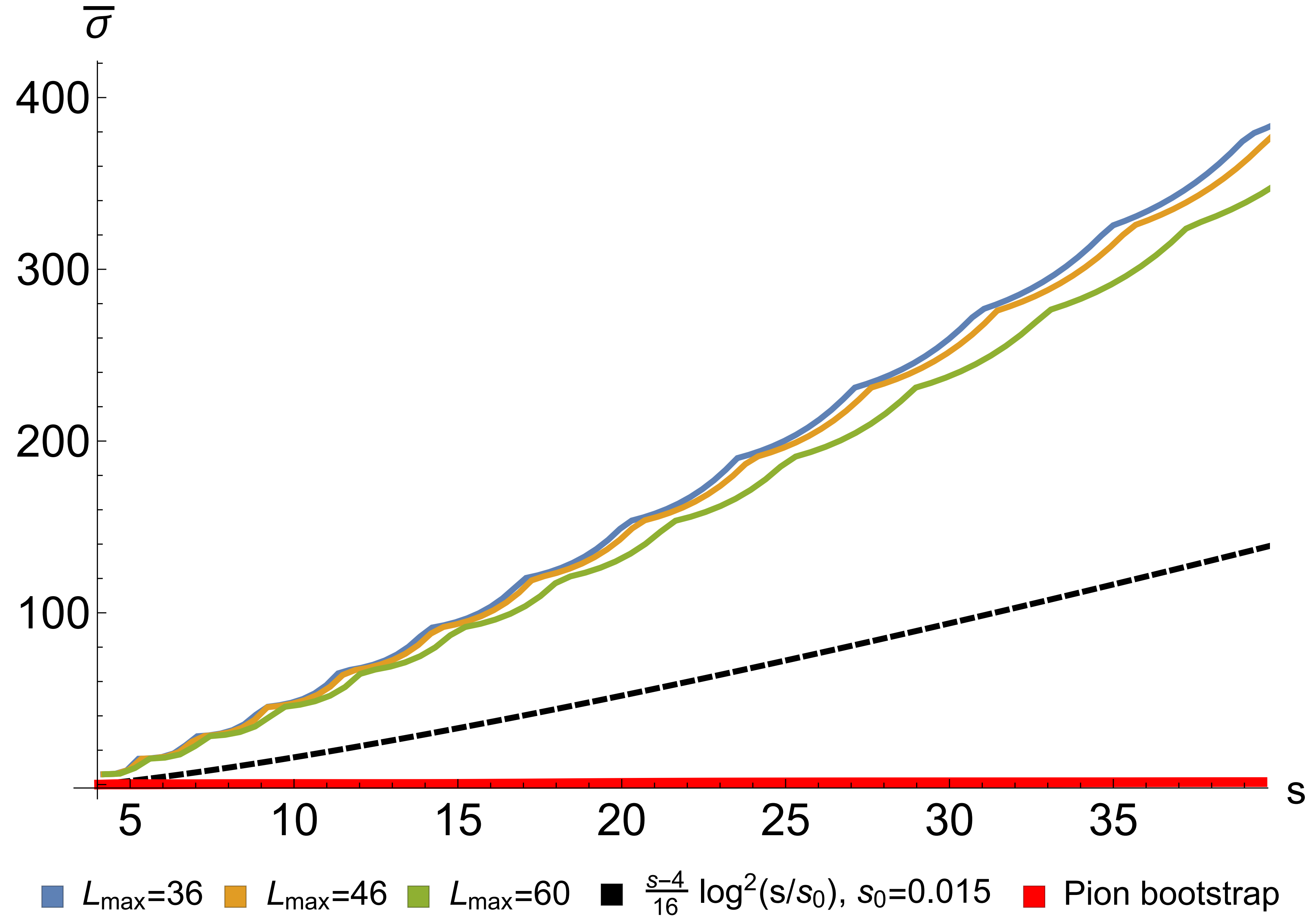
- I believe a better result is possible.
- To do that we need to find a better version of the following step.
- We need this and some jiggery pokery black magic of the type $$P_\ell(x) \geq c(x + \sqrt{x^2 - 1})^\ell\,, \quad x > 1$$
- Adapted to the $B^{(\ell)}_{n,m}(s_1)$ which entered our game.

A new look

- What we have been able to show is a bound that should be for any s.

$$\int_4^{s_{max}} \sum_{\ell=4}^{\infty} (2\ell+1) a_\ell(s) < \lambda \frac{\pi}{360} \frac{(s_{max}-4)^4 (3s_{max}-4)^3}{9s_{max}^2 - 30 s_{max} + 32}$$

- Borne out by numerics



Derivation of the Polyakov bootstrap

- 2101.09017 with R. Gopakumar and A. Zahed

Key finding

$$\text{Crossing Constraints}_t \bigoplus \text{Polyakov}_t = \text{Locality Constraints}_c \bigoplus \text{Polyakov}_c$$

Benchmarking: Fixed-t dispersion relation

1912.11100: Penedones, Silva, Zhiboedov; 2008.04931: Caron-huot, Mazac, Rastelli, Simmons-duffin; 2009.13506: Carmi, Penedones, Silva, Zhiboedov;2011.02957: Caron-huot, Duong; 2011.02400: Tolley et al ....

- Mellin conventions:

$$\mathcal{G}(u,v)=\int\frac{ds_1}{2\pi i}\frac{ds_2}{2\pi i}u^{s_1+\frac{2\Delta_\phi}{3}}v^{s_2-\frac{\Delta_\phi}{3}}\mu(s_1,s_2)\mathscr{M}(s_1,s_2)$$

Two kinds of sum rules

- In these 2 papers*, there are 2 kinds of sum rules I) Which follow from just crossing symmetry. II) Which follow from the Polyakov conditions. Let us begin with I.

$$\frac{\mathcal{M}(s_1, s_2)}{s_1 s_3} = \frac{1}{2\pi i} \oint_{s_1} \frac{ds_1'}{s_1' - s} \frac{\mathcal{M}(s_1', s_2)}{s_1'(-s_1' - s_2)}$$

* 1912.11100: Penedones, Silva, Zhiboedov; 2009.13506: Carmi, Penedones, Silva, Zhiboedov

Crossing sum rules

- Now put in the conformal partial wave expansion

$$\mathcal{M}(s_1,s_2) = \underbrace{\mathcal{M}(0,s_2)}_{G(s_2^2)} + \sum_{\Delta,\ell,k}^{\infty} c^{(k)}_{\Delta,\ell} P_{\Delta,\ell}(\tau_k,s_2)\left(\frac{1}{\tau_k - s_1} - \frac{1}{\tau_k} + \frac{1}{\tau_k + s_1 + s_2} - \frac{1}{\tau_k + s_2}\right)$$

- Using crossing symmetry we get

$$\mathcal{M}(s_1,s_2) - \mathcal{M}(s_2,s_1) = \sum_{p,q} s_1^p s_2^q \mathcal{E}_{p,q} = 0$$

1912.11100: Penedones, Silva, Zhiboedov, not well studied so far

Crossing sum rules

- A similar story is also true for QFT. There we will replace with partial wave expansion in terms of Gegenbauers.

- These constraints in effective field theories were termed "null-constraints" by Caron-huot and Duong and lead to two sided bounds in EFTs.

2011.02957: Caron-huot, Duong; 2011.02400: Tolley et al

Polyakov conditions

- Sum rules II follow from Polyakov conditions (no operators with $\Delta = 2\Delta_\phi + \ell + 2n$ exactly). The kind we will focus on are:

$$\omega_{p_1,p_2,p_3}(s_2) \equiv \oint_{C_\infty} \frac{ds_1}{2\pi i} \mathcal{M}(s_1, s_2) \frac{2}{\left(\frac{\Delta_\phi}{3} + p_1 - s_1\right)\left(\frac{\Delta_\phi}{3} + p_2 - s_1\right)\left(\frac{\Delta_\phi}{3} + p_3 + s_1 + s_2\right)} = 0$$

2009.13506: Carmi, Penedones, Silva, Zhiboedov

Comments on locality

- Putting in partial wave expansion in $\dfrac{\mathcal{M}(s_1,s_2)}{s_1 s_3} = \dfrac{1}{2\pi i}\oint_{s_1} \dfrac{ds_1'}{s_1'-s}\dfrac{\mathcal{M}(s_1',s_2)}{s_1'(-s_1'-s_2)}$ assuming no bound state poles, we see that only positive powers of s_1, s_2 appear in $\mathcal{M}(s_1,s_2)$ expanded around $s_1=0, s_2=0$.

- This is what we will loosely refer to as locality. Fixed-t dispersion has locality in-built. But impose crossing.

RECALL: Crossing sum rules fixed-t

- Now put in the conformal partial wave expansion

$$\mathcal{M}(s_1,s_2)=\underbrace{\mathcal{M}(0,s_2)}_{G(s_2^2)}+\sum_{\Delta,\ell,k}^{\infty}c^{(k)}_{\Delta,\ell}P_{\Delta,\ell}(\tau_k,s_2)\left(\frac{1}{\tau_k-s_1}-\frac{1}{\tau_k}+\frac{1}{\tau_k+s_1+s_2}-\frac{1}{\tau_k+s_2}\right)$$

- Using crossing symmetry we get

$$\mathcal{M}(s_1,s_2)-\mathcal{M}(s_2,s_1)=\sum_{p,q}s_1^p s_2^q\mathcal{E}_{p,q}=0$$

These are precisely our locality constraints

1912.11100: Penedones, Silva, Zhiboedov

Some details

$$\Omega_{p_1,p_2,p_3} = -\frac{\mathcal{M}(\frac{\Delta_\phi}{3}+p_1,s_2)}{(p_1-p_2)(p_1+p_3+s_2+\frac{2\Delta_\phi}{3})} - \frac{\mathcal{M}(\frac{\Delta_\phi}{3}+p_2,s_2)}{(p_2-p_1)(p_2+p_3+s_2+\frac{2\Delta_\phi}{3})} - \frac{\mathcal{M}(\frac{\Delta_\phi}{3}+p_3,s_2)}{(p_1+p_3+s_2+\frac{2\Delta_\phi}{3})(p_2+p_3+s_2+\frac{2\Delta_\phi}{3})}$$

$$\Omega_{p_1,p_2,p_3}(s_2) = \sum_{r=0}^{\infty} s_2^r \Omega^{(r)}_{p_1,p_2,p_3}$$

—crossing symmetric sum rules

$$\omega_{p_1,p_2,p_3}(s_2) = \sum_{r=0}^{\infty} s_2^r \omega^{(r)}_{p_1,p_2,p_3}$$

—fixed t sum rules, Sum rules "II"

$$\omega^{(i)}_{p_1,p_2,p_3} = \Omega^{(i)}_{p_1,p_2,p_3}, \quad i = 1,\cdots,5$$

—trivial equivalence

$$\omega^{(6)}_{p_1,p_2,0} = \Omega^{(6)}_{p_1,p_2,0} + \#\mathscr{W}_{-1,2}$$

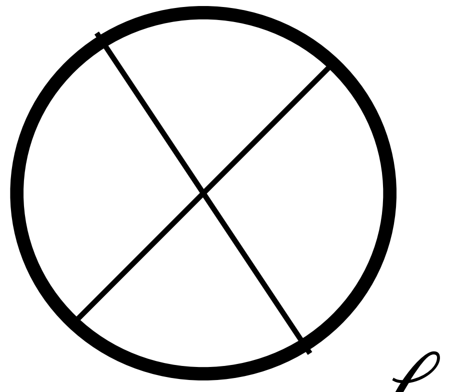
—equivalence on imposing locality constraints/sum rules "I"

The partial wave expansion is convergent for $-\tau^{(0)}/3 < a < 2\tau^{(0)}/3$

* α_0 cancels ** $\mathscr{W}_{-p,q}$**s have nice properties.**

Contact terms and connection to previous work

- A natural question from the purview of crossing symmetric dispersion: Can we expand in a basis where the locality constraints are inbuilt?
- Ans: Yes. These are the "Witten" blocks where all $\ell \geq 2$ contact terms are fixed.

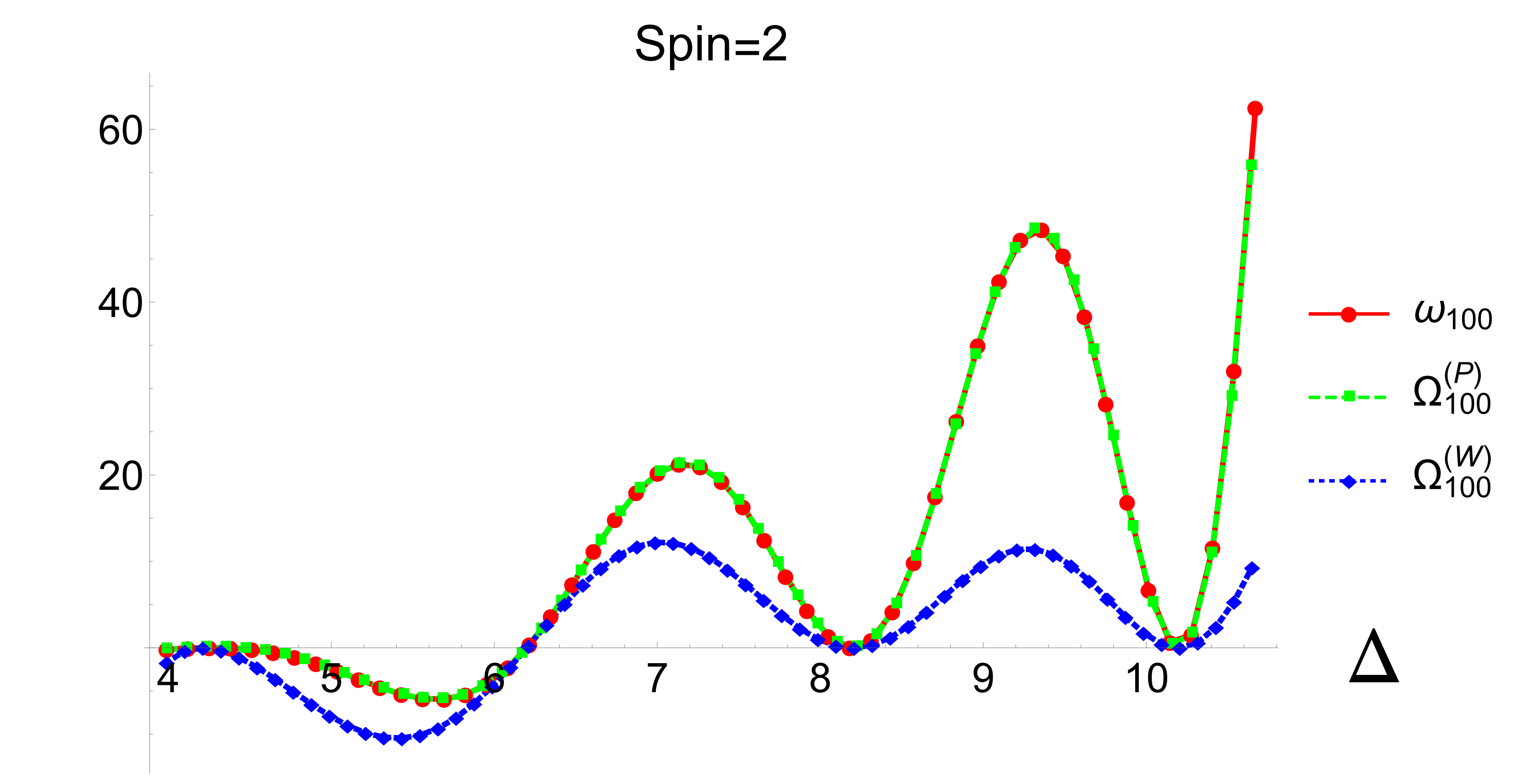
Eg  $\Big|_{\ell=2} = \sum_{\Delta,k} c^{(k)}_{\Delta,2}\left(\frac{2xb^{(2)}_{2,0}}{\tau_k} - \frac{yb^{(2)}_{0,2}}{\tau_k^2}\right)$

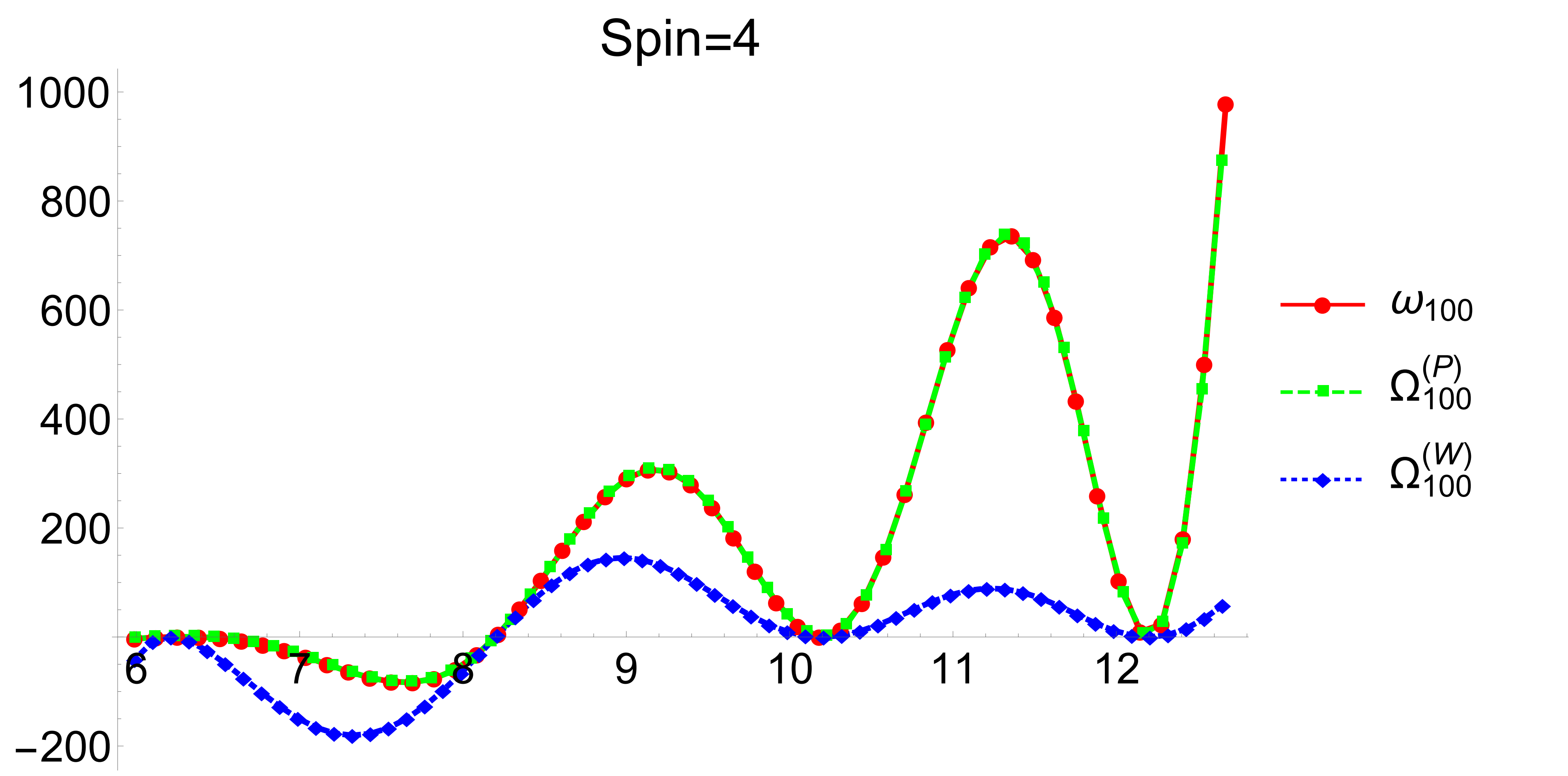
.....similar (complicated) polynomial forms for higher spins; explicit forms we have worked out till spin 10.

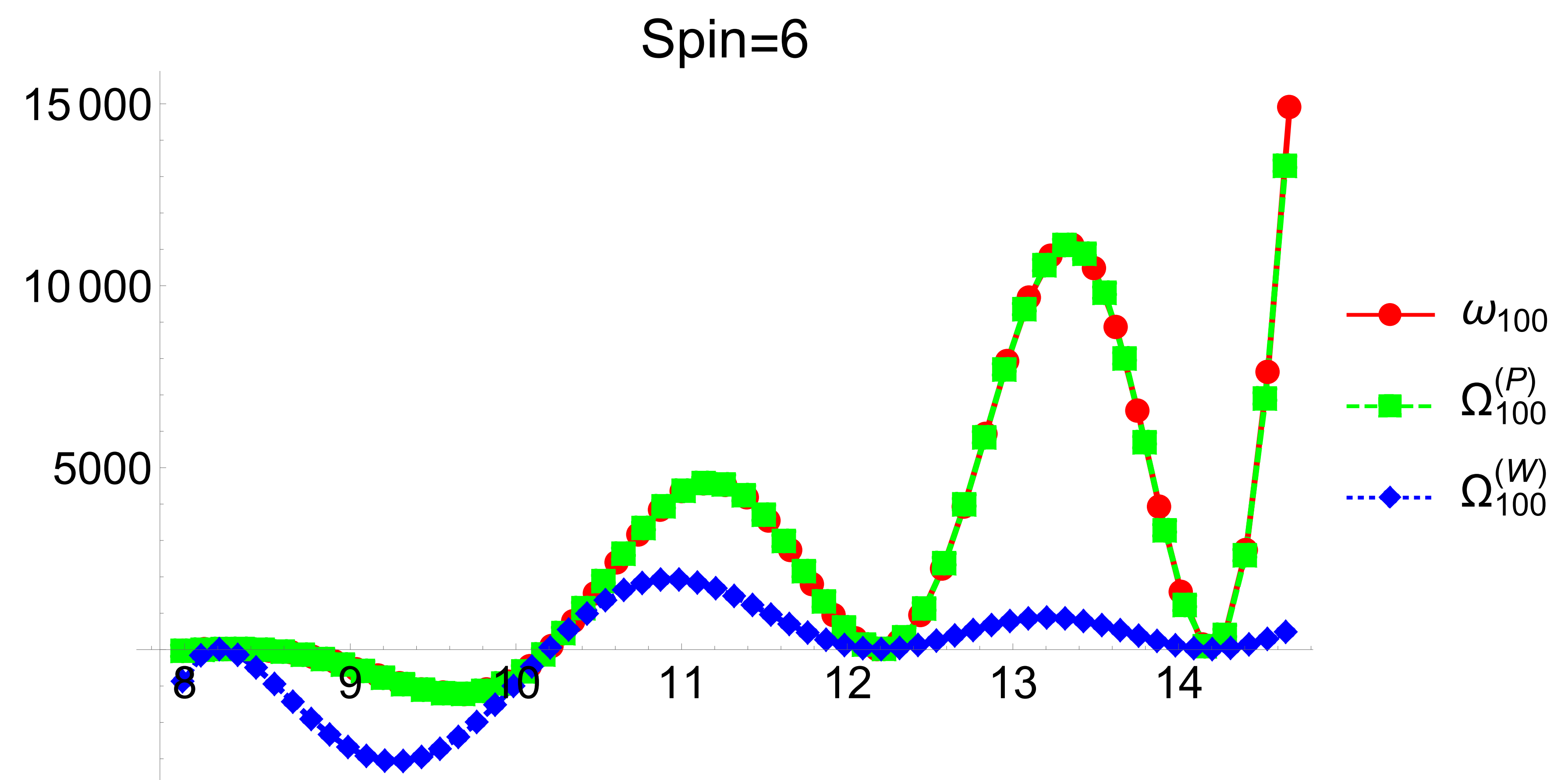
$$\sum_{\{m,n\}=0}^{m+n=\ell/2} a_{mn}x^m y^n$$

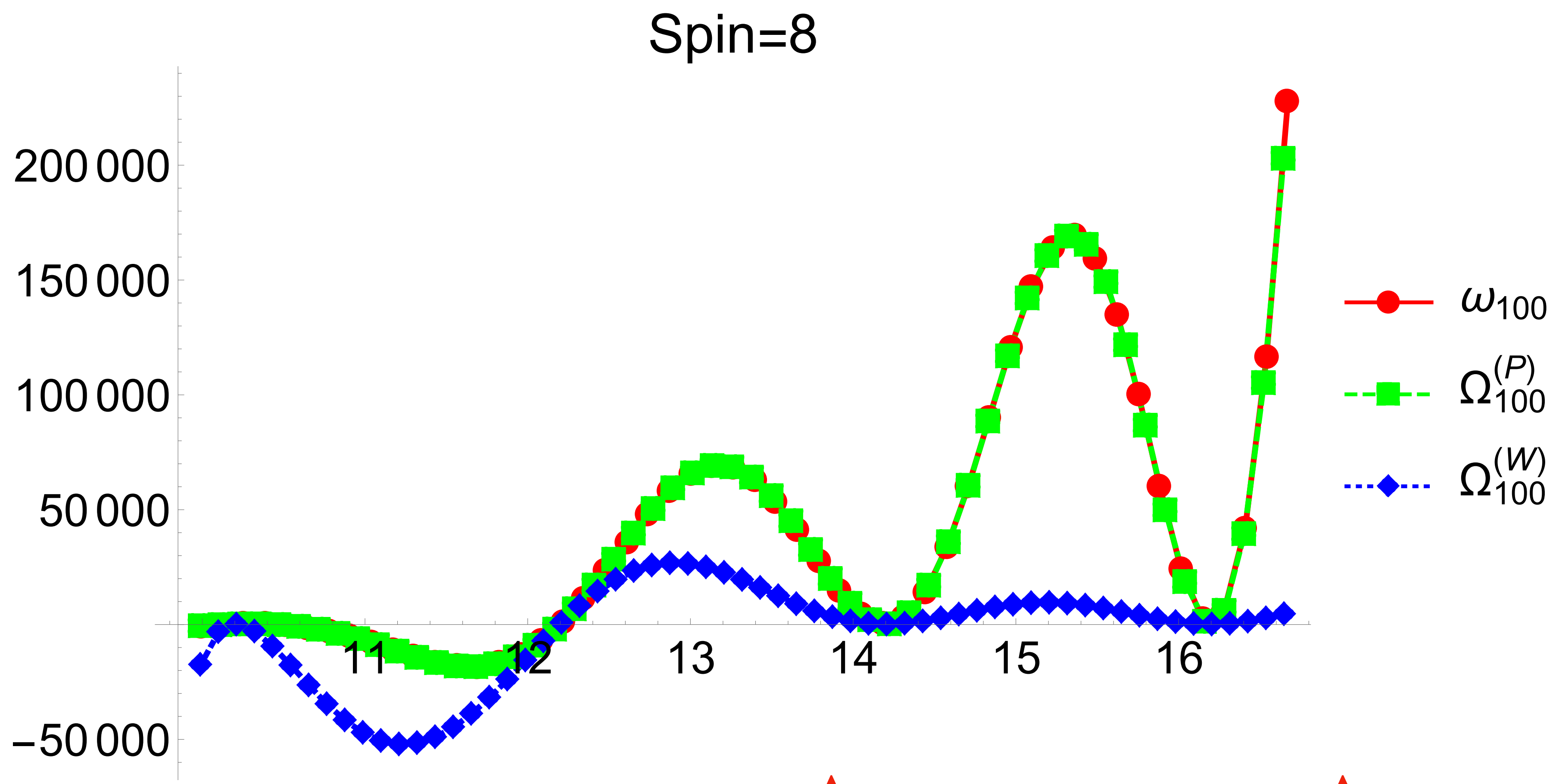
Gopakumar, AS '18; Heemskerk, Penedones, Polchinski, Sully

Analytic functionals (d=4)





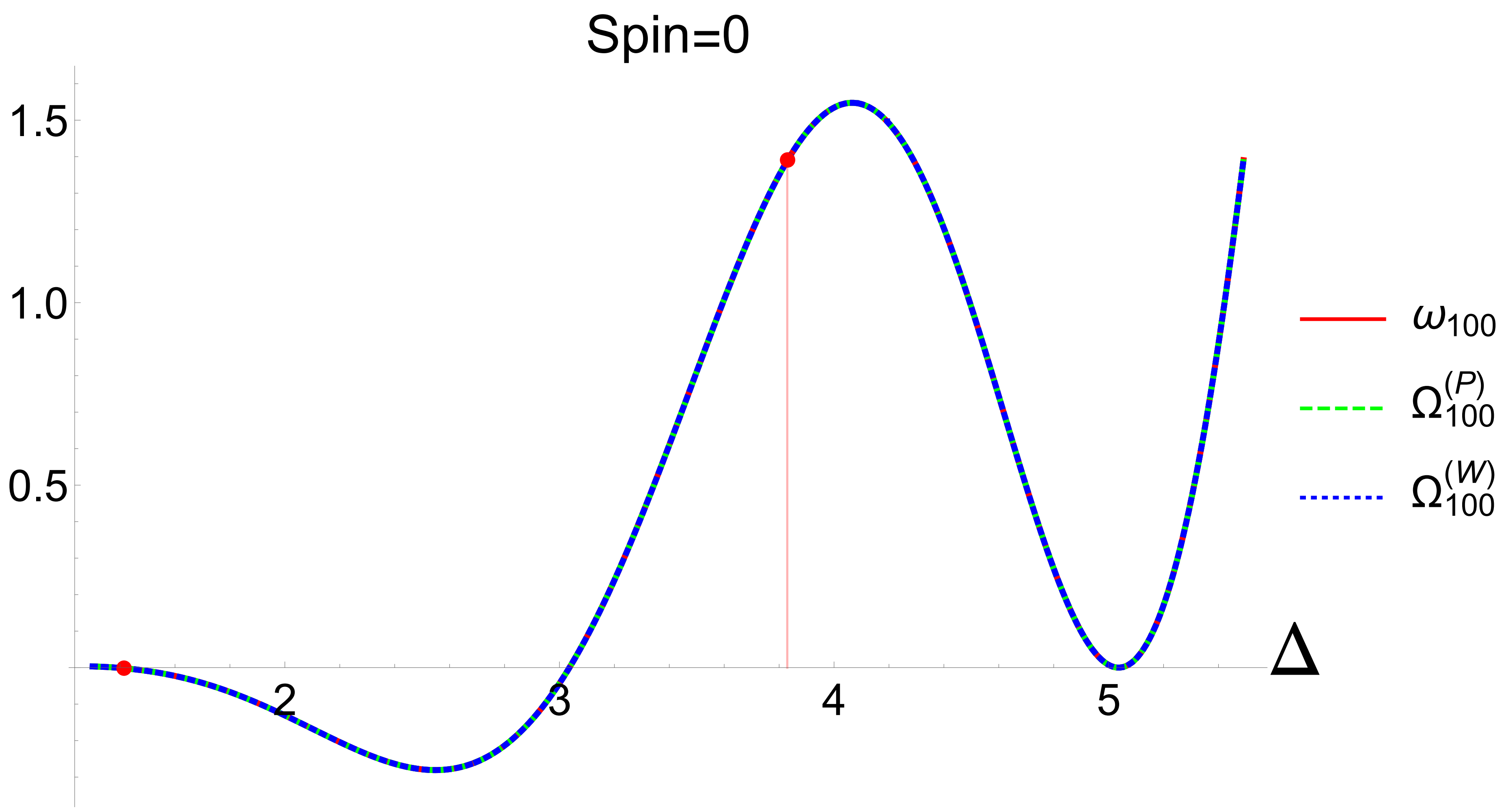


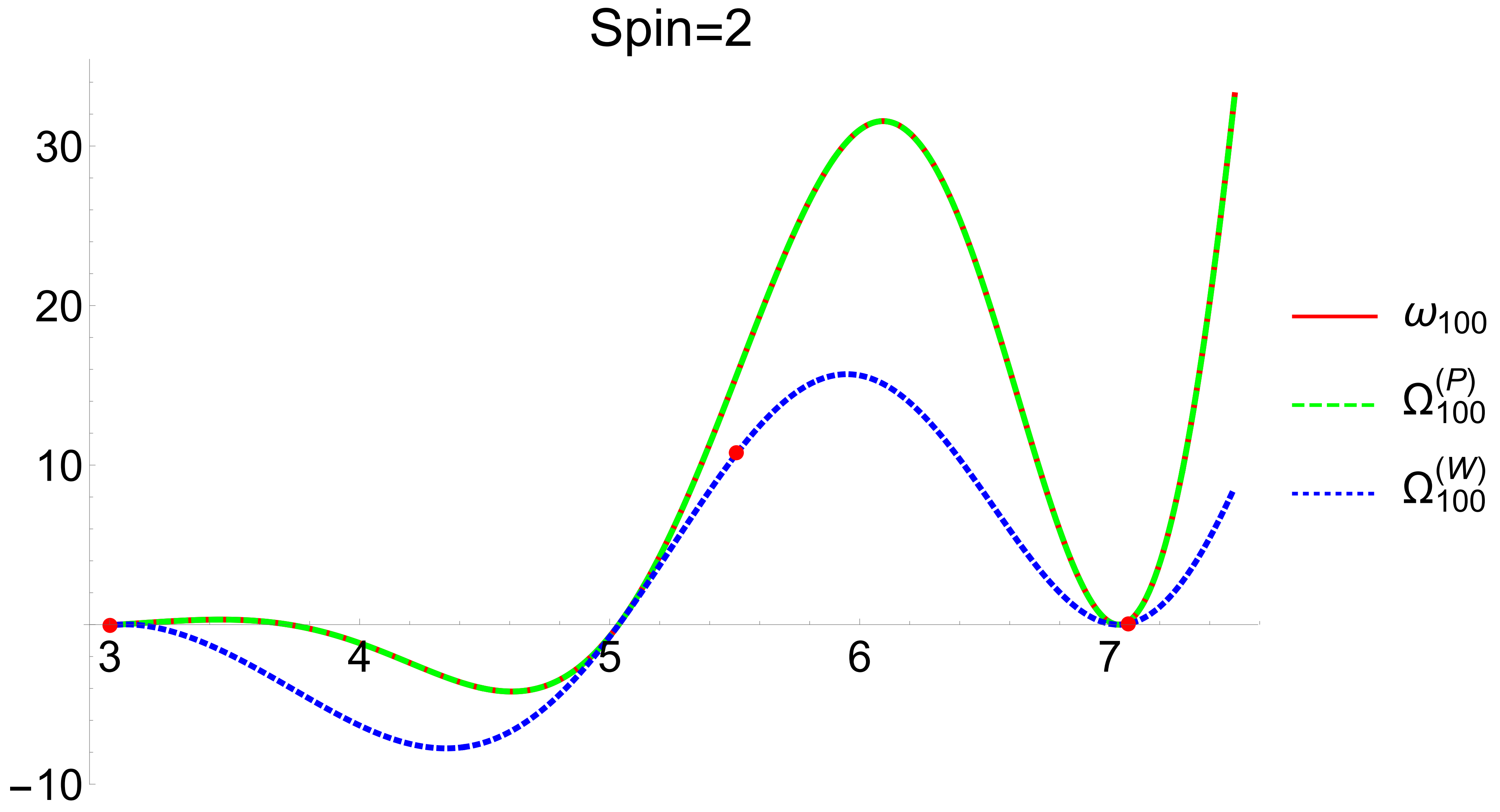


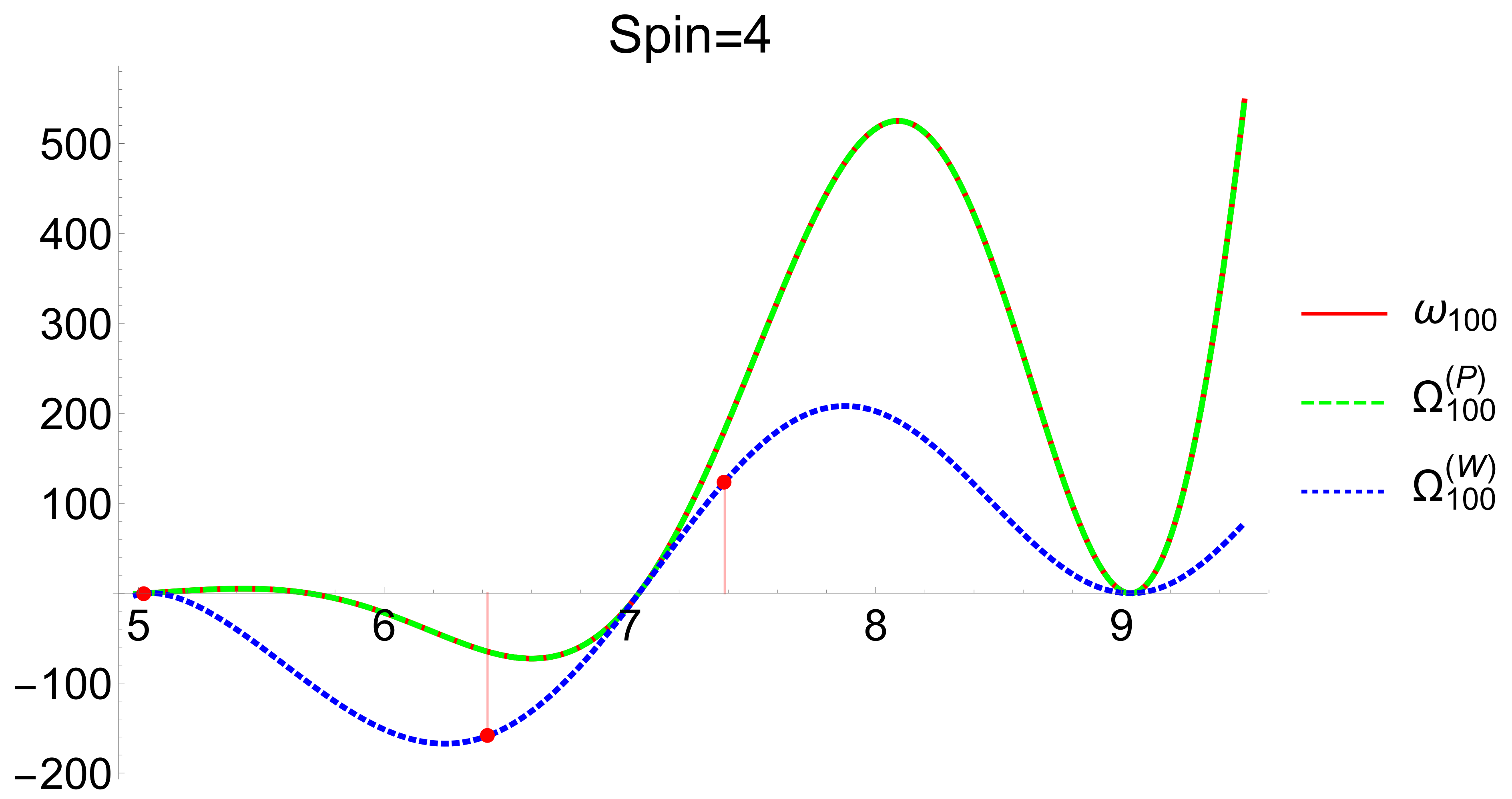
ω_{100} **in 2009.13506: Carmi, Penedones, Silva, Zhiboedov**

$* \; \Omega_{p1,0,0} \propto \mathcal{M}(\frac{\Delta_\phi}{3} + p_1, s_2) - \mathcal{M}(\frac{\Delta_\phi}{3}, s_2)$

Analytic functionals (d=3)







Final comments

- Crossing symmetric dispersion relation is proving quite handy/powerful in examining EFTs.
- In cases which do not have 3 channel crossing symmetry, we can develop similar methods (in progress with Raman). We believe this will be handy in processes like Moller/Bhabha scattering etc. We are trying to connect with the EFT-hedron story.
- We put Polyakov bootstrap on firm footing.
- Further applications in CFTs are expected (eg. two sided bounds on CFT correlator in position space by Paulos).

Thank you