How Different Is More?
Precision Correlators at Large R-Charge

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S.H. & Maeda, arXiv:1710.07336
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Rencontres Théoriciennes
The Internet, Cyber-Space, May 6, 2021

The Large Quantum Number Expansion

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The Large Quantum Number Expansion

By "otherwise strongly coupled" I'll mean outside of any simplifying limit where the theory becomes semiclassical for other reasons or possibly in a simplifying limit but with the quantum number taken so large that the system behaves differently than you might have expected despite being weakly coupled.

The Large Quantum Number Expansion

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The primary question in such a talk is, is this even a subject?

The Large Quantum Number Expansion

The Large Quantum Number Expansion

The answer is, yes, and in some sense it's an old one; many examples have appeared in the literature going far back into the past. Recently there have been a number of groups focusing on systematizing this point of view and applying it more broadly.

The Large Quantum Number Expansion

Pre-history:

- Atomic hypothesis [Democritus]
- Correspondence principle [Bohr]
- Large spin in hadron spectrum [Regge]
- Macroscopic limit [Deutsch] [Srednicki]

History:

- $\mathcal{N} = 4$ SYM at large R-charge [Bernstein, Maldacena, Nastase]
- and large spin [Belistsky, Basso, Korchemsky, Mueller], [Alday, Maldacena]
- Large-spin expansion in general CFT from light-cone bootstrap [Komargodski-Zhiboedov], [Fitzpatrick, Kaplan, Poland, Simmons-Duffin], [Alday 2016]
- Large-spin expansion in hadrons [SH, Swanson], [SH, Maeda, Maltz, Swanson], [Caron-Huot, Komargodski, Sever, Zhiboedov], [Sever, Zhiboedov]

The Large Quantum Number Expansion

Modern:

- Large-charge expansion in generic systems with abelian global symmetries: [SH, Orlando, Reffert, Watanabe 2015], [Monin 2016], [Monin, Pirtskhalava, Rattazzi, Seibold 2016], [Loukas 2016]
- Nonabelian symmetries: [Alvarez-Gaume, Loukas, Orlando, Reffert 2016], [Loukas, Orlando, Reffert 2016], [SH, Kobayashi, Maeda, Watanabe 2017], [Loukas 2017], [SH, Kobayashi, Maeda, Watanabe 2018]
- Charge AND spin: [Cuomo, de la Fuente, Monin, Pirtskhalava, Rattazzi 2017]
- Topological charge: [Pufu, Sachdev 2013] [Dyer, Mezei, Pufu, Sachdev 2015], [de la Fuente 2018]
- EFT connection with bootstrap: [Jafferis, Mukhametzhanov, Zhiboedov 2017]
- Large charge limit in gravity: [Nakayama, Nomura 2016], [Loukas, Orlando, Reffert, Sarkar 2018]

The Large Quantum Number Expansion

Vacuum manifolds $\Leftrightarrow$ chiral rings at large-R-charge:

- $D = 3$, $\mathcal{N} \geq 2$ theories : [SH, Maeda, Watanabe 2016]
- $D = 4$, $\mathcal{N} \geq 2$ theories : [SH, Maeda 2017], [SH, Maeda, Orlando, Reffert, Watanabe 2017]
- Double-scaling limit in lagrangian $\mathcal{N} \geq 2$ theories: [Bourget, Rodriguez-Gomez, Russo 2018]

The Large Quantum Number Expansion

- In addition, there has been a great deal of fascinating work in this area in the past few years that I don't have the space to do justice to in the references here.
- A sampling includes: [Favrod, Orlando, Reffert 2018] [Loukas,Orlando, Reffert, Sarkar 2018] [Kravec, Pal 2018] [Bourget, Rodriguez-Gomez, Russo 2018] [Badel, Cuomo, Monin, Rattazzi 2019] [Alvarez-Gaume, Orlando, Reffert 2019] [Arias-Tamargo, Rodriguez-Gomez, Russo 2019] [Grassi, Komargodski, Tizzano 2019] [Badel, Cuomo, Monin, Rattazzi 2020] [Delacretaz 2020] [Cuomo, Esposito, Gendy, Khmelnitsky, Monin, Rattazzi 2020] [Cuomo 2020] [Orlando, Reffert, Sannino 2020] [Antipin,Bersini, Sannino, Wang, Zhang 2020] [Komargodski, Mezei, Pal, Raviv-Moshe 2021] [Cuomo, Delacretaz, Mehta 2021] [Cassani, Komargodski 2021]

Large-Scale Structure of Theory Space

- The goals of the LQNE are largely to answer the same questions as the conformal bootstrap:
- Learn to systematically and efficiently analyze QFT (in practice usually CFT) that have no exact solution in terms of explicit functions.

Large-Scale Structure of Theory Space

- We'd all like to know "what does theory space look like": Generic theories, generic amplitudes.
- This is a very consequential question for field theory, mathematics, quantum gravity, and cosmology.
- Most theories are not integrable, and we need to learn how to attack them in general circumstances.
- "Direct" numerical bootstrap methods are remarkably efficient, power-law in number of operators.

Critique of Pure Bootstrap

- Since number of operators grows exponentially with dimension / central charge / other quantum number, direct numerical attack is still intractable in extreme limits.
- Fortunately, known "extreme limits" appear to have simplifying limits in many (all?) known circumstances. This is broadly a generalization of the notion of "duality".
- In the case of large spin in a single plane, the limit has been analyzed within the bootstrap itself.
- The relative ease of this is related to the fact that the conformal blocks themselves carry the quantum number.

Critique of Pure Bootstrap

- For other quantum numbers, this is not the case. For instance, there is no known analytic bootstrap method to attack the case of large spin in multiple planes in $D \geq 4$.
- The same is true* for internal global symmetries of various kinds.
- (*) (Though see [Jafferis, Mukhamezhanov, Zhiboedov 2017].)

Bootstrap-EFT duality?

- In many cases such limits are accessible to some new kinds of EFT in regions where bootstrap methods slow down.
- As we'll see, there's also a excellent agreement for one prediction where the two methods overlap .
- Where does this leave us? What do we hope to accomplish ?

Squad Goals

- (*) Most modestly: Translate EFT behavior into bootstrap terms, say what it means for CFT data. Operator dimensions and OPE coefficients.
- (***) Most grandiosely: Derive EFT behavior from bootstrap equations, and use it to solve everything in every limit where direct numerical methods break down.
- (**) Intermediate: Use some small subset of EFT inputs, and obtain some subset of CFT data not directly numerically accessible.
- Grandiose goal (***) appears out of reach for now. (I tried!)
- For progress on the intermediate goal (**) see [Jafferis-Mukhametzhanov-Zhiboedov 2017].
- This talk is about progress on modest goal (*).

Large charge J in the $O(2)$ model

- Simplest example: The conformal Wilson-Fisher $O(2)$ model at large O(2) charge J.
- Canonical question: What is the dimension Δ_J of the lowest operator $\mathcal{O}_J$ at large J?
- Translated via radial quantization: Energy of lowest state of charge J on unit S^2?
- Renormalization-group analysis reveals the low-lying large-charge sector is described by an EFT of a single compact scalar χ, which can be thought of as the phase variable of the complex scalar ϕ in the canonical UV completion of the $O(2)$ model.

Large charge J in the $O(2)$ model

- The leading-order Lagrangian of the EFT is remarkably simple:

$$\mathcal{L}_{\text{leading}-\text{order}} = b|\partial\chi|^3$$

- The coefficient b is not something we know how to compute analytically; nonetheless the simple structure of this EFT has sharp and unexpected consequences.
- The immediate consequence of the structure of the EFT is that the lowest operator is a scalar, of dimension

$$\Delta_J \simeq c_{\frac{3}{2}} J^{\frac{3}{2}} \ ,$$

where $c_{\frac{3}{2}}$ has a simple expression in terms of b.

Large charge J in the $O(2)$ model

- The leading-order EFT predicts more than just the leading power law, because quantum loop effects in the EFT are suppressed at large J, so the EFT can be quantized as a weakly-coupled effective action with effective loop-counting parameter $J^{-\frac{3}{2}}$.
- For instance we can compute the entire spectrum of low-lying excited primaries.
- The dimensions, spins, and degeneracies of the excited primaries, are those of a Fock space of oscillators of spin ℓ, with $\ell \geq 2$.

Large charge J in the $O(2)$ model

- The propagation speed of the χ-field is equal to $\frac{1}{\sqrt{2}}$ times the speed of light.
- So the frequencies of the oscillators are

$$\omega_\ell = \frac{1}{\sqrt{2}}\sqrt{\ell(\ell+1)}\,, \qquad \ell \geq 1\,.$$

- The $\ell = 1$ oscillator is also present, but exciting it only gives descendants; the leading-order condition for a state to be a primary is that there be no $\ell = 1$ oscillators excited.
- So for instance, the first excited primary of charge J always has spin $\ell = 2$ and dimension $\Delta_J^{(1)} = \Delta_J + \sqrt{3}$.

Large charge J in the $O(2)$ model

- Subleading terms can be computed as well.
- These depend on higher-derivative terms in the effective action with powers of $|\partial\chi|$ in the denominator .
- These counterterms have a natural hierarchical organization in J :

Large charge J in the $O(2)$ model

- At any given order in derivatives, there are only a finite number of such terms.
- As a result, at a given order in the large-J expansion, only a finite number of these terms contribute.
- Since there are far more observables than effective terms, there are an infinite number of theory-independent relations among terms in the asymptotic expansions of various observables.

Large charge J in the $O(2)$ model

- Our gradient-cubed term is the only term allowed by the symmetries at order $J^{\frac{3}{2}}$, and there is only one other term contributing with a nonnegative power of J, namely

$$\mathcal{L}_{J^{+\frac{1}{2}}} = b_{\frac{1}{2}} \left[|\partial\chi| \, \mathtt{Ric}_3 + 2 \, \frac{(\partial \, |\partial\chi|)^2}{|\partial\chi|} \right]$$

- In particular, there are no terms in the EFT of order J^0, with the result that the J^0 term in the expansion of Δ_J is calculable, independent of the unknown coefficients in the effective lagrangian.

Large charge J in the $O(2)$ model

- Specifically, the formula for Δ_J takes the form

$$\Delta_J = c_{\frac{3}{2}} \, J^{+\frac{3}{2}} \quad + \quad c_{\frac{1}{2}} \, J^{+\frac{1}{2}} \quad -0.0937256 \cdots$$

up to terms vanishing at large J.

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Large charge J in the $O(2)$ model

- This universal term and the other universal large-J relations in the $O(2)$ model don't have any fudge factors or adjustable parameters;
- Given the identification of the universality class, these values and relations are universal and absolute;
- Similar predictions have been made for OPE coefficients [Monin, Pirtskhalava, Rattazzi, Seibold 2016]

FAQ

- You might think that there is something *"weird"* or *"inconsistent"* or *"uncontrolled"* about a Lagrangian like $\mathcal{L} = |\partial\chi|^3$.
- So, let me anticipate some frequently asked questions:

FAQ

- Q: Isn't this Lagrangian singular?? It is a nonanalytic functional of the fields, so when you expand it around $\chi = 0$, you will get ill-defined amplitudes.
- A: Yes, but you aren't supposed to use the Lagrangian there. It is only meant to be expanded around the large charge vacuum , which at large J is the classical solution

  $$\chi = \mu t,$$

  with

  $$\mu = O(\sqrt{\rho}) = O(J^{\frac{1}{2}}) \ .$$

- The expansion into vev and fluctuations carries a suppression of μ^{-1} or more for each fluctuation.

FAQ

- (parenthetical comment:) There are already many well-known effective actions of this kind, including the Nambu-Goto action.

FAQ

- Q: Isn't this effective theory ultraviolet-divergent ? That means that loop corrections are incalculable and observables are meaningless beyond leading order.
- A: No. The EFT is quantized in a limit where loop corrections are small . Our UV cutoff Λ for the EFT is taken to satisfy

$$E_{\rm IR} = R_{S^2}^{-1} \quad \ll \quad \Lambda \quad \ll E_{\rm UV} = \sqrt{\rho} \propto J^{+\frac{1}{2}} R_{S^2}^{-1}$$

- Loop divergences go as powers of $\Lambda^3/\rho^{\frac{3}{2}} \ll 1$, and are proportional to nonconformal local terms which are to be subtracted off to maintain conformal invariance of the EFT.

FAQ

- Q: OK but then don't the counterterms ruin everything? Don't they render the theory incalculable?
- A: No. As usual in EFT the counterterm ambiguities of subtraction correspond one-to-one with terms in the original action allowed by symmetries;
- As we've mentioned there are only a finite and small number of those contributing at any given order in the expansion, and at some orders there are no ambiguities at all.

FAQ

- Q: You're saying that every CFT with a conserved global charge has this exact same asymptotic expansion . But here's a counterexample! ⟨ `describes theory SH didn't say anything about` ⟩ Doesn't that mean your theory is all wrong?
- A: No. I didn't make any claim that broad. Our RG analysis applies to many but not all CFT with a conserved global charge. More generally, CFT can be organized into large-charge universality classes.
- For instance, free complex fermions as well as free complex scalars in $D = 3$ are in different large-J universality classes.
- The large-J universality class of the O(2) model contains many other interesting theories, such as
  - The $\mathbb{CP}(n)$ models at large topological charge ;
  - The $D = 3, \mathcal{N} = 2$ superconformal fixed point for a chiral superfield with $W = \Phi^3$ superpotential, at large R-charge;
  - Probably others ∘ ∘ ∘

Other large-J universality classes

- Many other interesting universality classes in $D = 3$:
- Large Noether charge in the higher Wilson-Fisher $O(N)$ [Alvarez-Gaumé , Loukas, Reffert, Orlando 2016] and $U(N)$ models;
- Also the $\mathbb{CP}(n)$ [de la Fuente] and higher Grassmanian models real and complex ; [Loukas, Reffert, Orlando 2017]
- Large baryon charge in the $SU(N)$ Chern-Simons-matter theories;
- Large monopole charge in the $U(N)$ Chern-Simons-matter theories;
- Of course these last two are dual to one another and would be interesting to investigate.

Vacuum moduli spaces and the large-R-charge limit

- Among the most tractable universality classes are large R-charge in extended superconformal theories with moduli spaces of supersymmetric vacua.
- Simplest case is the $\mathcal{N} = 2,\ D = 3$ superconformal fixed point of three chiral superfields with superpotential $W = XYZ$.
- Its vaccum manifold has three one-complex-dimensional branches: $X, Y, Z \neq 0$.
- WLOG consider the $X-$branch.

Vacuum moduli spaces and the large-R-charge limit

- The X-branch has coordinate ring spanned by X^J, $J \geq 0$.
- These BPS scalar chiral primary operators are the (X-branch part of the) chiral ring of the theory.
- The dimension of X^J is exactly equal to its R-charge J and protected from all quantum corrections: In this case the formula for the dimension Δ_J is boring :

$$\Delta_J = 1 \cdot J \qquad \Leftarrow \quad \text{BORING!}$$

Vacuum moduli spaces and the large-R-charge limit

- The formula for the dimension of the second-lowest primary of $J_{\mathrm{R}} = J_{\mathrm{X}} = J$ is also boring; it lies an a protected scalar semishort representation with only 12 Poincaré superpartners:

$$\Delta_J^{(+1)} = 1 \cdot J + 1 \qquad \Leftarrow \quad \text{also boring!}$$

- Nonetheless we would like to see this explicitly in a large-J expansion, and also be able to compute non-protected large-J quantities such as third-lowest operator dimensions and also OPE coefficients.

Vacuum moduli spaces and the large-R-charge limit

- The effective theory describing the lowest state of $J_X = J_R = J$, is simply the moduli space effective action, appearing in the same role as the gradient-cubed theory for the $O(2)$ model.
- Unlike the $O(2)$ model EFT, here the leading effective action is simply free:

$$\mathcal{L} = \int d^2\theta \, d^2\bar{\theta} \, \Phi^\dagger \Phi \,, \qquad \Phi = (\text{const.}) \times X^{\frac{3}{4}} + \cdots \,,$$

where the $\cdots$ are higher-derivative D-terms.

Vacuum moduli spaces and the large-R-charge limit

- To compute operator dimensions, quantize the theory around the lowest classical solution with given large J on an S^2 spatial slice:
- Here, the classical solution is

$$\phi = v \exp\left(i\mu t\right) \ ,$$

$$\mu = \frac{1}{2R} \ , \qquad v = \sqrt{\frac{J}{2\pi R}} \ .$$

- Note here the frequency of the solution (chemical potential) is determined by supersymmetry (the BPS bound on operator dimensions) rather than the unknown coefficients in the Lagrangian.

Vacuum moduli spaces and the large-R-charge limit

- The results of the direct diagrammatic quantization are as follows, for the lowest and second-lowest states:

$$\Delta_J = J$$

$$+0 \times J^0 + 0 \times J^{-1} + 0 \times J^{-2} + 0 \times J^{-3}$$

$$+O(J^{-4}) \qquad \Leftarrow \text{three loops!}$$

$$\Delta_J{}^{(+1)} = J + 1 \times J^0$$

$$+0 \times J^{-1} + 0 \times J^{-2} + 0 \times J^{-3}$$

$$+O(J^{-4}) \qquad \Leftarrow \text{two loops!} \,,$$

confirming the predictions of supersymmetry to the order we can calculate .

Vacuum moduli spaces and the large-R-charge limit

- The third-lowest primary is a non-BPS scalar, with dimension

$$\Delta_J^{(+2)} = J + 2 \cdot J^0$$

$$+0 \times J^{-1} + 0 \times J^{-2}$$

$$-\kappa \times 192\,\pi^2 \times J^{-3}$$

$$+O(J^{-4}) \qquad \Leftarrow \text{one loop!}\ ,$$

where κ the coefficient of the leading interaction term in the *EFT*.

Vacuum moduli spaces and the large-R-charge limit

- The form of the leading interaction term is a D-term, consisting of a four-derivative bosonic component

$$\mathcal{L}_{-1} \equiv +4\,\kappa_{\rm FTP}\,\frac{|\partial\phi|^4}{|\phi|^6}\ ,$$

  plus conformally and superconformally completing terms worked out by many authors [Fradkin, Tseytlin] [Paneitz] [Riegert] [Kuzenko].

- We don't know the value of κ for the XYZ model, but we do know its sign :

$$\kappa > 0 \qquad\qquad \text{(superluminality constraint)}$$

  [Adams, Arkani-Hamed, Dubovsky, Nicolis]

- So the first nonprotected operator dimension gets a contribution of order J^{-3} with a negative coefficient of unknown magnitude .

Vacuum moduli spaces and the large-R-charge limit

- It is more fun to compute quantities which are both nontrivial in the large-J expansion and checkable in principle by exact supersymmetric methods.
- One nice example is the two-point functions of chiral primary operators in 8-supercharge theories.
- The technically simplest class of examples are the chiral primaries spanning the Coulomb branch chiral ring in $D = 4,\ \mathcal{N} = 2$ theories, in the special case the gauge group has rank one .

Vacuum moduli spaces and the large-R-charge limit

- Examples include
  - $\mathcal{N} = 4$ SYM with $G = SU(2)$,
  - $\mathcal{N} = 2$ SQCD with $N_c = 2$, $N_f = 4$,
  - Many rank-one nonlagrangian Argyres-Douglas theories with one-dimensional Coulomb branch,
  - including the recently discovered $\mathcal{N} = 3$ examples.
- Some of these are Lagrangian theories with marginal coupling, and some of them are non-Lagrangian theories with more abstract descriptions, but we can treat them all on an equal footing.

Vacuum moduli spaces and the large-R-charge limit

- The Coulomb branch chiral ring in a rank-one theory is spanned by

$$\mathcal{O}_{\mathcal{J}} \equiv \mathcal{O}_{\Delta}^{n} , \qquad \mathcal{J} = n\Delta ,$$

where the[(*)] generator $\mathcal{O}_{\Delta}$ of the chiral ring has $U(1)_R$-charge $J_R = \Delta$.

- [(*)] This assumes the chiral ring is freely generated; there are no known counterexamples, but see recent work [Argyres, Martone 2018] for counterexamples in higher rank.
- At large charge in radial quantization these correspond to classical solutions on the sphere where the Coulomb branch scalar $\hat{a}$ gets a vev proportional to $\sqrt{J}/R$.

Vacuum moduli spaces and the large-R-charge limit

- For Lagrangian theories the generator $\mathcal{O}$ is $\mathtt{tr}(\hat{\phi}^2)$ and $\Delta = 2$.
- For non-Lagrangian theories the dimension Δ of the generator can take certain other values.
- These are constrained to some extent and recently it was proven that Δ is always rational [Argyres, Martone 2018]
- We can write the large-$\mathcal{J}$ effective action in terms of an effective field $\phi \equiv (\mathcal{O}_\Delta)^{\frac{1}{\Delta}}$. The singularity in the change of variables is invisible in large-$\mathcal{J}$ perturbation theory because the quantum state field is supported far away from $\phi = 0$.

Vacuum moduli spaces and the large-R-charge limit

- The leading-order action is again the free action for ϕ, and the leading interaction term is the anomaly term compensating the difference in Weyl a- anomaly and $U(1)_R$-anomalies between the underlying interacting SCFT and the free vector multiplet.
- The leading interaction term is

$$\mathcal{L}_{\text{anom}} \equiv \alpha \int d^4\theta \, d^4\bar{\theta} \, \texttt{log}(\phi) \, \texttt{log}(\bar{\phi})$$

$$+(\text{curvature and } U(1)_{\text{R}} \text{ connection terms}) \, ,$$

- where the coefficient α is proportional to the Weyl-anomaly mismatch:

$$\alpha = +2 \, (a_{\text{CFT}} - a_{\text{EFT}})^{[\text{AEFGJ units}]}$$

Vacuum moduli spaces and the large-R-charge limit

- Some comments on this interaction term:
- It was first written down by [Dine, Seiberg 1997] as the unique four-derivative term in the Coulomb branch EFT of an N = 2 gauge theory;
- It is formally an $\mathcal{N} = 2$ D- term, *i.e.* a full-superspace integrand $\cdots$
- $\cdots$ but only formally, since it is non-single-valued; its single-valued version can be obtained as an F -term, *i.e.* an integral over only the θ's and not the $\bar{\theta}$'s.
- Its bosonic content comprises the famous Wess-Zumino term for the Weyl a-anomaly that was used [Komargodski, Schwimmer] to prove the a-theorem in four dimensions.
- This is why its coefficient α is proportional to the a-anomaly mismatch.

Vacuum moduli spaces and the large-R-charge limit

- One other remarkable fact about rank-one theories, is that the anomaly term is that it is unique as a (quasi-)F-term on conformally flat space.
- That is, there are an infinite number of higher-derivative D-terms, but there are no higher-derivative F-terms one can construct out of a single vector multiplet in a superconformal $\mathcal{N} = 2$ theory.
- The simple explanation: An $\mathcal{N} = 2$ superconformal theory is super-Weyl invariant, with the super-Weyl transformation parametrized by a chiral superfield Ω:

$$\phi \to \exp(\Omega) \cdot \phi \; .$$

- In the regime of the validity of the effective theory, ϕ has a nonzero vev, and in flat space we can super-Weyl transform the vector multiplet to 1.

- The EFT is therefore$^{(*)}$

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{anomaly}} + \mathcal{L}_{\text{higher D-term}}$$

- For quantities insensitive to D-terms, this simple, two-term effective action, can be quantized meaningfully, and gives unambiguous answers to all orders in $\frac{1}{J}$ perturbation theory.
- Note that the dimension Δ of the generator of the chiral ring does not enter into the EFT at all, nor does the marginal coupling τ or any other parameter .
- In other words, any purely F-term-dependent observable has a large-J expansion that is uniquely determined by the anomaly coefficient α and nothing else, for a one-dimensional Coulomb branch of an $\mathcal{N} = 2$ gauge theory.

Vacuum moduli spaces and the large-R-charge limit

- One set of such observables are the Coulomb branch correlation functions

$$\exp\left(q_n\right) \equiv Z_n \equiv Z_{S^4} \times |x-y|^{2\mathcal{J}} \left\langle \left(\mathcal{O}(x)_\Delta\right)^n \left(\overline{\mathcal{O}}(y)_\Delta\right)^n \right\rangle_{S^4}$$

- The insertions $\phi^{\mathcal{J}}(x)$ and $\bar{\phi}^{\mathcal{J}}(y)$ can be taken into the exponent as

$$S_{\text{sources}} \equiv -\mathcal{J} \, \mathtt{log}\left[\phi(x)\right] - \mathcal{J} \, \mathtt{log}\left[\bar{\phi}(y)\right]$$

- This quantity $Z_n = \exp\left(q_n\right)$ is partition function of the EFT with sources:

$$Z_n = \int \mathcal{D}\Phi \, \mathcal{D}\Phi^\dagger \exp\left(-S_{\text{EFT}} - S_{\text{sources}}\right)$$

Vacuum moduli spaces and the large-R-charge limit

- This quantity is scheme-dependent, and dependent on the normalization of $\mathcal{O}_\Delta$, but these dependences cancel out in the double difference observables

$$\frac{Z_{n+1}\, Z_{n-1}}{Z_n^2} = \exp\left(q_{n+1} - 2q_n + q_{n-1}\right) \ .$$

- These can now in principle be evaluated straightforwardly as functions of $\mathcal{J}$ and α using Ferynman diagrams, with no further input from the underlying CFT, as long as we are in large-$\mathcal{J}$ perturbation theory.

Vacuum moduli spaces and the large-R-charge limit

- The form of the expansion is

$$q_n = \mathbf{A}\, n + \mathbf{B} + \mathcal{J}\,\texttt{log}(\mathcal{J}) + \left(\alpha + \frac{1}{2}\right)\texttt{log}(\mathcal{J}) + \sum_{m\geq 1} \frac{\hat{K}_m(\alpha)}{\mathcal{J}^m}\,.$$

- The first two terms are the scheme and normalization ambiguities, the third term is the classical value of the source term, one loop free term, and classical anomaly term contributions.
- The last is the series of power-law corrections coming from loop diagrams with interaction vertices coming from the source term and the anomaly term, with the anomaly term vertices carrying powers of α.
- The structure of the EFT makes the polynomials $\hat{K}_m(\alpha)$ a polynomial in α of order $m+1$:

$$\hat{K}_m(\alpha) = \sum_{\ell=0}^{m+1} \hat{K}_{m,\ell}\,\alpha^{\ell}\,.$$

Vacuum moduli spaces and the large-R-charge limit

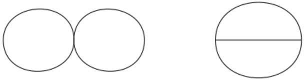
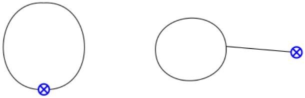
description	term	diagrams
Two-loop with no α-vertices	$\hat{K}_{1,0}$	
One-loop with one α-vertex	$\hat{K}_{1,1}\alpha$	
Tree-level with two α-vertices	$\hat{K}_{1,2}\alpha^2$	

Table 1 – Diagrams appearing at order $1/\mathcal{J}$.

Vacuum moduli spaces and the large-R-charge limit

- Of course, actually directly evaluating multiloop diagrams in an EFT is hard ;
- To evaluate the power-law corrections, my collaborators and I used a combination of
  - Direct evaluation of some low-order diagrams;
  - Use of known data for some theories such as the free vector multiplet and $\mathcal{N} = 4$ SYM ;
  - Supersymmetric recursion relations [Papadodimas 2009];
  - Embedding of the Coulomb-branch EFT into nonunitary UV completions invoving ghost hypermultiplets to apply the recursion relations to arbitrary values of α.

Vacuum moduli spaces and the large-R-charge limit

- With this combination of tricks, we were able to solve all the power-law corrections for any value of α, with the result:

$$q_n = \mathbf{A}\, n + \mathbf{B} + \texttt{log}\left[\Gamma\left(\mathcal{J} + \alpha + 1\right)\right]$$

$$+\text{smaller than any power of } \mathcal{J}\ .$$

- I'll comment on those exponentially small corrections in a moment.

Confirmation of the large-$\mathcal{J}$ expansion

- But first, let me talk about some evidence for this picture of large-J self-perturbatization of strongly coupled theories.
- Starting with our predictions for the $O(2)$ model, where we predicted a formula

$$\Delta_J = \Delta_J = c_{\frac{3}{2}} J^{+\frac{3}{2}} + c_{\frac{1}{2}} J^{+\frac{1}{2}} -0.0937256\cdots$$

- It would be good to compare with bootstrap calculations in the $O(2)$ model; at the moment bootstrap methods can only reach $J \leq 2$ with any precision. [Kos, Poland, Simmons-Duffin 2013].
- It would be good if bootstrap methods could be developed to the point of being able to confirm our results, or add something substantial to them.
- But at the moment that hasn't happened, so let's move on to other avenues of confirmation.

Confirmation of the large-$\mathcal{J}$ expansion

- The first really nontrivial confirmation came from a Monte Carlo analysis up to $J = 15$ in the $O(2)$ model, independently computing charged operator dimensions and estimating the leading Lagrangian coefficient b from the energies of charged ground states on the torus.
- These results are from a PRL by [Banerjee, Orlando, Chandrasakhran 2017].

Monte Carlo numerics [Banerjee, Chandrasekharan, Orlando 2017]

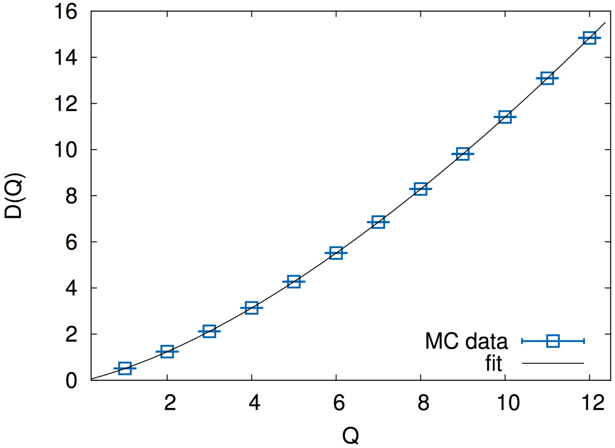


Figure: Operator dimensions with the $c_{3/2}, c_{1/2}$ coefficients in the EFT prediction fit to data, giving $c_{3/2} = 1.195/\sqrt{4\pi}$ and $c_{1/2} = 0.075\sqrt{4\pi}$.

Monte Carlo numerics [Banerjee, Chandrasekharan, Orlando 2017]

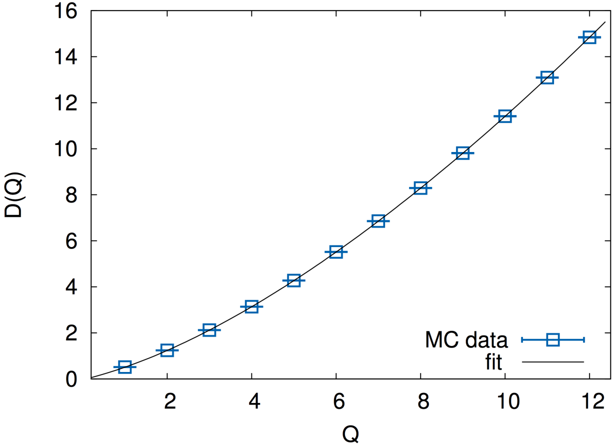


Figure: Note the coefficients are fit with high-J data for operator dimensions and torus energies, and yet the leading-order prediction extrapolates extremely well down to $J = 2$.

Confirmation of the large-$\mathcal{J}$ expansion

- Though precise bootstrap results only exist up to $J = 2$, note that the values of the EFT parameters calculated from Monte Carlo calculation give

  $$\Delta_{J=2} = 1.236(1) \qquad \text{[Monte Carlo + large − J]}$$

  which one can compare to the bootstrap result

  $$\Delta_{J=2} = 1.236(3) \qquad \text{[bootstrap]}\ .$$

- There are other high-precision agreements between large-J theory and MC simulation in [Banerjee, Chandrasekharan, Orlando 2017].

Confirmation of the large-$\mathcal{J}$ expansion

- Moving beyond the $O(2)$ case to other models in the same large- J universality class, one can look at dimensions of operators carrying topological charge J in the $\mathbb{CP}(n)$ models.
- This analysis was done by [de la Fuente 2018], using a combination of large-N methods and numerical methods, with the result

$$\Delta_J^{\mathbb{CP}(n)} = c_{\frac{3}{2}}(n)\, J^{\frac{3}{2}} + c_{\frac{1}{2}}(n)\, J^{\frac{1}{2}} + c_0 + O(J^{-\frac{1}{2}}) \ ,$$

where the first two coefficients depend on the n of the model, but the J^0 term does not; in particular he finds

$$c_0 = -0.0935 \pm 0.0003 \ ,$$

as compared to the EFT prediction

$$c_0 = -0.0937 \cdots \ .$$

- So the error bars are less than one percent , and the EFT prediction sits inside of them.

Confirmation of the large-$\mathcal{J}$ expansion

- Now let's move on to our predictions for $D = 4, \mathcal{N} = 2$ superconformal theories with one-dimensional Coulomb branch.
- For the case of free Abelian gauge theory and $\mathcal{N} = 4$ SYM with $G = SU(2)$ our all-orders-in-J formula agrees with the exact expression:

$$Z_n^{(\mathrm{EFT})} = Z_n^{(\mathrm{CFT})} = n! \ , \qquad \text{free vector multiplet} \ ,$$

$$Z_n^{(\mathrm{EFT})} = Z_n^{(\mathrm{CFT})} = (2n+1)! \ , \qquad \mathcal{N} = 4 \ \mathrm{SYM} \ .$$

In these cases, there are no exponentially small corrections to the formula.

Confirmation of the large-$\mathcal{J}$ expansion

- For other cases, the correlation functions are D-term independent and can be evaluated by exact supersymmetric methods involving localization [Pestun 2007] and supersymmetric recursion relations [Papadodimas 2009], [Gerchkovitz, Gomis, Komargodski 2014] $\cdots$
- $\cdots$ though at present these methods are limited to theories with a marginal coupling.
- Even using these methods, the recursion relations grow more challenging in application to compute corelators of higher J owing to the complication of the sphere partition function as a function of the coupling .
- Nonetheless we have been able to carry the recursion relations to $J \sim 76$ in the case of $\mathcal{N} = 2$ SQCD with $N_c = 2, \ N_f = 4$.

Numerics (Localization)

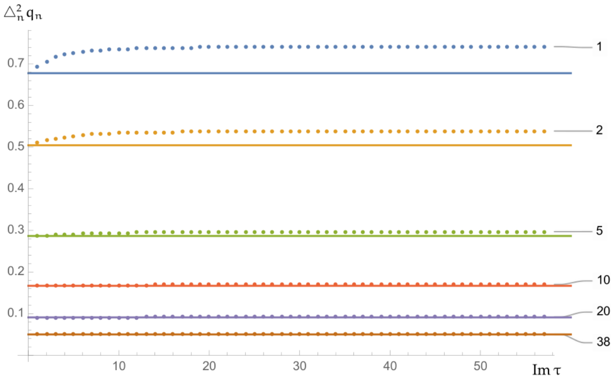


Figure 4.1 – Second difference in n for $\triangle_n^2 q_n^{(\mathrm{loc})}$ (dots) and for $\triangle_n^2 q_n^{\mathrm{EFT}}$ (continuous lines) as function of $\mathrm{Im}\,\tau$ at fixed values of n. The numerical results quickly reach a τ-independent value that is well approximated by the asymptotic formula when n is larger than $n \gtrsim 5$.

Confirmation of the large-$\mathcal{J}$ expansion

- It is interesting to try to understand the disagreement between the all-orders-$\frac{1}{J}$ formula and the exact localization results.
- Our framework for large-J analysis dictates that any disagreement must be smaller than any power of J and associated with a breakdown of the Coulomb-branch EFT .
- The natural candidate for such an effect would be propagation of a massive particle over the infrared scale $R = |x - y|$.
- Therefore we would expect the leading difference between the localization result and the EFT prediction, to be of the form

$$q_n^{(\mathrm{loc})} - q_n^{(\mathrm{EFT})}$$

$$\sim \mathrm{const.} \times \exp\left(-M_{\mathrm{BPS\ particle}} \times R\right)$$

$$= \mathrm{const.} \times \exp\left(-(\mathrm{const.})\sqrt{\frac{\mathcal{J}}{\mathrm{Im}(\tau)}}\right) .$$

Confirmation of the large-$\mathcal{J}$ expansion

- We compared the difference between EFT and exact results in the scaling limit of [Bourget, Rodriguez-Gomez, Russo 2018], where J is taken large with this exponent held fixed and fit it to this virtual-BPS-dyon ansatz for the exponentially small correction .
- We found the difference $q_n^{(\text{loc})} - q_n^{(\text{EFT})}$ fits very well to

$$q_n^{(\text{loc})} - q_n^{(\text{EFT})} \simeq 1.6\, e^{-\frac{1}{2}\sqrt{\pi\,\lambda}}\ ,$$

$$\lambda \equiv 2\pi\mathcal{J}/\mathtt{Im}(\tau)\ .$$

Numerics (Localization)

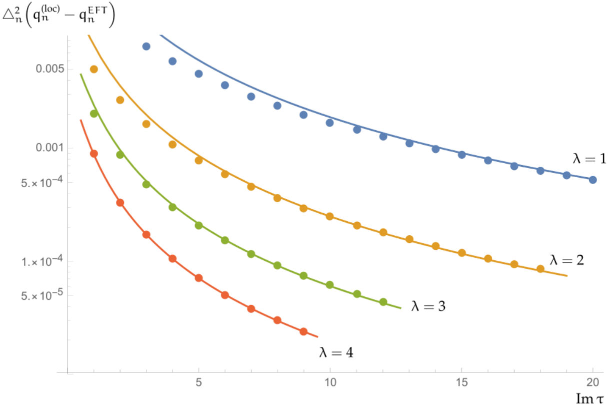


Figure 6.1 – Second difference in n for the discrepancy between localization and EFT results $\triangle_n^2\left(q_n^{(\mathrm{loc})} - q_n^{\mathrm{EFT}}\right)$ (dots) compared to $\triangle_n^2\left(1.6\, e^{-\sqrt{\pi\lambda}/2}\right)$ (continuous lines) as functions of $\operatorname{Im}\tau$ at fixed values of $n/\operatorname{Im}\tau = \lambda/(4\pi)$. The agreement is quite good already for $\lambda = 3$.

Confirmation of the large-$\mathcal{J}$ expansion

- So this is a rather interesting situation.
- Due to the magic of supersymmetry, not only can we compute all power-law corrections exactly modulo the scheme-dependent coefficients, we are actually able to compare to exact results to a precision where we can see the qualitative breakdown of the effective theory that we used to generate the all orders approximation .
- Seeing this, one is naturally tempted to try and go further and compare the exponentially small correction with physcial expectations at a precision level as well.

Confirmation of the large-$\mathcal{J}$ expansion

- In order to do this, one really has to take on the "non-(super-)universal"$^{(*)}$ coefficients A and B.
- The sum rules are fine for checking power law corrections, where all three adjacent terms in the sum rule have the same order of magnitude,
- but when checking exponentially small corrections which are rapidly decreasing as a function of n, the sum rule tends to introduce large relative errors and one would like to do better by deriving the actual value of the coefficients A and B.

Confirmation of the large-$\mathcal{J}$ expansion

- The main challenge in doing this, is that the A and B coefficients are not only dependent on the marginal parameter τ, they are also scheme dependent .
- Often in the literature, including in the literature on supersymmetric localization, a "scheme dependent" coefficient is often treated as synonymous with an "inherently ambiguous" coefficient.
- This point of view is often used as a rationale for not doing certain kinds of computations, but it is simply wrong .
- Having a scheme-dependent coefficient in a microscopic or effective lagrangian, just means that you have to be careful about how operationally you are defining your renormalized lagrangian paramters relative to the UV completion or renormalization procedure being used.

Confirmation of the large-$\mathcal{J}$ expansion

- For generic theories with marginal parameters this is often a bit involved; but
- for theories with extended supersymmetry the scheme dependence can often be reduced to an ambiguity by a holomorphic function of the complex coupling constant; and
- for theories such as SQCD which have an S-duality symmetry, even the holomorphic ambiguity can be reduced to a finite parameter , which
- can then be eliminated altogether by matching with perturbation theory .
- So, that is the course we are going to take here.

The holomorphic reparametrization scheme-dependence

- The first scheme dependence to discuss is the one that affects the A coefficient.
- It is a kind of "classical" scheme dependence having to do with the parametrization of the holomorphic gauge coupling .
- The Coulomb-branch chiral primary $\mathcal{O} \equiv \mathcal{O}_2 \equiv \mathrm{Tr}(\hat{\phi}^2)$ is uniquely defined up to an overall normalization, characterized by its supersymmetry properties and by its dimension and R-charge .
- However the overall normalization is exactly what matters so we have to specify it.

The holomorphic reparametrization scheme-dependence

- In the literature the way mostly used to normalize $\mathcal{O}$ is by its relation to a marginal operator.
- After all, $\mathcal{O}$ can be thought of as the $\mathcal{N} = 2$ F-term superspace integrand over all four positively R-charged Grassman coordinates θ_+ to generate the holomorphic half of the marginal operator that adjusts the gauge coupling $\tau \equiv \frac{4\pi i}{g^2_{\mathrm{YM}}} + \frac{\theta}{2\pi}$:

$$\int d^4\theta_+ \, \mathcal{O}_2 = [\mathrm{theory-independent\ constant}] \times \mathrm{Tr}(F_+^2) + \cdots \ ,$$

  where F_+ is the self-dual piece of the Yang-Mills field strength and the $\cdots$ are the kinetic terms for the scalars and fermions .
- So the normalization of $\mathcal{O}$ is related to the normalization of the dimension-two chiral primary operator $\mathcal{O}$ is naturally linked to the normalization of the marginal operator that is a superconformal descendant in the same multiplet .

The holomorphic reparametrization scheme-dependence

- However this doesn't resolve the question because a marginal operator doesn't have a universal natural normalization either.
- Rather, a (chiral half of a complex) marginal operator transforms under reparametrizations of the coupling constant as a section of the holomorphic cotangent bundle of theory space .
- That is, it transforms as

$$\mathcal{M}_{[\tau]} = \frac{d\tau'}{d\tau}\mathcal{M}_{[\tau']} \ , \qquad \mathcal{M} \equiv \mathrm{Tr}(F_+^2) + \cdots$$

and the chiral primary $\mathcal{O}$ has the same transformation, since its normalization is canonically related to the normalization of $\mathcal{M}$:

$$\mathcal{O}_{[\tau]} = \frac{d\tau'}{d\tau}\mathcal{O}_{[\tau']} \ ,$$

under a holomorphic reparametrization $\tau' = f(\tau)$.

The holomorphic reparametrization scheme-dependence

- Under this coupling reparametrization scheme transformation, the exponentiated $A-$coefficient transforms as the norm-squared of the chiral primary itself

$$\exp\left(A_{[\tau]}\right) = \left|\frac{d\tau'}{d\tau}\right|^2 \exp\left(A_{[\tau']}\right)$$

- We will exploit this transformation law to solve for A in a particularly simple holomorphic coordinate and then write the transformation law in any other holomorphic coordinate including the natural Lagrangian parameter τ.

Euler-counterterm ambiguity

- There is a second, less obvious scheme ambiguity related to the Euler-density counterterm E_4.
- First of all it is very non-obvious why this counterterm should even be relevant at all for the computation of two-point functions!
- But some elementary deduction shows that it is.
- After all, two-point functions on flat space are conformally equivalent to two-point functions on the four-sphere, and
- the four-sphere has a nonzero Euler number .

Euler-counterterm ambiguity

- So the sphere partition function transforms multiplicatively under an additive shift of the coefficient of the Euler counterterm .
- Supersymmetry does allow the Euler counterterm to appear in the action.
- However this term is in some sense an $\mathcal{N} = 2$ $F-$term, so it can only appear with a (holomorphic) + (antiholomorphic) dependence on the holomorphic gauge coupling .
- Since the $Z_n = e^{q_n}$ are unnormalized partition functions with sources, they are affected by the same counterterm ambiguity as the sphere partition function without sources.

Euler-counterterm ambiguity

- The B coefficient is the n^0 term in the large-n expansion of the q_n, so e^B transforms the same way under the Euler-counterterm ambiguity as does the sphere partition function :
-

$$\mathcal{L} \to \mathcal{L} - \mathrm{Re}[\mathrm{Log}[P(\tau)]]\, E_4 \ ,$$

$$Z_{S^4} \to |P(\tau)|^2\, Z_{S^4} \ , \qquad\qquad e^B \to |P(\tau)|^2\, e^B \ .$$

- This transformation law means we must assign B a scheme label as well:

$$\exp\left(B_{\mathrm{scheme}\ 2}\right) = \frac{Z_{\mathrm{scheme}\ 2}}{Z_{\mathrm{scheme}\ 1}} \exp\left(B_{\mathrm{scheme}\ 1}\right)$$

S-duality

- Fixing the scheme-ambiguities is greatly simplified in a theory with an S-duality .
- In terms of the exponentiated gauge coupling

$$q \equiv e^{2\pi i\tau} \ ,$$

the S-duality symmetry acts as:

$$S: \qquad q \to 1 - q \ , \qquad\qquad T: \qquad q \to \frac{q}{q-1} \ .$$

- This is not quite the familiar fractional linear transformation by which the S-duality acts in $\mathcal{N} = 4$ super-Yang-Mills.

S-duality

- The infrared effective Abelian gauge coupling σ is the one that transforms in the familiar way by fractional linear transformations,

$$\sigma \to \frac{a\sigma + b}{c\sigma + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

with the generators acting by

$$S: \qquad \sigma \to -\frac{1}{\sigma} \ ,$$

$$T: \qquad \sigma \to \sigma + 1 \ .$$

- The relationship between the two couplings is given by the modular Lambda function

$$q = e^{2\pi i \tau} = \lambda(\sigma) \ ,$$

$$\sigma = 2\tau + \frac{4i}{\pi}\log[2] - \frac{i}{\pi}\left[\frac{q}{2} + \frac{13}{64}q^2 + \frac{23}{192}q^3 + \frac{2,701}{32,768}q^4 + \cdots\right]$$

S-duality

- Given our transformation law for coupling reparametrizations we can take modular transformations as a special case .
- It follows that the chiral marginal operator $\mathcal{M}_{[\sigma]}$ and the chiral primary $\mathcal{O}_{[\sigma]}$ in the $\sigma-$frame, transform as holomorphic modular forms of weight 2 .
- From there we can see that the A- coefficient transforms as a nonholomorphic modular form of weights $(2,2)$.

Recursion relations and their duality-covariant solution

- The next ingredient is the recursion relations discovered by [Papadodimas 2009] as a generalization of the tt^* equations to $D = 4$.
- These relations say that

$$\partial_\sigma \partial_{\bar{\sigma}}\, q_n = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}}$$

- When applied to the power law corrections they uniquely fix the form of q_n to be the $\Gamma-$function $\Gamma(2n + \frac{5}{2})$ up to the terms $An + B$.

Recursion relations and their duality-covariant solution

- They also give equations for the coupling dependence of the A- and $B-$ terms.
- For the A- function they give

$$\partial_\sigma \partial_{\bar{\sigma}} A_{[\sigma]} = 8\, e^{A_{[\sigma]}}$$

- For the B- function they give

$$\partial_\sigma \partial_{\bar{\sigma}} (B - A) = 0\ .$$

- Note that these equations are covariant under both the holomorphic reparametrization scheme-dependence, and under the Euler counterterm scheme dependence, both of which shift A and/or B by a holomorphic plus antiholomorphic function of the complex coupling .

Recursion relations and their duality-covariant solution

- That means that we can solve these equations in any scheme we like and transform it to whatever other scheme we like.
- It is simplest to solve in the $\sigma-$coordinate.
- In the $\sigma-$coordinate, the Liouville equation , the modular property , and the correct match with tree-level double-scaled perturbation theory uniquely fix the result to be

$$e^{A_{[\sigma]}} = \frac{16}{[\mathrm{Im}[\sigma]]^2} \ .$$

Recursion relations and their duality-covariant solution

- To specify the Euler counterterm scheme choice, we will compare with the scheme in which the sphere partition function was originally calculated by Pestun using the $U(2)$ instanton partition function computed by Nekrasov .
- The partition function as computed in this scheme has a derivable duality transformation given by:

$$q \to 1-q: \qquad \exp\left(B_{\substack{\text{Pestun}-\\ \text{Nekrasov}}}[1-q]\right) = \frac{|q|^2}{|1-q|^2} \exp\left(B_{\substack{\text{Pestun}-\\ \text{Nekrasov}}}[q]\right) ,$$

$$q \to \frac{1}{q}: \qquad \exp\left(B_{\substack{\text{Pestun}-\\ \text{Nekrasov}}}[\frac{1}{q}]\right) = |q|^{-4} \exp\left(B_{\substack{\text{Pestun}-\\ \text{Nekrasov}}}[q]\right)$$

Recursion relations and their duality-covariant solution

- I say "derivable" rather than "derived" because the transformation does not appear AFAIK in the literature.
- In order to find it, it was essential to relate the Pestun-Nekrasov scheme to a slightly different scheme used by [Alday-Gaiotto-Tachikawa 2009] in which the duality transfomation law is more manifest by its relation to the crossing-symmetry transformation of a four-point function in two-dimesional Liouville theory under the well-known AGT correspondence .

Recursion relations and their duality-covariant solution

- With this transformation law for the $B-$coefficient in the Pestun-Nekrasov scheme, and the general constraint from the recursion relations

$$\exp\left(B\right) = |\text{some holomorphic function}|^2 \times \exp\left(A_{[\sigma]}\right) \;\; ,$$

- we have the solution

$$\exp\left(B_{\substack{\text{Pestun}-\\ \text{Nekrasov}}}\right) = [\text{const.}] \times \frac{|\lambda(\sigma)|^{+\frac{2}{3}}\,|1-\lambda(\sigma)|^{+\frac{8}{3}}}{|\eta(\sigma)|^{8}\,[\text{Im}(\sigma)]^2}$$

Recursion relations and their duality-covariant solution

- Again we have an ambiguity by a coupling-independent constant which we can fix again by matching with double-scaled perturbation theory .
- This time it is simpler to match at strong double-scaled coupling λ.
- We are able to do this by making use of the exact solution to the one-loop double-scaled coupling dependence found by [Grassi, Komargodski, Tizzano 2019].
- The result is

$$\exp\left(B_{\substack{\text{Pestun}-\\ \text{Nekrasov}}}\right) = \gamma_{\mathrm{G}}^{+12}\, e^{-1}\, 2^{-\frac{9}{2}}\pi^{-\frac{3}{2}}\, \frac{|\lambda(\sigma)|^{+\frac{2}{3}}\, |1-\lambda(\sigma)|^{+\frac{8}{3}}}{|\eta(\sigma)|^{8}\, [\mathrm{Im}(\sigma)]^{2}}$$

Exponentially small corrections

- So now we have solved for the full EFT approximation to the correlation functions:

$$Z_n^{(\mathrm{exact})} = Z_n^{(\mathrm{eft})} \times Z_n^{(\mathrm{mmp})} ,$$

$$Z_n^{(\mathrm{eft})} = e^{q_n^{(\mathrm{eft})}} = e^{An+B} \times \Gamma(2n + \frac{5}{2}) ,$$

  where the factor $Z_n^{(\mathrm{mmp})} = e^{q_n^{(\mathrm{mmp})}}$ is the set of exponentially small corrections describing massive macroscopic propagation of virtual BPS particles .
- We can get a handle on these too by the same strategy.

Exponentially small corrections

- Here's how we do it .
- We use the fact that the connected MMP term $q_n^{(\mathrm{mmp})}$ is exactly what is left over when we take the full connected partititon function with sources $q_n = \mathrm{Log}[Z_n]$ and subtract the connected EFT contribution $q_n^{(\mathrm{eft})}$ for which we now have an exact formula :

$$q_n^{(\mathrm{mmp})} \equiv q_n - \mathbf{A}\, n - \mathbf{B} - \mathrm{Log}\left[\Gamma(2n + \frac{5}{2}) \right]$$

- Using this identity we can rewrite the recursion relation for the full connected amplitude with sources q_n as a recursion relation for the macroscopic massive propagation contribution $q_n^{(\mathrm{mmp})}$.

Exponentially small corrections

- The resulting equation of variation for $q_n^{(\mathrm{mmp})}$ takes the form

$$[\mathrm{LHS}] = [\mathrm{RHS}]$$

- $\cdots$ with

$$[\mathrm{LHS}] \equiv 16\,\mathrm{Im}[\sigma]^2\,\partial_\sigma \bar{\partial}_{\bar{\sigma}} q_n^{(\mathrm{MMP})}$$

- $\cdots$ and

$$[\mathrm{RHS}] \equiv (2n+\frac{7}{2})(2n+\frac{5}{2})\left[\frac{Z_{n+1}^{(\mathrm{MMP})}}{Z_n^{(\mathrm{MMP})}} - 1\right] - (2n+\frac{3}{2})(2n+\frac{1}{2})\left[\frac{Z_n^{(\mathrm{MMP})}}{Z_{n-1}^{(\mathrm{MMP})}} - 1\right]$$

where $Z_n^{(\mathrm{mmp})} \equiv e^{q_n^{(\mathrm{mmp})}}$

Exponentially small corrections

- The recursion relation is one input.
- The next input is the structure of the asymptotic expansion as dictated by effective field theoretic considerations.

Exponentially small corrections

- To understand the structure of the expansion, we have to think carefully about what it is the MMP corrections represent.
- Mathematically they are the sum of all connected diagrams minus the EFT contributions $\cdots$
- $\cdots$ with the latter representing propagation of virtual massless particles together with microscopic propagation of massive particles, which are absorbed into effective couplings of the EFT .
- After these are subtracted we are left with connected diagrams which each have at least one macroscopic propagator for a massive particle.

Exponentially small corrections

- The lightest massive particle is the doublet hypermultiplet .
- In terms of the R-charge and gauge coupling its mass is given by

$$M_{\text{hyper}} = \frac{1}{R}\sqrt{\frac{\mathcal{J}}{\pi\ \mathtt{Im}[\sigma]}} = \frac{1}{R}\sqrt{\frac{2n}{\pi\ \mathtt{Im}[\sigma]}}$$

  where R is the radius of the three-sphere in radial quantization .
- Remember we are always working in the limit

$$E_{\text{IR}} = R_{\text{sphere}}^{-1} \ll \Lambda \ll E_{\text{UV}} = \sqrt{\rho} \propto \frac{\sqrt{\mathcal{J}}}{R} \propto \frac{\sqrt{n}}{R} \propto M_{\text{hyper}}\ ,$$

  so at fixed coupling and large R-charge $\mathcal{J}$ the mass of the hyper is parametrically above the cutoff Λ.

Exponentially small corrections

- There is nothing inconsistent about including heavy particles above the cutoff in an effective field theory ⋯
- ⋯ so long as we do it consistently!
- Actually such treatments of heavy supercutoff objects in EFT are well-understood and familiar in many contexts where the heavy state is stable or approximately stable .
- Examples include heavy quark effective theory [Isgur-Wise] effective string theory [Lüscher, Symanzik, Weisz][Polchinski, Strominger], the D-brane action [Born, Infeld][Leigh, Polchinski], gapped goldstones [Brauner, Murayama, Watanabe], [Alvarez-Gaume, Loukas, Orlando, Reffert], [Cuomo, Esposito, Gendy, Khmelnitsky, Monin, Rattazzi] and other examples .
- These examples can all be described in terms of a second quantized Hilbert space coupled to a first quantized dynamics of motions of the heavy particle .

Exponentially small corrections

- As we have seen, large $R-$charge is a semi-classical limit $\cdots$
- $\cdots$ so we expect the leading contribution of the virtual massive particle at large R-charge to come from a classical configuration of the action for a massive BPS partcle coupled to masless vector multiplet .
- The heavy particle is conformal and gets its mass strictly from the magnitude of the vev of the vector multiplet, which is consant in the conformal frame of the cylinder .
- So we have to look for finite action classical trajectories of a particle of constant mass in on the cylinder .

Exponentially small corrections

- This narrows it down a lot because there aren't very many finite action trajectories for a massive hyper on the cylinder.
- In fact the only such trajectories are great circles of the spatial S^3 at a fixed value of the radial time coordinate.

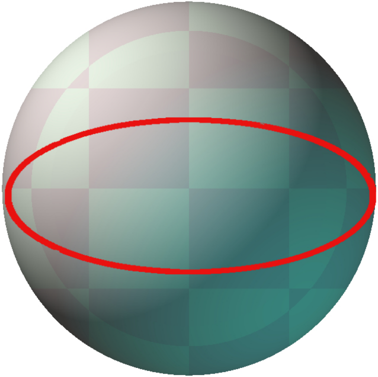


Figure: The leading contribution to the connected MMP function.

Exponentially small corrections

- This means that the MMP function $q_n^{(\mathrm{mmp})}$ has the asymptotic expansion at fixed coupling and large charge that is of the form $q_n = e^{-\mathfrak{W}}$ with

$$\mathfrak{W} = [\text{worldline instanton action } S_{\mathrm{WLI}}] + [\text{parametrically smaller in } n]$$

- Using the BPS mass formula we find that the worldline instanton action is

$$S_{\mathrm{WLI}} = \sqrt{\frac{8\pi n}{\mathrm{Im}[\sigma]}}$$

- The first subleading correction is given by the quantum fluctuation determinant of the geometric worldline action about the classical trajectory. It contributes to $\mathfrak{W}$ proportional to $\mathrm{Log}[n]$.

Exponentially small corrections

- There are also contributions from the classical and quantum back-reaction of the massive hyper on the degrees of freedom of the massless abelian vector multiplet .
- These contribute to $\mathfrak{W}$ with an $n-$dependence of at most $n^{-\frac{1}{2}}$.
- Incorporating additional higher-order geometric fluctuations of the massive trajectory and additional loops of the massless vector multiplet gives contributions which are suppressed by further powers of $n^{-\frac{1}{2}}$.
- So we have an asymptotic expansion of the form

$$-\mathrm{Log}[q_n] \equiv \mathfrak{W} = \sqrt{\frac{8\pi n}{\mathrm{Im}[\sigma]}} + \gamma[\sigma]\,\mathrm{Log}[n]$$

$$+\sum_{p\geq 0} w_p[\sigma]\, n^{-\frac{p}{2}}$$

Exponentially small corrections

- At this point we could in principle just calculate all these terms directly in the effective theory of a geometric fluctuations of a massive worldline coupled to massless fields in the bulk about a nontrivial classical solution .
- But it turns out we have to do very little caculation.
- The recursion relations give PDE s for the $\sigma-$dependence of the functions $\gamma[\sigma], w_p[\sigma]$ at each order , and we have enough information about boundary conditions to find the physically correct solution to each PDE .

Exponentially small corrections

- For instance, the recursion relation at order $\mathrm{Log}[n]$ gives

$$(\partial_\sigma - \partial_{\bar{\sigma}})\gamma[\sigma] = 0 \; .$$

- This means γ can depend only on the real part of σ which is proportional to the infrared θ−angle θ_{IR}.
- But the dynamics must be independent of θ_{IR} at weak coupling, so γ must be independent of σ identically:

$$\gamma[\sigma] = (\sigma - \mathrm{independent}) = \gamma \; .$$

Exponentially small corrections

- To find the actual value of γ we must match with double-scaled perturbation theory again.
- Taking the double-scaling limit of $\mathfrak{W}$ and then taking the strong coupling expansion of that double-scaling limit we find

$$\gamma = \lim_{\substack{n\to\infty \\ \lambda \text{ fixed}}} \mathfrak{W}\Big|_{\lambda \text{ term}} = F^{(\text{inst})}[\lambda]\Big|_{\lambda \text{ term}} = -\frac{1}{4} \, .$$

- Here the quantity $F^{(\text{inst})}[\lambda]$ is [Grassi, Komargodski, Tizzano]'s "worldline instanton partition function " which sums up all the terms scaling as n^0 in the double scaling limit of the MMP function, without any terms of order n^{-1} or smaller, and also without any EFT contributions .
- The function $F^{(\text{inst})}[\lambda]$ can be thought of as the sum over massive macroscopic worldlines and the first-quantized quantum fluctuations of their worldlines about the classical trajectory while discarding all quantum fluctuations of the massless fields .

Exponentially small corrections

- The functions $w_p[\sigma]$ can be found similarly.
- At each p the recursion relation gives a first-order PDE for σ;
- The boundary condition at weak coupling forces the correct solution to depend on $s \equiv \mathrm{Im}[\sigma]$ only;
- This determines the solution up to a single $\sigma-$ independent constant $\cdots$
- $\cdots$ which can be fixed by taking the double scaling limit and matching with the function $F^{(\mathrm{inst})}[\lambda]$ of [Grassi, Komargodski, Tizzano].

Exponentially small corrections

- The first few terms are:

$$w_1 = \frac{1}{48(s/2\pi)^{3/2}} + \frac{1}{\sqrt{(s/2\pi)}} - \frac{11\sqrt{(s/2\pi)}}{16}$$

$$w_2 = -\frac{1}{4} - \frac{1}{64(s/2\pi)} + \frac{19(s/2\pi)}{64}$$

$$w_3 = -\frac{1}{5120(s/2\pi)^{5/2}} - \frac{1}{96(s/2\pi)^{3/2}}$$

$$-\frac{119}{512\sqrt{(s/2\pi)}} + \frac{11\sqrt{(s/2\pi)}}{32} - \frac{527(s/2\pi)^{3/2}}{3072}$$

Exponentially small corrections

- $$w_4 = \frac{119}{1024} + \frac{1}{2048(s/2\pi)^2}$$
$$+\frac{1}{64(s/2\pi)} - \frac{19(s/2\pi)}{64} + \frac{235(s/2\pi)^2}{2048}$$
$$w_5 = \frac{1}{229,376(s/2\pi)^{7/2}} + \frac{3}{10,240(s/2\pi)^{5/2}}$$
$$+\frac{737}{98,304(s/2\pi)^{3/2}} + \frac{101}{1024\sqrt{(s/2\pi)}}$$
$$-\frac{8,155\sqrt{(s/2\pi)}}{32,768} + \frac{527(s/2\pi)^{3/2}}{2048}$$
$$-\frac{14,083(s/2\pi)^{5/2}}{163,840}$$

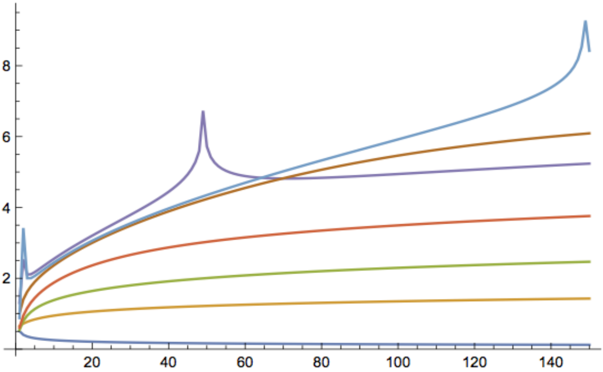


Figure: Plot giving the accuracy of the fixed-coupling large-charge estimates of the MMP function through N^{6}LO, plotted as the number of digits of accuracy of each of the estimates, as a function of n. The quantity being plotted is $-\frac{1}{\mathrm{Log}[10]}$ the logarithm of the relative error in the estimate of the MMP function. The horizontal axis is n, and the vertical axis is $-\frac{1}{\mathrm{Log}[10]}\,\mathrm{Log}\left|\frac{q_n^{(\mathrm{MMP})}-(q_n^{(\mathrm{MMP})})_{\mathrm{estimate}}}{q_n^{(\mathrm{MMP})}}\right|$. The LO, NLO, N^{2}LO, N^{3}LO, N^{4}LO, N^{5}LO and N^{6}LO estimates are given by the blue, yellow, green, red, and purple, brown, and light blue curves respectively, which are in ascending order on the chart for $n \gtrsim 65$.

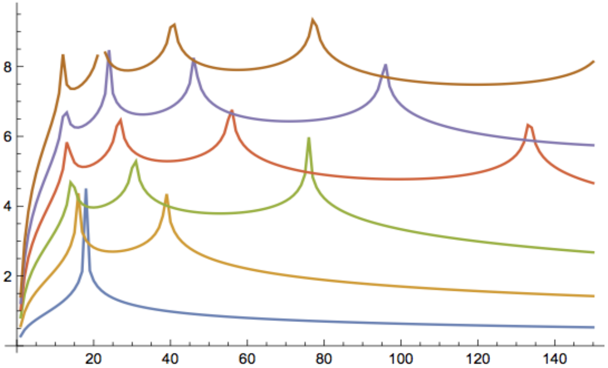


Figure: Plot of the giving the accuracy of the double-scaled large-charge estimates of the MMP function through N^5LO. The quantity being plotted is $-\frac{1}{\mathrm{Log}[10]}$ the logarithm of the relative error in the estimate of the MMP function. The horizontal axis is n, and the vertical axis is $-\frac{1}{\mathrm{Log}[10]} \mathrm{Log}\left[\frac{|q_n^{(\mathrm{MMP})} - (q_n^{(\mathrm{MMP})})_{\mathrm{estimate}}|}{|q_n^{(\mathrm{MMP})}|} \right]$ The LO, NLO, N^2LO, N^3LO, N^4LO and N^5LO double-scaled estimates are given by the blue, yellow, green, red, and purple, and brown dots respectively.

Conclusions

- The large-J expansion gives an analytically controlled way to compute CFT data outside of any other sort of simplifying limit, particularly illuminating simple behavior in regimes where numerical bootstrap methods cannot currently access, despite formal similarity of the expansions.
- The large- J predictions in cases such as the $O(2)$ model and various $D = 4$, $\mathcal{N} = 2$ superconformal theories with one-dimensional Coulomb branch, agree extremely well even at low J with Monte Carlo, bootstrap, and exact supersymmetric methods.
- These results have greatly improved our quantitative control and conceptual understanding of even the simplest strongly-coupled CFT.
- Analysis of more examples is sure to yield further interesting surprises about the large-scale structure of theory space .
- Thank you.