A scattering amplitude in conformal field theory

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Based on arXiv:2003.07361
with Marco Meineri and João Penedones

Also 2012.09825, 1909.00878, 1807.07003,
and earlier work with X. Lu and M. Luty

Rencontres Théoriciennes ⋄ (virtual) Paris ⋄ April 22, 2021

Disclaimer

This talk is **not** about scattering of wavepackets in CFT

Instead, it is about defining an CFT observable that is:

- Derived from time-ordered correlator ($\sim$ LSZ reduction)
- Crossing-symmetric
- Analytic up to branch cuts on the real axis
- Positive imaginary part in the forward limit, sum of lower-point functions

$\Rightarrow$ legitimate to call it a "scattering amplitude"?

Outline

An axiomatic approach to CFT

The fundamental observables in CFT are correlation functions:

- 2- and 3-point functions fixed in terms of CFT data

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}}$$

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{|x_1 - x_2|^{\Delta_{12|3}}|x_1 - x_3|^{\Delta_{13|2}}|x_2 - x_3|^{\Delta_{23|1}}}$$

- 4- and higher-point functions computable using the OPE

$$\phi_1(x_1)\phi(x_2) = \sum_k f_{12k}\left(x_1 - x_2, \frac{\partial}{\partial x_2}\right)\phi_k(x_2)$$

Rapidly convergent over most of the configuration space

Fourier transform to momentum space

$$\phi(p) \stackrel{?}{=} \int d^d x \, e^{ip\cdot x} \phi(x)$$

Bad idea in Euclidean space:

- Need to define correlators at coincident points
- Spoils the convergence of the OPE

Good idea in Minkowski space:

- Wightman correlators are tempered distributions

Kravchuk, Qiao, Rychkov 2020 & 2021

- Orthogonality of momentum eigenstates

$\Rightarrow$ conceptual simplicity

The 2-point function in momentum space

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}}$$

Step 1: analytic continuation from Euclidean to Minkowski

$$\langle 0|\, \phi(x_1)\phi(x_2)\, |0\rangle = \frac{1}{\left[-(x_1^0 - x_2^0 - i\epsilon)^2 + (\mathbf{x}_1 - \mathbf{x}_2)^2\right]^{\Delta}}$$

(not symmetric $1 \leftrightarrow 2$)

Step 2: Fourier transform

$$\langle 0|\, \phi(p_1)\phi(p_2)\, |0\rangle = \frac{2^{2d-2\Delta+1}\pi^{3d/2+1}}{\Gamma(\Delta)\Gamma\left(\Delta - \frac{d}{2} + 1\right)} \qquad (\geq 0)$$

$$\times\, \delta^d(p_1 + p_2)\Theta(p_2^0)\Theta(-p_2^2)\,(-p_2^2)^{\Delta - d/2}$$

Translation symmetry in momentum space

Translation symmetry $\Leftrightarrow$ momentum conservation

Notation:

$$\langle 0|\, \phi_1(p_1)\cdots\phi_n(p_n)\, |0\rangle = (2\pi)^d\delta^d(p_1+\ldots+p_n)\langle\!\langle\phi_1(p_1)\cdots\phi_n(p_n)\rangle\!\rangle$$

Example: scalar 2-point function

$$\langle\!\langle\phi(-p)\phi(p)\rangle\!\rangle = C_\phi\Theta(p^0)\Theta(-p^2)\,(-p^2)^{\Delta-d/2}$$

Spectral condition:

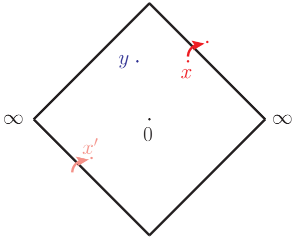
- vanishes outside forward light-cone in p
- positive: norm of the state $\phi(p)\,|0\rangle$

Special conformal symmetry in momentum space

Most general ansatz consistent with
Poincaré + scale symmetry + spectral condition

$$\langle\langle \phi_1(-p)\phi_2(p)\rangle\rangle \stackrel{?}{=} C\,\Theta(p^0)\Theta(-p^2)(-p^2)^{(\Delta_1+\Delta_2-d)/2}$$

Cannot use exponentiated form
of special conformal transformation
in Minkowski space-time



Infinitesimal form is a 2nd-order
partial differential equation in p:

$$\left[\frac{1}{2}p^\mu \frac{\partial^2}{\partial p^\nu \partial p_\nu} - p^\nu \frac{\partial^2}{\partial p^\nu \partial p_\mu} + (\Delta_2 - d)\frac{\partial}{\partial p_\mu}\right]\langle\langle \phi_1(-p)\phi_2(p)\rangle\rangle = 0$$

$\Rightarrow$ gives the constraint $\Delta_1 = \Delta_2$

Conformal 3-point function in momentum space

Very difficult to perform Fourier transform directly

Using Poincaré + scale symmetry:

$$\langle\langle \phi_1(p_1)\phi_2(p_2)\phi_3(p_3)\rangle\rangle = (p_2^2)^{(\Delta_1+\Delta_2+\Delta_3-2d)/2}\, F\left(\frac{p_1^2}{p_2^2}, \frac{p_3^2}{p_2^2}\right)$$

(choice of p_2^2 as reference scale is arbitrary)

F satisfies a system of 2nd-order PDE of Appell F_4 type

Coriano, Delle Rose, Mottola, Serino 2013

Bzowski, McFadden, Skenderis 2013

- Crossing-symmetric solution for Euclidean 3-point function
- Boundary condition given by OPE for Wightman function

The scalar 3-point function

OPE condition: $\langle\!\langle \phi_1(p_1)\cdots\rangle\!\rangle \sim (-p_1^2)^{\Delta_1-d/2}$ as $p_1^2 \to 0_-$

MG 1909.00878

$$\langle\!\langle \phi_1(p_1)\phi_2(p_2)\phi_3(p_3)\rangle\!\rangle = C_{123}\Theta(-p_1^0)\Theta(-p_1^2)\Theta(p_3^0)\Theta(-p_3^2) \times \frac{(-p_1^2)^{\Delta_1-d/2}(-p_3^2)^{\Delta_3-d/2}}{(p_2^2)^{(\Delta_1+\Delta_3-\Delta_2)/2}} F_4\left(\frac{p_1^2}{p_2^2}, \frac{p_3^2}{p_2^2}\right)$$

F_4: Appell double hypergeometric

$$F_4(x,y) = \sum_{i,j=0}^{\infty} \frac{\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}\right)_{i+j}\left(\frac{\Delta_1+\Delta_2+\Delta_3-d}{2}\right)_{i+j}}{i!j!\left(\Delta_1-\frac{d}{2}+1\right)_i\left(\Delta_3-\frac{d}{2}+1\right)_j} x^i y^j$$

(valid for p_2 space-like: $p_2^2 > 0$)

The Lorentzian OPE

OPE ⇔ Hilbert space completeness relation

$$\langle 0|\phi(x_1)\phi(x_2)\underset{\textstyle\sum|\psi\rangle\langle\psi|}{\uparrow}\phi(x_3)\phi(x_4)|0\rangle$$

Convergent independently of the Lorentzian ordering

⇒ compatible with Fourier transform!

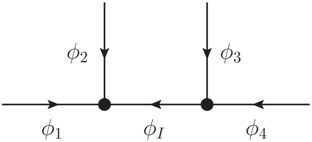
OPE for product of operators acting on the vacuum:

$$\phi(p_1)\phi(p_2)\,|0\rangle = \sum_{\psi} \frac{\langle\!\langle \psi(-p_1-p_2)\phi(p_1)\phi(p_2)\rangle\!\rangle}{\langle\!\langle \psi(-p_1-p_2)\psi(p_1+p_2)\rangle\!\rangle}\,\psi(p_1+p_2)\,|0\rangle$$

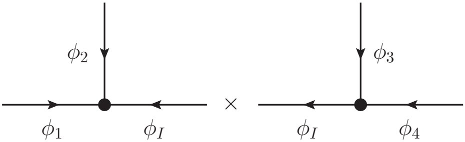
Feynman diagrams for Wightman functions

4-point functions factorize into products of 3-point functions:

$$\langle\!\langle \phi_1\phi_2\phi_3\phi_4 \rangle\!\rangle = \sum_{\phi_I}$$



$$= \sum_{\phi_I}$$



Conformal blocks (partial waves) are simple to compute

Also true for higher-point Wightman functions

Adding spin

Spinning 2-point functions can be resolved using Ward identities
e.g. spin-one field: MG 1807.07003

$$\langle\!\langle V^\mu(-p)V^\nu(p)\rangle\!\rangle = C_V\Theta(p^0)\Theta(-p^2)(-p^2)^{\Delta-d/2}\left(\eta^{\mu\nu} - \frac{2\Delta-d}{\Delta-1}\frac{p^\mu p^\nu}{p^2}\right)$$

In the center-of-mass frame $p=(1,0,0,\ldots)$, decompose into irreducible representations of $SO(d-1)$:

$$\langle\!\langle \mathcal{O}^{\mu_1\ldots\mu_\ell}(-p)\mathcal{O}^{\nu_1\ldots\nu_\ell}(p)\rangle\!\rangle = C_{\mathcal{O}}\,\Theta(p^0)\Theta(-p^2)(-p^2)^{\Delta-d/2} \times\sum_{m=0}^{\ell}\frac{(\Delta-\ell-d+2)_{\ell-m}}{(\Delta+m-1)_{\ell-m}}\,\varepsilon_m^{\mu_1\ldots\mu_\ell}(-p)\varepsilon_m^{\nu_1\ldots\nu_\ell}(p)$$

see "*Conformal partial waves in momentum space*" MG 2012.09825

Disconnected correlators

Gaussian theories are unreasonably complicated in position space:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{(x_{12}^2 x_{34}^2)^\Delta} + \frac{1}{(x_{13}^2 x_{24}^2)^\Delta} + \frac{1}{(x_{14}^2 x_{23}^2)^\Delta}$$

$$= \frac{1}{(x_{12}^2 x_{34}^2)^\Delta} \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 G_{\mathcal{O}}\left(\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\right)$$

$\mathcal{O}$: composite operators $[\phi\partial^\ell\phi]$ with even spin ℓ

Not so in momentum space:

$$\langle\!\langle\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\rangle\!\rangle = 0 \qquad \text{unless } p_i + p_j = 0$$

Question: Why is most of the CFT literature in position space?

Properties of the Lorentzian OPE

Two important Euclidean properties are lost:

- Only distributional convergence, not absolute convergence
  Example: Gaussian theory

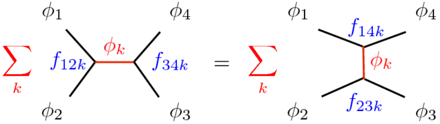
$$\langle\!\langle \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\rangle\!\rangle = \left(p_1^2p_2^2p_3^2p_4^2\right)^{(2\Delta_\phi-d)/4} \times \Big[\delta^d(p_1+p_2) + \underbrace{\delta^d(p_1+p_3) + \Theta(p_2^0)\Theta(-p_2^2)\delta^d(p_1+p_4)}_{\text{not point-wise convergent}}\Big]$$

$\rightarrow$ point-wise convergence depending on kinematics?

MG, Lu, Luty, Mikaberidze 2019

- Single OPE channel

  Wightman functions are **not** crossing-symmetric



Summary

	Euclidean	Minkowski
Position space	Ordinary conformal bootstrap → absolutely convergent OPEs overlapping	Analytic bootstrap • Light-cone limit • Regge limit • OPE inversion formula
Momentum space	Polyakov bootstrap Isono, Noumi, Shiu Cosmological bootstrap Arkani-Hamed, Baumann, Duaso Pueyo, Joyce, Lee, Pimentel	Momentum-space bootstrap? → orthogonal basis of states

Outline

A conformal bootstrap in momentum space?

The Lorentzian dilemma:

- OPE only for Wightman functions

$$\langle 0|\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)|0\rangle$$

- Crossing symmetry only for time-ordered functions

$$\langle 0|T\{\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\}|0\rangle$$

In QFT, problem solved by scattering amplitudes:

- Overlap of in and out states, hence admits unitarity cuts (Wightman function)

- Crossing symmetry following from LSZ reduction (time-ordered product)

Divergences in the Fourier transform?

Divergence in the 2-point function as $p^2 \to 0$ when $\Delta < \frac{d}{2}$

$$\langle\!\langle T\{\phi(-p)\phi(p)\}\rangle\!\rangle = \frac{C_\phi}{2i\sin\left[\pi\left(\frac{d}{2}-\Delta\right)\right]}(p^2 - i\epsilon)^{\Delta-d/2}$$

Where does it come from?

- Finite contribution at large $\vec{x}$
- Only possible divergence from late/early times $x^0 \to \pm\infty$ (depending on the sign $p^0 \to \mp|\mathbf{p}|$)

$\Rightarrow$ divergence selects a particular ordering of operators

$$\langle\!\langle T\{\phi(-p)\phi(p)\}\rangle\!\rangle \overset{p^0\to|\mathbf{p}|}{\sim} \langle\!\langle \phi(-p)\phi(p)\rangle\!\rangle$$

LSZ reduction in CFT

Define $$|\phi(\mathbf{p})\rangle = \lim_{p^0\to|\mathbf{p}|_+} (-p^2)^{d/2-\Delta}\phi(p)|0\rangle$$

(scattering state: $\langle 0|\,\phi(x)|\phi(\mathbf{p})\rangle \propto e^{ip\cdot x}$)

Then:

$$\lim_{p^0\to|\mathbf{p}|}(p^2-i\epsilon)^{d/2-\Delta}\,\langle 0|\,T\{\phi(p')\phi(p)\}\,|0\rangle = Z_\phi\,\langle 0|\phi(p')|\phi(\mathbf{p})\rangle$$

$\to$ can be generalized to higher-point functions

LSZ reduction for one operator

$$\lim_{p^0\to|\mathbf{p}|}(p^2-i\epsilon)^{d/2-\Delta}\langle 0|T\{\phi(p)\mathcal{O}_1\ldots\mathcal{O}_n\}|0\rangle$$
$$= Z_\phi\,\langle 0|T\{\mathcal{O}_1\ldots\mathcal{O}_n\}|\phi(\mathbf{p})\rangle$$

LSZ reduction iterated

One operator at a time, using the Wightman 3-pt function, gives:

LSZ reduction for two operators

$$\lim_{\substack{p_1^0\to|\mathbf{p_1}|\\ p_2^0\to|\mathbf{p_2}|}} (p_1^2-i\epsilon)^{d/2-\Delta_\phi}(p_2^2-i\epsilon)^{d/2-\Delta_\phi}\langle\langle T\{\phi(p_1)\phi(p_2)\mathcal{O}_1\ldots\mathcal{O}_n\}\rangle\rangle$$

$$=\sum_\psi \lambda_{\phi\phi\psi}\frac{\pi^{d/2-1}\Gamma\left(\frac{d}{2}-\Delta_\phi\right)^2\Gamma(\Delta_\psi)\Gamma\left(\Delta_\psi-\frac{d}{2}+1\right)}{2^{2\Delta_\phi-\Delta_\psi-d+1}\Gamma\left(\frac{\Delta_\psi}{2}\right)^2 s^{(2\Delta_\phi+\Delta_\psi-d)/2}}$$

$$\times\, e^{i\pi(2\Delta_\phi-\Delta_\psi)}\langle\langle T\{\mathcal{O}_1\ldots\mathcal{O}_n\}\psi(p_1+p_2)\rangle\rangle$$

$+$ contributions from spinning operators

where $s=-(p_1+p_2)^2$ (note crossing symmetry $1\leftrightarrow 2$)

The CFT scattering amplitude

$$iA = \left(\prod_{i=1}^{4} \lim_{p_i^2 \to 0} (p_i^2 - i\epsilon)^{d/2-\Delta_\phi} \right) \langle\langle T\{\phi(p_1)\cdots\phi(p_4)\}\rangle\rangle$$

Function of:

- center-of-mass energy s and scattering angle θ $\left(\cos\theta = \frac{u-t}{u+t}\right)$
- scaling dimension Δ_ϕ of the external operator
- scaling dimension Δ and spin ℓ of the exchanged operator
- OPE coefficients $\lambda_{\phi\phi\mathcal{O}}$

$$iA = s^{d/2-2\Delta_\phi} \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 \left[e^{i\pi(2\Delta_\phi-\Delta-\ell)} - 1 \right] g_{\Delta,\ell}(\cos\theta)$$

Conformal block expansion for the CFT amplitude

$$iA = s^{d/2-2\Delta_\phi} \sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} \left[e^{i\pi(2\Delta_\phi-\Delta-\ell)} - 1\right] g_{\Delta,\ell}(\cos\theta)$$

- $g_{\Delta,\ell}$ is a polynomial of degree ℓ in $\cos\theta$, interpolating between
  - ◇ SO$(d-1)$ partial waves when $\Delta = d-2+\ell$ (conserved)
  - ◇ SO(d) partial waves when $\Delta \to \infty$
- Zero in generalized free field theory ($\Delta = 2\Delta_\phi + \ell + 2n$)
- $\mathrm{Im}\, A \geq 0$ in the forward limit $\theta \to 0$

Form factor: one off-shell leg

Sufficient for LSZ to have 3 on-shell momenta out of 4

$\Rightarrow$ $F(s,t,u)$ function of 3 Mandelstam invariants

$$F = (-s-t-u)^{-\Delta_\phi} \sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} F_{\Delta,\ell}\left(w = \tfrac{s}{s+t+u}, \cos\theta = \tfrac{u-t}{u+t}\right)$$

Properties:

- $F_{\Delta,\ell}$ given in terms of hypergeometric functions in w and Gegenbauer polynomials of degree $\leq \ell$ in $\cos\theta$
- Recover amplitude A in the limit $w \to -\infty$
- Dominated by low-twist intermediate operators when $w \to 0$:

$$F_{\Delta,\ell} \approx w^{(\Delta-\ell)/2-\Delta_\phi}\, \mathcal{C}_\ell^{(\Delta-\ell-1)/2}(\cos\theta)$$

Analyticity and crossing

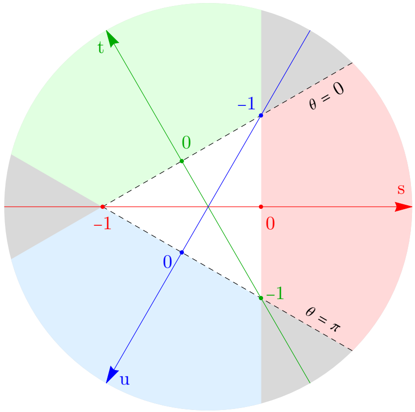
$F_{\Delta,\ell} \propto (-s - i\epsilon)^{(\Delta-\ell)/2-\Delta_\phi}$

$\rightarrow$ complex phase at $s \geq 0$

Suggestive of simple analytic continuation to real F for $s < 0$

(with $-1 \leq \cos\theta \leq 1$)

$\rightarrow$ crossing symmetry?



$s + t + u = -1$

Mellin representation

Mellin representation for the form factor:

$$F(s,t,u) \propto \int [d\gamma] M(\gamma_{ij}) \frac{\Gamma(\gamma_{12})\Gamma(\gamma_{13})\Gamma(\gamma_{14})}{(-s)^{\gamma_{12}}(-t)^{\gamma_{13}}(-u)^{\gamma_{14}}}$$

Derived taking the Fourier transform of the Mellin representation for the Euclidean 4-point function $(\Rightarrow \ s,t,u<0)$

- Manifest crossing symmetry $s \leftrightarrow t \leftrightarrow u$
- Analyticity: only branch cuts for positive Mandelstam variables

Outline

Perturbative CFT

CFT amplitude matches QFT scattering amplitude when

$$\Delta_\phi = \frac{d}{2} - 1 + \gamma\,\varepsilon \qquad\qquad \varepsilon \ll 1$$

Relation with usual LSZ:

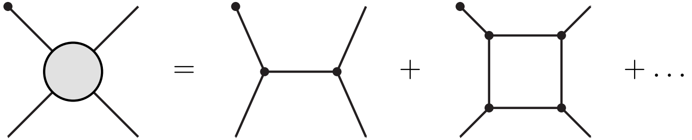
$$(-p^2)^{\Delta_\phi - d/2} = \frac{1}{-p^2}\,\underbrace{\left[1 + \gamma\,\varepsilon\,\log(-p^2) + \ldots\right]}_{\text{Wave-function renormalization } Z}$$

$\rightarrow$ rule: amputate external legs

Simplest example: ϕ^3 theory in $d = 6 + \varepsilon$ dimensions

$$\mathscr{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{g}{3!}\,\phi^3 \qquad \text{with fixed point at } \frac{g_*}{(4\pi)^3} = \frac{2}{3}\,\varepsilon$$

ϕ^3 theory: form factor



$$= \frac{g^2}{s+t+u}\left[\frac{1}{s}+\frac{1}{t}+\frac{1}{u}-\frac{g^2}{(4\pi)^3}\frac{1}{s}\,\mathrm{Li}_2\left(-\frac{s}{t}\right)+\dots\right]$$

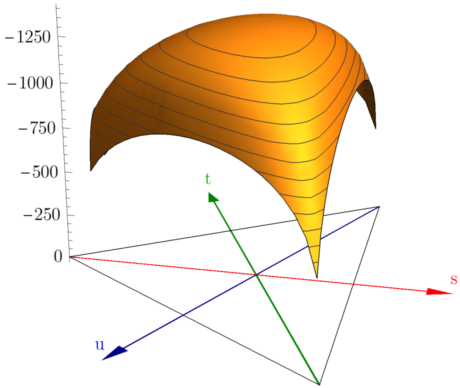
Expansion in s, in units where $s+t+u=-1$:

- s^{-1} term: $\frac{g^2}{s}+\dots \Rightarrow$ one scalar operator at twist 2: ϕ
- s^0 term: $g^2\left(\frac{4}{\sin^2\theta}-6\right)+\dots \Rightarrow$ twist 4 operators $[\phi\partial^\ell\phi]$
  with $\gamma = -\frac{4}{3(\ell+1)(\ell+2)}$
- s^1 term $\Rightarrow$ twist 6 operators, with computable γ $\leftarrow$ new!

Nonperturbative example: Ising model in $d = 3$

Using the data for ~ 100 operators in $\sigma \times \sigma$ OPE

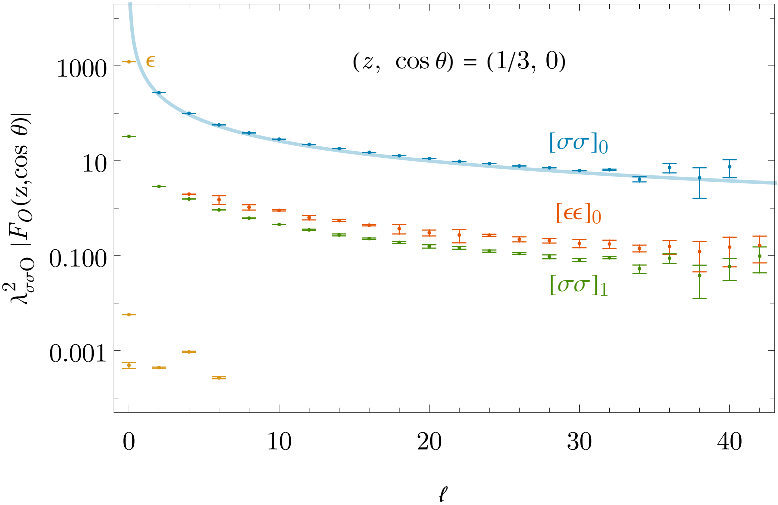
Simmons-Duffin 1612.08471



$\leftarrow$ Form factor computed using s-channel OPE

Crossing symmetry satisfied at the per-mille level!

OPE convergence in the Ising model



No absolute convergence along leading Regge trajectory ($\ell^{-0.98}$)

Applications

Overlapping s-, t- and u-channel OPEs $\Rightarrow$ bootstrap?

- Not a well-posed semi-definite programming problem: oscillatory asymptotic behavior of the form factor
- Gliozzi's bootstrap?
- Relax kinematic assumptions to make form factor symmetric?
- Partial wave unitarity? S-matrix interpretation?

Summary and outlook

Two convenient frameworks to study conformal field theory:

- Euclidean position space $\rightarrow$ absolutely convergent OPE
- Minkowski momentum space $\rightarrow$ orthogonality of states

  $\rightarrow$ conceptually simple conformal partial waves

An approach to crossing symmetry in momentum space using a generalization of the LSZ reduction formula:

- Perturbative CFT data from Feynman diagrams
- Momentum-space bootstrap?