The Coulomb branch of 5d supeconformal field theories on a circle

Cyril Closset

University of Birmingham & Royal Society URF

Rencontres Théoriciennes, IHP, Paris
16 Septembre 2021

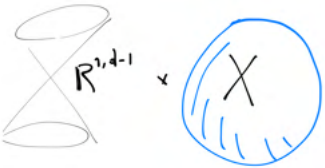
based on 2107.03509 with H. Magureanu

QFT and geometry

Motivation and context, in a nutshell:

- QFT is hard (and mathematically shaky...).
- Supersymmetry often gives rise to more geometric approaches to quantum fields.
- We can engineer SQFT in string theory (using open and/or closed strings).

Recall the basic string-theory framework from the point of view of 10d (or 11d) space-time:

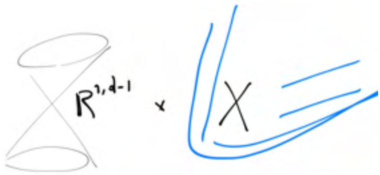


We have a low-energy effective quantum field theory in d-dimensional space-time coupled to gravity, with an effective Newton constant:

$$G_N \sim \frac{1}{\mathrm{vol}(\mathbf{X})}$$

QFT and geometry

Consider $\mathbf{X}$ being a non-compact variety. Then gravity decouples.



Specifically, in this talk:

- we will take $\mathbf{X}$ to be a canonical threefold singularity in the sense of [Reid, 1983].
- we place ourselves in 11-dimensional M-theory. Note the identity:

$$11 = 5 + 6$$

- We then obtain a quantum field theory in five dimensions.

[Seiberg, 1996; Morrison, Seiberg, 1996; Morrison, Seiberg, Intriligator 1996]

Introduction: the SW solution for SQCD

Rational elliptic surfaces

4d SQFTs of rank one, revisited

Modularity

5d SCFTs on a circle and geometric engineering

The U-plane of the E_n 5d SCFTs

Introduction: the SW solution for SQCD

arXiv:hep-th/9407087v1 15 Jul 1994

hep-th/9407087, RU-94-52, IAS-94-43

Electric-Magnetic Duality, Monopole Condensation, And Confinement In $N=2$ Supersymmetric Yang-Mills Theory

N. Seiberg

Department of Physics and Astronomy
Rutgers University, Piscataway, NJ 08855-0849, USA

and

Institute for Advanced Study
Princeton, NJ 08540, USA

and

E. Witten

Institute for Advanced Study
Princeton, NJ 08540, USA

We study the vacuum structure and dyon spectrum of $N=2$ supersymmetric gauge theory in four dimensions, with gauge group $SU(2)$. The theory turns out to have remarkably rich and physical properties which can nonetheless be described precisely; exact formulas can be obtained, for instance, for electron and dyon masses and the metric on the moduli space of vacua. The description involves a version of Olive-Montonen electric-magnetic duality. The "strongly coupled" vacuum turns out to be a weakly coupled theory of monopoles, and with a suitable perturbation confinement is described by monopole condensation.

6/94

...099v1 17 Aug 1994

Monopoles, Duality and Chiral Symmetry Breaking in N=2 Supersymmetric QCD

N. Seiberg

Department of Physics and Astronomy
Rutgers University, Piscataway, NJ 08855-0849, USA

and

Institute for Advanced Study
Princeton, NJ 08540, USA

and

E. Witten

Institute for Advanced Study
Princeton, NJ 08540, USA

...e study four dimensional $N=2$ supersymmetric gauge theories with matter multiplets. ...r all such models for which the gauge group is $SU(2)$, we derive the exact metric on moduli space of quantum vacua and the exact spectrum of the stable massive states. ...mber of new physical phenomena occur, such as chiral symmetry breaking that is ...by the condensation of magnetic monopoles that carry global quantum numbers. ...e cases in which conformal invariance is broken only by mass terms, the formal... ...natically gives results that are invariant under electric-magnetic duality. In one ...his duality is mixed in an interesting way with $SO(8)$ triality.

4d $\mathcal{N} = 2$ SQCD

We are interested in 4d $\mathcal{N} = 2$ supersymmetric gauge theories. For simplicity, focus on SQCD with $SU(2)$ gauge group:

- Vector multiplet for gauge group $SU(2)$:

$$\mathcal{V} = (\phi, A_\mu, \lambda_I, \bar{\lambda}^I, D_{IJ})$$

  Scalar potential includes term $V = \left|[\bar{\phi}, \phi]\right|^2 \geq 0$.

- N_f 'flavors': hypermultiplets in the fundamental, $\mathbf{2} \oplus \bar{\mathbf{2}}$, with masses m_i.
- Flavour symmetry algebra $\mathfrak{g}_F$: $\mathfrak{so}(2N_f)$ if $m_i = 0$, $\forall i$, $\mathfrak{u}(N_f)$ if $m_i = m$, and $\mathfrak{u}(1)^{N_f}$ with generic masses.
- Asymptotic freedom implies $N_f \leq 4$. The theory with $N_f = 4$ and $\mathfrak{g}_F = \mathfrak{so}(8)$ is a 4d SCFT with an exactly marginal gauge coupling.

4d $\mathcal{N} = 2$ SQCD

- Generic vacuum is on the **Coulomb branch:**

$$\phi = -\frac{i}{\sqrt{2}} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} , \qquad SU(2) \to U(1)$$

The SW solution gives the exact low-energy effective action for the IR $U(1)$:

$$S = \int d^4x \, \mathrm{Im}\big(\tau(a)\big) \left(F_{\mu\nu} F^{\mu\mu} + \partial_\mu a \partial^\mu a + \cdots\right)$$

By supersymmetry, the CB metric is determined by an holomorphic function, the prepotential:

$$\tau = \frac{\partial^2 \mathcal{F}}{\partial a^2}$$

The CB is parameterised by the gauge-invariant parameter:

$$u = \langle \mathrm{Tr}(\phi^2) \rangle \approx -a^2 + \cdots$$

The CB of 4d $\mathcal{N} = 2$ SQCD is 'the u-plane'.
The point at infinity, $u = \infty$, is the weak coupling point.

The u-plane of SQCD

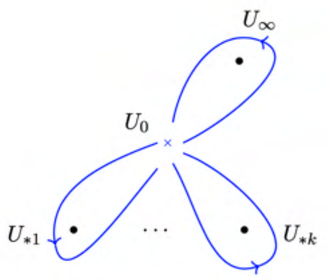
Electric-magnetic duality of a $U(1)$ vector multiplet:

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \mathbb{M}_* \begin{pmatrix} a_D \\ a \end{pmatrix} , \quad \mathbb{M}_* \in \mathrm{SL}(2,\mathbb{Z}) \cong \left\langle S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$$

with $SL(2,\mathbb{Z})$ monodromies of the 'electromagnetic periods' (modulo constant shifts if $m_i \neq 0$). We have:

$$a_D = \frac{\partial \mathcal{F}}{\partial a} , \qquad \tau = \frac{\partial a_D}{\partial a} .$$

For fixed masses, the u-plane has the form:

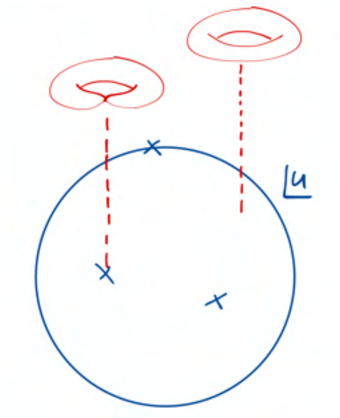


- paths γ_v, $v = 1, \cdots, k$, and $v = \infty$.
- $\gamma_\infty = -(\gamma_1 + \cdots + \gamma_k)$
- If m_i generic, $k = N_f + 2$.
- $\mathbb{M}_\infty \prod_{l=1}^{k} \mathbb{M}_{*l} = \mathbf{1}$.

We will think of the u-plane as a projective plane, $\mathbb{P}^1 \cong \{u\}$ with a distinguished point $u = \infty$.

The SW solution

Postulate that τ with $\mathrm{Im}(\tau) \geq 0$ is the modular parameter of an elliptic curve, E_u:



- We then have:

$$\tau = \frac{\omega_D}{\omega_a} = \frac{\partial a_D}{\partial a} ,$$

$$\omega_D = \frac{da_D}{du} = \int_{\gamma_B} \boldsymbol{\omega} ,$$

$$\omega_a = \frac{da}{du} = \int_{\gamma_A} \boldsymbol{\omega} .$$

- The SW solution is a specific **elliptic fibration over the CB.** The one-parameter family of curves E_u is usually called ‘the SW curve’.
- The ‘Seiberg-Witten geometry’ is the total space of the SW fibration over the u-plane.
- It necessarily has singular fibers.
  Kodaira classification.

The SW solution

- **Singularity at infinity** determined at weak coupling (1-loop β-function):

$$I^*_{4-N_f} \quad : \qquad \mathbb{M}_\infty = -T^{4-N_f}$$

- **Simple singularities in the interior:** I_n singularity (multiplicative fiiber):

$$I_n \quad : \qquad \mathbb{M}_* = T^n$$

The actual monodromy is conjugate to T^n.
If a single dyon of charge (m, q) becomes massless at $u = u_*$:

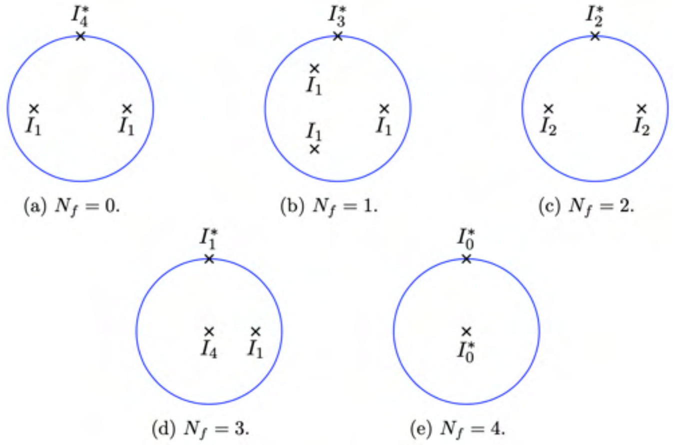
$$\mathbb{M}_*^{(m,q)} = B^{-1} T B = \begin{pmatrix} 1+mq & q^2 \\ -m^2 & 1-mq \end{pmatrix} \, .$$

- **Other possibilities,** from the Kodaira classification of singular elliptic fibers:

$$II \ : \qquad \mathbb{M}_* = (ST)^{-1} \, , \qquad\qquad II^* \ : \qquad \mathbb{M}_* = ST \, ,$$
$$III \ : \qquad \mathbb{M}_* = S^{-1} \, , \qquad\qquad III^* \ : \qquad \mathbb{M}_* = S \, ,$$
$$IV \ : \qquad \mathbb{M}_* = (ST)^{-2} \, , \qquad\qquad IV^* \ : \qquad \mathbb{M}_* = (ST)^2 \, .$$

The u-plane of massless SQCD

For massless SQCD, we have: [Seiberg, Witten, 1994]



I_n singularity: n mutually local particles become massless.

The symmetry group of 4d $\mathcal{N} = 2$ SQCD

The (global) symmetry group of a theory is, by definition, the group that acts effectively on gauge-invariant states. In particular, we must quotient by gauge redundancies.

The global symmetry of **massless** SQCD is easily determined **in the UV:**

$$G_F = SO(2N_f)/\mathbb{Z}_2$$

We also write this as:

N_f	0	1	2	3	4
G_F	-	$U(1)$	$(SU(2)/\mathbb{Z}_2)\times(SU(2)/\mathbb{Z}_2)$	$SU(4)/\mathbb{Z}_4$	$\mathrm{Spin}(8)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

The pure $SU(2)$ gauge theory ($N_f = 0$) has a one-form symmetry:

[Gaiotto, Kapustin, Seiberg, Willett, 2014]

$$\mathcal{Z}^{[1]} = \mathbb{Z}_2$$

which acts on Wilson loops in the fundamental (*i.e.* background quark worldlines):

$$\mathbb{Z}_2 \; : \; W \to -W$$

The symmetry group of 4d $\mathcal{N} = 2$ SQCD

We would like to determine the symmetry directly **in the IR**.

Let us start with a partial answer:

Claim: The semi-simple part of the flavor symmetry algebra $\mathfrak{g}_F^{\rm NA} = {\rm Lie}(G_F)^{\rm NA}$ is given in terms of the Kodaira singularities in the interior:

$$\mathfrak{g}_F^{\rm NA} = \bigoplus_{v=1}^{k} \mathfrak{g}_v \ ,$$

with:

F_v	I_n	I_m^*	II	III	IV	II^*	III^*	IV^*
$\mathfrak{g}_v$	$\mathfrak{su}(n)$	$\mathfrak{so}(8+2m)$	$-$	$\mathfrak{su}(2)$	$\mathfrak{su}(3)$	$\mathfrak{e}_8$	$\mathfrak{e}_7$	$\mathfrak{e}_6$

We will soon explain how to determine G_F itself, directly from the SW geometry.

Rational elliptic surfaces

SW curve and periods: generalities

It is convenient to bring the SW curve into the Weierstrass normal form:

$$y^2 = 4x^3 - g_2(u,m)x - g_3(u,m)$$

The singular fibers are located along the zeros of the discriminant:

$$\Delta(u) = g_2(u)^3 - 27g_3(u)^2$$

For SQCD, this is a polynomial of order $N_f + 2$. At generic masses, we have $N_f + 2$ simple roots in u (giving rise to I_1 singularities).

Example: For pure $SU(2)$, we have:

$$g_2(u) = \frac{4u^2}{3} - 4\Lambda^4 \,, \qquad g_3(u) = -\frac{8u^3}{27} + \frac{4}{3}u\Lambda^4 \,,$$

and the discriminant:

$$\Delta = 16\Lambda^8 \left(u^2 - 4\Lambda^4\right)$$

SW curve and periods: generalities

Kodaira's classification of singularities of elliptic fibrations:

$$g_2 \sim (u-u_*)^{\mathrm{ord}(g_2)} , \qquad g_3 \sim (u-u_*)^{\mathrm{ord}(g_3)} , \qquad \Delta \sim (u-u_*)^{\mathrm{ord}(\Delta)} .$$

fiber	τ	$\mathrm{ord}(g_2)$	$\mathrm{ord}(g_3)$	$\mathrm{ord}(\Delta)$	M_*	flavor
I_k	$i\infty$	0	0	k	T^k	$\mathfrak{su}(k)$
I_k^*	$i\infty$	2	3	$k+6$	$-T^k$	$\mathfrak{so}(2k{+}8)$
I_0^*	τ_0	≥ 2	≥ 3	6	$-\mathbf{1}$	$\mathfrak{so}(8)$
II	$e^{\frac{2\pi i}{3}}$	≥ 1	1	2	$(ST)^{-1}$	-
II^*	$e^{\frac{2\pi i}{3}}$	≥ 4	5	10	(ST)	$\mathfrak{e}_8$
III	i	1	≥ 2	3	S^{-1}	$\mathfrak{su}(2)$
III^*	i	3	≥ 5	9	S	$\mathfrak{e}_7$
IV	$e^{\frac{2\pi i}{3}}$	≥ 2	2	4	$(ST)^{-2}$	$\mathfrak{su}(3)$
IV^*	$e^{\frac{2\pi i}{3}}$	≥ 3	4	8	$(ST)^2$	$\mathfrak{e}_6$

SW curve and periods: generalities

We are interested in the ‘physical periods’:

$$a_D = \int_{\gamma_B} \lambda_{\rm SW} \;, \qquad\qquad a = \int_{\gamma_A} \lambda_{\rm SW} \;.$$

with the Seiberg-Witten differential such that:

$$\frac{d\lambda_{\rm SW}}{du} = \boldsymbol{\omega} \;, \qquad\qquad \boldsymbol{\omega} \equiv \frac{dy}{x}$$

Thus, we can find the physical periods from the ‘geometric periods’:

$$\omega_D = \int_{\gamma_B} \boldsymbol{\omega} \;, \qquad\qquad \omega_a = \int_{\gamma_A} \boldsymbol{\omega} \;.$$

At any fixed m, they satisfy a standard Picard-Fuchs equation:

$$\Delta(u)\frac{d^2\omega}{du^2} + P(u)\frac{d\omega}{du} + Q(u)\,\omega = 0$$

SW geometry and rational elliptic surface

The low-energy physics on the CB is determined by the (affine) bundle:

$$\mathbb{C}^2 \to (\text{SW geom}) \to \overline{\mathcal{B}} \cong \{u\}$$

with the fibers given by the periods (a_D, a).

Once we geometrize the periods by introducing the SW curve E_u, we have:

$$E \to \mathcal{S} \to \overline{\mathcal{B}}$$

We compactify the base by adding the point at infinity:

$$\overline{\mathcal{B}} \cong \{u\} \cong \mathbb{P}^1$$

The SW geometry $\mathcal{S}$ is then a **rational elliptic surface (RES) with a section.**

Note: Any (resolved) RES $\tilde{\mathcal{S}}$ can be obtained as a blow up of the projective plane at 9 points, $dP_9 = \mathrm{Bl}_9(\mathbb{P}^9)$. This is also called 'half-K3 surface' by string theorists. A deep fact is then that:

$$H_2(\tilde{\mathcal{S}}, \mathbb{Z}) \cong \langle (O), E \rangle \oplus (-E_8)$$

with E_8 denoting the E_8 lattice, for the 2-cycles with the intersection pairing.

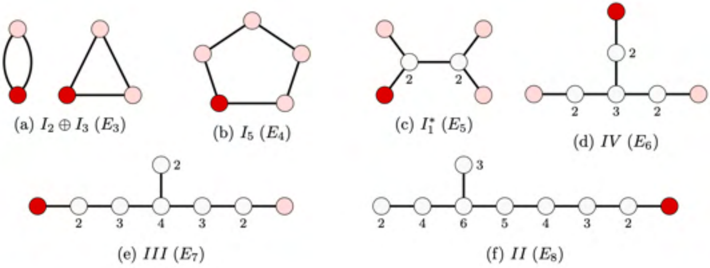
SW geometry and rational elliptic surface

The singular fibers lead to ADE singularities on $\mathcal{S}$, in correspondence with the ADE 'flavor' type.

They admit a standard resolution, $\tilde{\mathcal{S}} \to \mathcal{S}$. (Kodaira-Neron model.)

$$\pi^{-1}(U_{*,v}) = F_v \cong \sum_{i=0}^{m_v-1} \widehat{m}_{v,i}\Theta_{v,i} \ ,$$

Example: The E_n family.



(a) $I_2 \oplus I_3$ (E_3) (b) I_5 (E_4) (c) I_1^* (E_5) (d) IV (E_6) (e) III (E_7) (f) II (E_8)

The Mordell-Weil group of rational section

Elliptic curves are additive groups:

$$P_1 + P_2 = P_3$$

Given an elliptic fibration $E \to \mathcal{S} \to \mathbb{P}^1$, there may exist non-trivial rational sections. In Weierstrass form:

$$P = (x(u), y(u)) \ , \qquad x(u),\ y(u) \in \mathbb{C}(u)$$

They form a finitely generated abelian group, the Mordell-Weil group:

$$\Phi = \mathrm{MW}(\mathcal{S}) \cong \mathbb{Z}^{\mathrm{rk}(\Phi)} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_t} \ .$$

The number of free generators, $\mathrm{rk}(\Phi) \geq 0$, is called the rank of the MW group.

The trivial element in Φ is the zero section, $O = (\infty, \infty)$.

Importantly, the MW group can have non-trivial torsion elements, $k_i P_{\mathrm{tor}} = O$.

The classification of rational elliptic surfaces

Rational elliptic surfaces $\mathcal{S}$ are fully classified. [Persson, 1990; Miranda, 1990]
They are characterised by:

- A set of 'allowed' singular fibers, (F_v).
- The MW group Φ.

In fact, in most cases, the set of singular fibers fully determines $\mathcal{S}$.

A basic but powerful global constraint is:

$$\sum_v \mathrm{ord}(\Delta)|_{U_{*v}} = 12$$

where the sum includes '$v = \infty$'. There is thus a finite set of allowed singularities. Additional considerations show that these are the following 20:

$$I_1, \cdots, I_9 \ , \quad I_0^*, \cdots, I_4^* \ , \quad , II \ , \ III \ , \ IV \ , \ II^* \ , \ III^* \ , \ IV^* \ .$$

Total number of distinct RES: **289**.

4d SQFTs of rank one, revisited

Fixing the fiber at infinity

The RES perspective, and Persson's classification, gives us a bird's-eye view of rank-one 4d $\mathcal{N}=2$ theories.

The basic idea, generalising [Caorsi, Cecotti, 2018], is that the UV $\mathcal{N}=2$ SQFT is determined by the fiber at infinity:

$$\mathcal{T}_{F_\infty} \qquad \longleftrightarrow \qquad \{\mathcal{S} \mid \pi^{-1}(\infty) = F_\infty\} \ .$$

F_∞	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9
$D_{S^1}\mathcal{T}_{5d}$	E_8	E_7	E_6	E_5	E_4	E_3	E_2	E_1 or $\widetilde{E}_1$	E_0
$\#\mathcal{S}$	227	140	77	51	26	16	6	$2+2$	1
F_∞	II	III	IV	I_0^*	I_1^*	I_2^*	I_3^*	I_4^*	
$\mathcal{T}_{4d}$	E_8	E_7	E_6	D_4	D_3	D_2	$N_f=1$	$N_f=0$	
$\#\mathcal{S}$	137	93	49	19	13	6	2	1	
F_∞					IV^*	III^*	II^*		
$\mathcal{T}_{4d}$					$A_2\ (H_2)$	$A_1\ (H_1)$	$-\ (H_0)$		
$\#\mathcal{S}$					8	4	2		

MN theories

AD theories

SQCD

Fixing the fiber at infinity

Some comments:

- Fixing F_∞, the list of distinct RES with such a fiber gives the number of distinct CB configurations for $\mathcal{T}_{F_\infty}$, which we denote by:

$$\mathcal{S} \cong (F_\infty, F_1, \cdots, F_k)$$

  For instance, pure $SU(2)$ has a single CB configuration, $\mathcal{S} \cong (I_4^*, I_1, I_1)$.
- The above 'periodic table' includes the 3 'classic AD SCFTs [Argyres, Douglas, 1995] and the 3 E_n MN theories [Minahan, Nemeschansky, 1996].
- It does not include the other 4d SCFTs [Argyres, Wittig, 2007; Argyres, Lotito, Lu, Martone, 2016] with enhanced CB (although, see [Caorsi, Cecotti, 2016]).
- *Conjecture (?):* the table gives the full list of CB configurations for rank-one 4d $\mathcal{N} = 2$ SQFTs with a 'trivial' CB (*i.e.* with only a $U(1)$ vector multiplet).
- The top row corresponds to 5d SCFTs on $\mathbb{R}^4 \times S^1$, as we will show.
- If we choose $F_\infty = I_0$ (the trivial fiber), we get the E-string on $\mathbb{R}^4 \times T^2$. There are therefore 289 distinct CB configurations for that theory.

Symmetry group and rational sections

We claimed above that the non-abelian part of the flavour symmetry was captured by the singular fibers (in the interior), $F_{v\neq\infty}$.

We also claim that each generator of $\Phi_{\rm free} = \Phi/\Phi_{\rm tor}$ gives rise to a $U(1)$ flavor symmetry.

The full flavour symmetry algebra is then:

$$\mathfrak{g}_F = \bigoplus_{s=1}^{\mathrm{rk}(\Phi)} \mathfrak{u}(1)_s \oplus \bigoplus_{v=1}^{k} \mathfrak{g}_v \ ,$$

One can also show that:

$$\mathrm{rank}(\mathfrak{g}_F) = 8 - \mathrm{rank}(\mathfrak{g}_\infty) \ .$$

Example: $SU(2)$, $N_f = 1$. The massless CB configuration is $\mathcal{S} \cong (I_3^*, 3I_1)$. In that case, one indeed finds $\Phi \cong \mathbb{Z}$, in agreement with $\mathfrak{g}_F = \mathfrak{u}(1)$.

Symmetry group and rational sections

The **global form of flavour group** can be determined by analysing the full MW group. For simplicity, assume that $\mathrm{rk}(\Phi) = 0$, so that G_F is semi-simple:

$$\Phi = \Phi_{\rm tor} = \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_t}$$

Let $\tilde{G}_F$ denote the simply-connected group such that $\mathfrak{g}_F = \mathrm{Lie}(G_F)$.

Define the subgroup of $\Phi_{\rm tor}$ of 'interior-narrow sections':

$$\mathcal{Z}^{[1]} = \left\{ P \in \Phi_{\rm tor} \;\middle|\; (P) \text{ intersects } \Theta_{v,0} \text{ for all } F_{v\neq\infty} \right\} ,$$

and denote by $\mathscr{F}$ the cokernel of the inclusion map $\mathcal{Z}^{[1]} \to \Phi_{\rm tor}$:

$$0 \to \mathcal{Z}^{[1]} \to \Phi_{\rm tor} \to \mathscr{F} \to 0 \; .$$

Then, we claim that:

- $G_F = \tilde{G}_F / \mathscr{F}$ is the flavour symmetry group.
- $\mathcal{Z}^{[1]}$ is the one-form symmetry group.

This is very similar to discussions of the gauge group in F-theory [Anspinwall, Morrison, 1998; Morrison, Park, 2012; ....] (not coincidentally).

Symmetry group and rational sections

Example: SQCD. For massless SQCD, one finds:

N_f	0	1	2	3	4
$\mathcal{S}$	$(I_4^*, 2I_1)$	$(I_3^*, 3I_1)$	$(I_2^*, 2I_2)$	(I_1^*, I_4, I_1)	(I_0^*, I_0^*)
Φ	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_4$	$\mathbb{Z}_2^2$

This matches the results expected from the UV:

- $N_f = 0$: we have $\Phi_{\rm tor} = \mathscr{Z}^{[1]} = \mathbb{Z}_2$, in agreement with known results.
- $N_f = 2$: we have $\Phi_{\rm tor} = \mathscr{F}$ and $G_F = SU(2) \times SU(2))/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.
- $N_f = 3$: we have $\Phi_{\rm tor} = \mathscr{F}$ and $G_F = SU(4)/\mathbb{Z}_4$.
- $N_f = 4$: we have $\Phi_{\rm tor} = \mathscr{F}$ and $G_F = \mathrm{Spin}(8)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Symmetry group and rational sections

The general result can also be applied to non-Lagrangian theories. We have the following interesting RES: [Miranda, Persson, 1986]

- $\mathcal{S} = (II, II^*)$, with $\Phi = 0$.
  - If $F_\infty = II^*$, we have the AD point H_0 with trivial flavour group.
  - If $F_\infty = II$, we have the E_8 MN SCFT, with $G_F = E_8$.
- $\mathcal{S} = (III, III^*)$, with $\Phi = \mathbb{Z}_2$.
  - If $F_\infty = III^*$, we have the AD point H_1 with flavour group $G_F = SO(3)$.
  - If $F_\infty = III$, we have the E_8 MN SCFT, with $G_F = \mathrm{E}_7/\mathbb{Z}_2$.
- $\mathcal{S} = (IV, IV^*)$, with $\Phi = \mathbb{Z}_3$.
  - If $F_\infty = IV^*$, we have the AD point H_2 with flavour group $G_F = PSU(3)$.
  - If $F_\infty = IV$, we have the E_8 MN SCFT, with $G_F = \mathrm{E}_6/\mathbb{Z}_3$.

All these flavour groups are centerless. For the MN theories, this determination reproduces recent results [Bhardwaj, 2021]. The H_1 flavour group was determined in [Buican, Jiang, 2021], and the H_2 flavour group is a new result.

Systematic analysis of CB configurations

Using the Persson classification and some direct computations, we can map out the full set of CB configurations of a given SQFT $\mathcal{T}_\infty$, in principle.

Example: $SU(2)$, $N_f = 3$. There are 13 allowed configurations:

	$\{F_v\}$	m_1	m_2	m_3	$\mathfrak{g}_F$	$\mathrm{rk}(\Phi)$	Φ_{tor}
massless →	I_1^*, I_4, I_1	0	0	0	A_3	0	$\mathbb{Z}_4$
u(3) sym. →	$I_1^*, I_3, 2I_1$	m_1	m_1	m_1	$A_2 \oplus \mathfrak{u}(1)$	1	–
AD point H_2 →	I_1^*, IV, I_1	$\Lambda/2$	m_1	m_1	$A_2 \oplus \mathfrak{u}(1)$	1	–
	I_1^*, I_3, II	$-\Lambda/16$	m_1	m_1	$A_2 \oplus \mathfrak{u}(1)$	1	–
	I_1^*, III, I_2	$\Lambda/4$	0	0	$2A_1 \oplus \mathfrak{u}(1)$	1	$\mathbb{Z}_2$
	$I_1^*, 2I_2, I_1$	m_1	0	0	$2A_1 \oplus \mathfrak{u}(1)$	1	$\mathbb{Z}_2$
	I_1^*, III, II	$-\frac{7}{4}\Lambda$	$i\sqrt{2}\Lambda$	m_1	$A_1 \oplus 2\mathfrak{u}(1)$	2	–
	$I_1^*, III, 2I_1$	$\frac{m_2^2}{\Lambda} + \frac{\Lambda}{4}$	m_2	m_1	$A_1 \oplus 2\mathfrak{u}(1)$	2	–
	I_1^*, II, I_2, I_1	m_1	$\frac{(4m_1+\Lambda)^{3/2}}{6\sqrt{3\Lambda}}$	m_1	$A_1 \oplus 2\mathfrak{u}(1)$	2	–
	$I_1^*, I_2, 3I_1$	m_1	m_2	m_1	$A_1 \oplus 2\mathfrak{u}(1)$	2	–
	$I_1^*, 2II, I_1$	$(-2T_2\Lambda + \frac{13}{8}\Lambda^3,\ 5T_2\Lambda^2 - \frac{57}{16}\Lambda^4)$			$3\mathfrak{u}(1)$	3	–
	$I_1^*, II, 3I_1$	$(\frac{1}{4}T_2\Lambda - \frac{1}{16}\Lambda^3,\ \frac{1}{2}T_2\Lambda^2 - \frac{3}{16}\Lambda^4)$			$3\mathfrak{u}(1)$	3	–
generic →	$I_1^*, 5I_1$	m_1	m_2	m_3	$3\mathfrak{u}(1)$	3	–

Modularity

Modularity of the u-plane

For any 4d $\mathcal{N} = 2$ SQFT with mass parameters m, we have an 'extended CB' where m are viewed as VEVs for background vector multiplets.

There are many 'special loci' on the extended Coulomb branch which have modular properties. More precisely, it can happen that, at some fixed values of the masses, **the u-plane is a modular curve:**

$$\overline{\mathcal{B}} \cong \mathbb{H}/\Gamma \ , \qquad\qquad \Gamma \subset SL(2, \mathbb{Z})$$

for some particular modular subgroup Γ. When this happens, the map:

$$u \ : \ \mathbb{H}/\Gamma \to \overline{\mathcal{B}} \ : \ \tau \mapsto u(\tau)$$

is an isomorphism. The Γ-invariant function $u(\tau)$ is called the Hauptmodul (or principal modular function) of Γ.

When the CB is modular, the singularities are in one-to-one correspondence with cusps and elliptic points of Γ. This simplifies the analysis of *e.g.* the monodromy group.

Note: even when the CB is not modular, it is advantageous to work on the τ-plane. See [Aspman, Furrer, Manschot, 2000, 2021] for recent discussions.

Modular curves for SQCD

Massless SQCD with $N_f \neq 1$ is modular: [Seiberg, Witten, 1994; Nahm, 1996]

Theory	$\Delta(u) = 0$	$F_{v\neq\infty}$	F_∞	*Modular Function*	*Monodromy*	*Cusps* τ
$N_f = 0$	$+1, -1$	I_1, I_1	I_4^*	$u(\tau) = 1 + \frac{1}{8}\left(\frac{\eta(\frac{\tau}{4})}{\eta(\tau)}\right)^8$	$\Gamma^0(4)$	$0, 2, i\infty$
$N_f = 1$	$u^3 = 1$	$3I_1$	I_3^*	$u^3 = \frac{2E_4(\tau)^{\frac{3}{2}}}{E_4(\tau)^{\frac{3}{2}} + E_6(\tau)}$	$\Gamma_{N_f=1}$	$0, 1, 2, i\infty$
$N_f = 2$	$+1, -1$	I_2, I_2	I_2^*	$u(\tau) = 1 + \frac{1}{8}\left(\frac{\eta(\frac{\tau}{2})}{\eta(2\tau)}\right)^8$	$\Gamma(2)$	$0, 1, i\infty$
$N_f = 3$	$0, 1$	I_4, I_1	I_1^*	$u(\tau) = -\frac{1}{16}\left(\frac{\eta(\tau)}{\eta(4\tau)}\right)^8$	$\Gamma_0(4)$	$0, -\frac{1}{2}, i\infty$

Note: Massless $N_f = 1$ is not modular.

Modular curves for SQCD

Example: pure $SU(2)$. Modular curve for $\Gamma^0(4)$. Two cusps of width 1.

$$u(\tau) = \frac{1}{8}\left(q^{-\frac{1}{4}} + 20q^{\frac{1}{4}} - 62q^{\frac{3}{4}} + 216q^{\frac{5}{4}} - 641q^{\frac{7}{4}} + 1636q^{\frac{9}{4}} + \mathcal{O}\left(q^{\frac{11}{4}}\right)\right) .$$

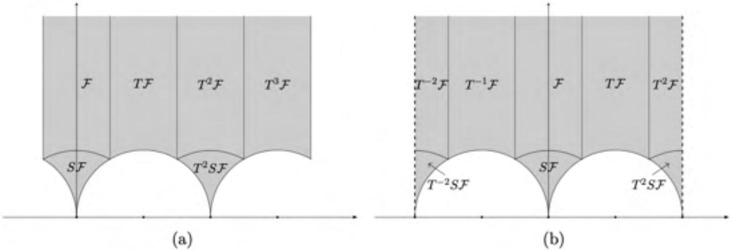


Figure 4: Fundamental domains for $\Gamma^0(4)$. Figure (a) shows a standard choice, with width one cusps at $\tau = 0$ and 2, while in figure (b) the cusp at $\tau = \pm 2$ is split, with the branch cut of the periods indicated by the dashed line.

Associated monodromies:

$$\mathbb{M}_{u=1} = STS^{-1} , \qquad \mathbb{M}_{u=-1} = (T^2S)T(T^2S)^{-1} , \qquad \mathbb{M}_{\infty} = PT^4 .$$

Modular curves for SQCD

Another example: $SU(2), N_f = 1$.

$\{F_v\}$	m_1	$\mathfrak{g}_F$	rk(Φ)	Φ_{tor}
$I_3^*, 3I_1$	m_1	$\mathfrak{u}(1)$	1	–
I_3^*, II, I_1	$m_1^3 = \frac{27}{16}\Lambda^3$	$\mathfrak{u}(1)$	1	–

Two configurations: massless one is not modular. The other is modular for $\Gamma = \Gamma^0(3)$:

$$u(\tau) = -\frac{5}{3} - \frac{1}{9}\left(\frac{\eta\left(\frac{\tau}{3}\right)}{\eta(\tau)}\right)^{12} ,$$

Note the AD points H_0 as an elliptic point: [Argyres, Plesser, Seiberg, Witten, 1995]

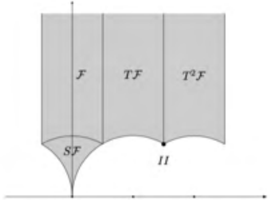


Figure 7: Fundamental domain for $\Gamma^0(3)$ corresponding to the configuration (I_3^*, I_1, II) on the CB of the 4d $SU(2)$, $N_f = 1$ theory. The marked point $\tau = 2 + e^{2i\pi/3}$ is the elliptic point of the congruence subgroup $\Gamma^0(3)$.

5d SCFTs on a circle and geometric engineering

5d SCFT from M-theory

M-theory is expected to define a surjective map:

$$\{\text{CY threefold singularity}\} \to \{\text{5d SCFTs}\} \ : \quad \mathbf{X} \mapsto \mathcal{T}_{\mathbf{X}}^{\text{5d}}$$

This is poorly understood in general.

Most basic quantity:

$$r = \text{rank}(\mathcal{T}_{\mathbf{X}}^{\text{5d}}) = \text{number of exceptional divisor in generic crepant resolution } \tilde{\mathbf{X}} \to \mathbf{X}$$

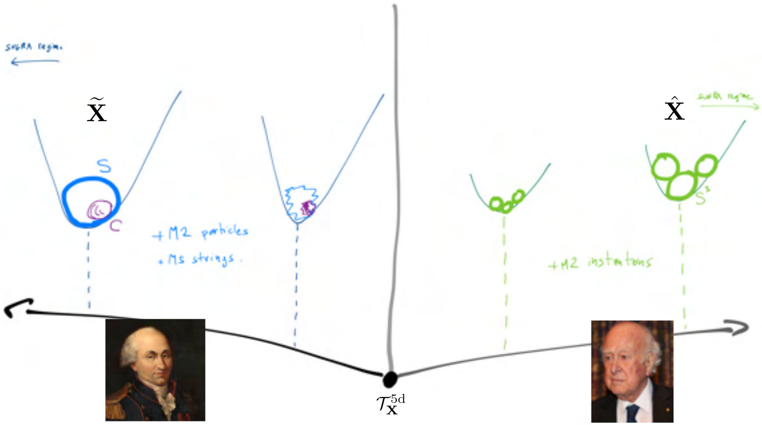
If $\tilde{\mathbf{X}}$, the $\mathcal{N} = 1$ SUGRA approximation is valid for large Kähler volumes. We then have a gauge theory $U(1)^r$ with 5d prepotential determined classically:

$$\mathcal{F}_{\text{5d}} = \frac{1}{6} \sum_{i,j,k} \mathcal{F}_{i,j,k} S_i \cdot S_j \cdot S_k$$

5d SCFT from M-theory

Geometric realization of the full moduli space of $\mathcal{T}_{\mathbf{X}}^{\rm 5d}$.

- Kähler moduli of $\tilde{\mathbf{X}}$ = ECB moduli
- $\mathbb{C}$-structure moduli of $\widehat{\mathbf{X}}$ = Higgs branch moduli (+ irrelevant couplings)



5d SCFT from M-theory: rank one

Here we focus on the simplest example, of **rank one:** [Morrison, Seiberg, 1996]

$$\tilde{\mathbf{X}} = \mathrm{Tot}(\mathcal{K} \to S) \ , \qquad S = \mathbb{F}_0 \text{ or } dP_n \ (n \neq 8)$$

Singularity $\mathbf{X}$: blow-down the zero section S, which is a Fano surface.

Two ways of describing the **del Pezzo surface:**

(i) $dP_n \cong \mathrm{Bl}_n(\mathbb{P}^2)$: blow up of $\mathbb{P}^2$ at n *generic* points.

(ii) $dP_n \cong \mathrm{Bl}_{n-1}(\mathbb{F}_0)$: blow up of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ at $n-1$ *generic* points.

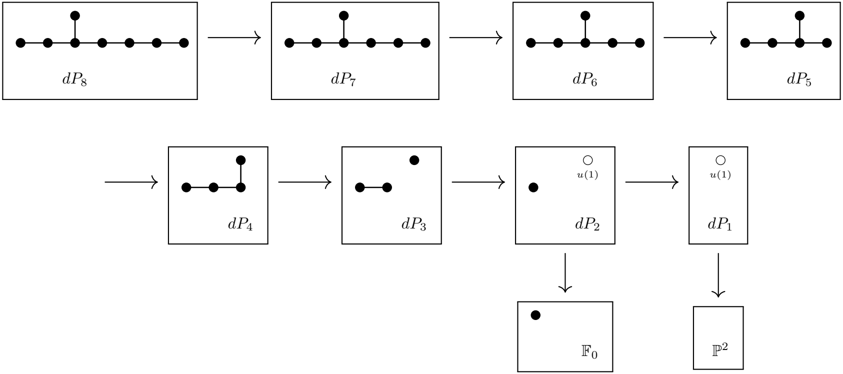
Intersection form $H_2(S, \mathbb{Z}) \times H_2(S, \mathbb{Z}) \to \mathbb{Z}$ can be written as:

$$\begin{pmatrix} 9-n & 0 \\ 0 & -A^{E_n}_{IJ} \end{pmatrix} \ , \quad I, J = 1, \cdots, n \ , \qquad 9 - n = \deg(S) = \mathcal{K} \cdot \mathcal{K}$$

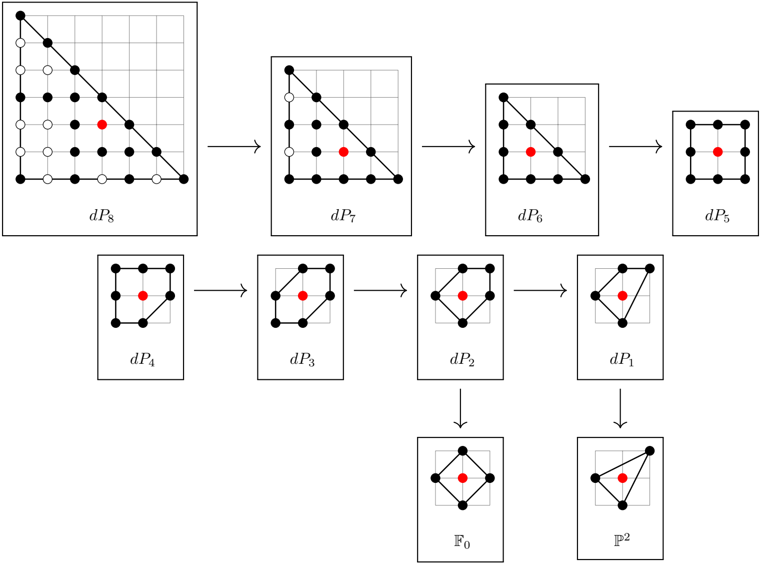
$\Rightarrow$ M2-brane particles on CB form representations of $E_n = \mathfrak{e}_n$ algebra.

E_n theories from del Pezzos

These SCFTs are all related by RG flows triggered by massive deformations:



E_n theories from del Pezzos



for 'generalized toric' (GTP) description, see [Benini, Benvenuti, Tachikawa, 2009]

The 5d gauge theory limit

- These 10 rank-one SCFTs were first discovered by Seiberg as UV fixed points of 5d $\mathcal{N}=1$ gauge theories. [Seiberg, 1996]
- Recall that 5d gauge theories are IR-free effective theories. The perturbative gauge-theory description is valid for RG scales:

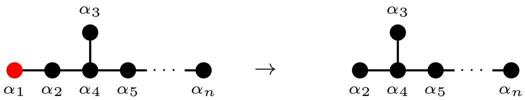
$$\mu \ll m_0 \equiv \frac{1}{g_{5\mathrm{d}}^2}$$

- $\mathcal{T}_{E_n}^{5\mathrm{d}}$ admits a mass deformation to a 5d $\mathcal{N}=1$ gauge theory in the IR:

$$E \ll m_0 = \frac{1}{g_{5\mathrm{d}}^2} \quad : \qquad \textbf{5d } \mathcal{N}=1 \; SU(2) \textbf{ with } N_f = n-1 \textbf{ fundamentals.}$$

This mass deformation breaks the flavor algebra as:

$$E_n \qquad \to \qquad \mathfrak{so}(2n-2) \oplus \mathfrak{u}(1)$$



5d theories on $\mathbb{R}^4 \times S^1$: generalities

Consider the SCFT $\mathcal{T}_{\mathbf{X}}^{\rm 5d}$ compactified on a finite-size circle, of radius β.

This gives us a **4d $\mathcal{N}=2$ supersymmetric** Kaluza-Klein (KK) field theory:

$$\boxed{D_{S^1}\mathcal{T}_{\mathbf{X}}^{\rm 5d} \text{ on } \mathbb{R}^4 \quad \cong \quad \mathcal{T}_{\mathbf{X}}^{\rm 5d} \text{ on } \mathbb{R}^4 \times S^1_\beta}$$

Note:

- $D_{S^1}\mathcal{T}_{\mathbf{X}}^{\rm 5d}$ is not conformal. Scale $\mu_{\rm KK} = \frac{1}{\beta}$.
- The Coulomb branch is complexified. In terms of the low-energy abelian vector multiplet:

$$\overline{\mathcal{B}} \quad : \quad \boxed{a = i(\varphi + iA_5)\ ,} \qquad e^{2\pi i A_5} \equiv e^{\int_{S^1} A}$$

IR vector multiplet is not globally defined on $\overline{\mathcal{B}}$, just like for any 4d $\mathcal{N}=2$ SQFT.

see *e.g.* [Nekrasov, 1996]

5d theories on $\mathbb{R}^4 \times S^1$: generalities

- At any point $U \in \overline{\mathcal{B}}$, we have massive half-BPS particle excitations. Their masses:

$$M_\gamma = |Z_\gamma| = |e^i a_i + m_i a_D^i + q^I \mu_I + n\mu_{\rm KK}| \,,$$

  are determined by their electro-magnetic charges:

$$\gamma = (e, m, q, n) \in \Gamma \subset \mathbb{Z}^{2r+f+1}$$

  To determine the **BPS spectrum** $\{\gamma\}$ is a complicated, unsolved problem in general. (Recent studies for 5d KK theories: [Eager, Selmani, Walcher, 2016; Banerjee, Longhi, Romo, 2019, 2020; CC, Del Zotto, 2019; Longhi, 2020; Mozgovoy, Pioline, 2020])

- Note: the charge lattice includes, electromagnetic charges, 5d flavor charges *and* the KK charge.

- Key features of any SW geometry are its singular loci: the complex-codim-1 loci on the CB where some BPS particles become massless.

The U-plane for a 5d SCFT on S^1

As a first approximation, let us think of our E_n theories as 5d $SU(2)$ gauge theories. The low-energy $U(1)$ scalar is:

$$a = i(\varphi + iA_5)\ , \qquad e^{2\pi i A_5} \equiv e^{\int_{S^1} A}$$

and the gauge-invariant order parameter is:

$$\boxed{U = \langle W \rangle = e^{2\pi i a} + e^{-2\pi i a} + \cdots}$$

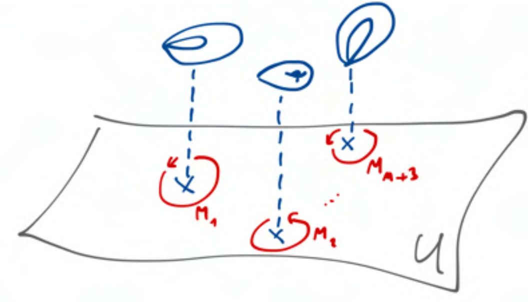
Here W is a supersymmetric Wilson line in 5d, wrapped along the S^1.

Similarly, the complexified mass parameters are flavor Wilson lines:

$$M_I = e^{2\pi i \mu_I} = e^{-\beta m_I + i\vartheta_I}$$

The U-plane for a 5d SCFT on S^1

At fixed M_I, the Coulomb branch is one-dimensional, with local coordinate $U \in \mathbb{C}$. This is **the U-plane.**



As in 4d, the low-energy physics is fully determined by some Seiberg-Witten geometry, which was derived in [Ganor, Morrison, Seiberg, 1996; Eguchi, Sakai, 2002].

All our computations are done using the E_n SW curves of [Eguchi, Sakai, 2002]. (Matches Hori-Vafa mirror in toric case.)

The U-plane from local mirror symmetry

The SW solution is essentially local **mirror symmetry:** [Katz, Mayr, Vafa, 1996]

$$\text{CB of } D_{S^1}\mathcal{T}_{\mathbf{X}}^{5\mathrm{d}} \quad \longleftrightarrow \quad \text{IIA string theory on } \mathbb{R}^4 \times \tilde{\mathbf{X}}$$
$$\longleftrightarrow \quad \text{IIB string theory on } \mathbb{R}^4 \times \hat{\mathbf{Y}}$$

We have the local mirror symmetry between smooth threefolds:

$$\tilde{\mathbf{X}} \quad \leftrightarrow \quad \hat{\mathbf{Y}}\,, \qquad D(\tilde{\mathbf{X}}) \quad \leftrightarrow \quad \mathrm{Fuk}(\hat{\mathbf{Y}})$$

In particular:

- U, M_I are complex structure parameters of $\widehat{\mathbf{Y}}$.
- a, μ_I are Kähler parameters of $\tilde{\mathbf{X}}$.
- The exact expression:

$$a(U) = \frac{1}{2\pi i}\log\frac{1}{U} + \sum_k c_k U^k$$

is the mirror map.

The U-plane for a 5d SCFT on S^1

For our local dP_n geometries, the mirror threefold $\widehat{\mathbf{Y}}$ can be written as the suspension of an affine elliptic curve, E:

$$v_1 v_2 + P(w,t) = 0 \; , \qquad E = \{(w,t) \in \mathbb{C}^* \times \mathbb{C}^* \mid P(w,t) = 0\}$$

- ◇ For the five toric geometries, E_0, $\tilde{E}_1$, E_1, E_2 and E_3, we have $P(w,t)$ equal to the Newton polygon of the toric diagram. [Chiang, Klemm Yau, Zaslow, 1999; Hori, Vafa, 2000]
- ◇ For the higher del Pezzos, the mirror curves are also known. They are all limits of the E-string theory Seiberg-Witten curve [Ganor, Morrison, Seiberg, 1996; Eguchi, Sakai, 2002]
- ◇ We have:

$$H_3(\widehat{\mathbf{Y}}, \mathbb{Z}) \cong \mathbb{Z}^{|Q_0|} \; , \qquad |Q_0| = 2r + f + 1 = n + 3$$

Large-volume perspective

Coming back to Type IIA on:

$$\tilde{\mathbf{X}} = \mathrm{Tot}\left(\mathcal{K} \to dP_n\right)\ , \qquad \boxed{dP_n \cong \mathrm{Bl}_{n-1}(\mathbb{F}_0)}$$

we have:

- A four-cycle $\mathcal{B}_4 = [dP_n]$. The D4-brane on $\mathcal{B}_4$ gives the **'monopole.'**
- A 2-cycle $\mathcal{C}_f$ such that $\mathcal{C}_f^2 = 0$ and $\mathcal{C}_f \cdot \mathcal{B}_4 = -2$.
  The D2-brane on $\mathcal{C}_f$ is **'the W-boson.**
- Exceptional 2-cycles E_a, $a = 1, c \ldots, n-1$: gives the hypermultiplets.

BPS particles are wrapped branes, with central charge:

$$Z = m\Pi_{\mathrm{D4}} + n_{\mathrm{D2_f}}\Pi_{D2_f} + \sum n_{\mathrm{D2_{E_a}}}\Pi_{D2_{E_a}} + n_{\mathrm{D0}}$$

The D-brane periods are well known *at large volume*, where the supergravity approximation holds (as an asymptotic expansion):

$$\Pi_{D2_{\mathcal{C}}} = \int_{\mathcal{C}} (B + iJ)\ ,$$

$$\Pi_{D4} = \frac{1}{2}\int_{\mathcal{B}_4} (B + iJ)^2 + \frac{\chi(\mathcal{B}_4)}{24} + \cdots\ .$$

Large-volume perspective

The exact periods receive worldsheet instanton corrections, which can be obtained using local mirror symmetry.

There are $n+3$ D-brane periods, but $n+1$ are known exactly (no quantum corrections):

$$\Pi_{\rm D0} = 1 \;, \qquad \Pi_{\mathcal{C}_I} = \frac{1}{2\pi i}\log(z_I) = \mu_I$$

in other words, $z_I = M_I$ for the flavor curves. These are such that $\mathcal{C}_I \cdot \mathcal{B}_4 = 0$.
The non-trivial periods are:

$$\Pi_{\mathcal{C}_f} \equiv 2a = \frac{1}{2\pi i}\log\left(\frac{1}{U^2}\right) + \cdots$$

and:

$$\Pi_{\rm D4} = a_D = \frac{1}{(2\pi i)^2}\log\left(\frac{1}{U^2}\right)\log\left(\frac{M_0}{U^2}\right) + \frac{\chi}{24} + \cdots$$

Note the normalisation of the a-period. (W-boson of electric charge 2 for $SU(2)$.)

Local mirror threefold and curve

In the mirror $\widehat{\mathbf{Y}}$, we have to compute classical periods:

$$a = \int_{S_a^3} \Omega \ , \qquad a_D = \int_{S_D^3} \Omega \ , \qquad S_a^3 \cdot S_D^3 = 1 \ .$$

The corresponding cohomology classes fit in a mixed Hodge structure:

$$H^3(\tilde{\mathbf{X}}) \cong H^{2,1} \oplus H^{1,2} \oplus H^{2,2} \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^{n+1}$$

This can be reduced to the periods of the elliptic (or affine) curve E:

$$a_D = \int_{\gamma_B} \lambda_{\rm SW} \ , \qquad\qquad a = \int_{\gamma_A} \lambda_{\rm SW} \ .$$

Thus, we may focus on that 'SW curve' to describe the mirror geometry.

BPS states and flavour symmetry

Write the 3-fold $\widehat{\mathbf{Y}}$ as a double-fibration on 'the W-plane':

$$E \times \mathbb{C}^* \to \widehat{\mathbf{Y}} \to \mathbb{C} \cong \{W\}$$

with:

$$F(x, y; W) = 0 \;, \qquad\qquad v_1 v_2 = U - W \;.$$

- At $U = W$, the $\mathbb{C}^*$ fiber degenerates.
- The elliptic fiber E degenerates at $W = U_*$.
- U is a complex structure parameter, and W and ambiant coordinate of the CY$_3$ geometry. But we can substitute one for the other in the obvious way.
- Thus, we can view the RES $\mathcal{S}$ as part of $\widehat{\mathbf{Y}}$ itself, as $E \to \mathcal{S} \to \{W\}$.
- We can build supersymmetric 3-cycles S^3_γ as torus fibrations over paths on the W-plane. [Hori, Vafa, 2000]

Let $\Gamma_2 \subset S^3_\gamma$ be the two-chain with boundary along $\gamma \in E_U$ above the fiber at $W = U$. We have:

$$\Pi_\gamma = \int_{S^3_\gamma \subset \widehat{\mathbf{Y}}} \Omega = \int_{\Gamma \subset \mathcal{S}} \Omega_2 = \int_{\gamma \in E} \lambda_{\rm SW} \;,$$

with $\partial\Gamma = \gamma$, provided that:

$$\Omega_2 = d\lambda_{\rm SW} = \boldsymbol{\omega} \wedge dU$$

inside $\mathcal{S}$.

BPS states and flavour symmetry

There is an F-theory perspective on this IIB geometry, by viewing τ itself as the axio-dilaton. In that interpretation:

- The singular fibers are 7-branes of type ADE.
- One interprets $W = U$ as the position of a single probe D3-brane.
- BPS states are string junctions between the D3-brane and the 7-branes.

To discuss the flavour group, we consider (formal) pure flavour states which are open strings between 7-branes. They correspond to closed 2-cycles $\Gamma \in \mathrm{NS}(\tilde{S})$. Their flavor weights under $\mathfrak{g}^{\mathrm{NA}} = \oplus_v \mathfrak{g}_v$ are determined by the intersection with the exceptional fibers:

$$w_i^{(\mathfrak{g}_v)}(\Gamma) = \Theta_{v,i} \cdot \Gamma$$

The abelian charges are given in terms of the so-called Shioda map: [Shioda, 1990]

$$q_s(\Gamma) \equiv \varphi(P_s) \cdot \Gamma \ .$$

Moreover, physical states should not intersect the fiber at infinity:

$$\Gamma \text{ physical} \qquad \Leftrightarrow \qquad w_i^{(F_\infty)}(\Gamma) = \Theta_{\infty,i} \cdot \Gamma = 0 \ ,$$

BPS states and flavour symmetry

Recall our definitions:

$$\mathscr{Z}^{[1]} = \left\{ P \in \Phi_{\rm tor} \;\middle|\; (P) \text{ intersects } \Theta_{v,0} \text{ for all } F_{v\neq\infty} \right\} ,$$

and:

$$0 \to \mathscr{Z}^{[1]} \to \Phi_{\rm tor} \to \mathscr{F} \to 0 \; .$$

The Shioda map. An important mathematical result [Shioda, 1990] is that there exists a group homomorphism:

$$\varphi \;:\; \Phi \to {\rm NS}(\tilde{\mathcal{S}}) \otimes \mathbb{Q} \; ,$$

which maps sections to horizontal divisors (with rational coefficients). This map is given explicitly by:

$$\varphi(P) = (P) - (O) - ((P)\cdot(O) + 1)F + \sum_v \sum_{i=1}^{{\rm rank}(\mathfrak{g}_v)} \lambda_{v,i}^{(P)} \Theta_{v,i} \; ,$$

with the rational coefficients:

$$\lambda_{v,i}^{(P)} = \sum_{j=1}^{{\rm rank}(\mathfrak{g}_v)} (A_{\mathfrak{g}_v}^{-1})_{ij} \, \Theta_{v,j} \cdot (P) \; ,$$

given in terms of the inverse of the Cartan matrix of $\mathfrak{g}_v$.

BPS states and flavour symmetry

Then, the argument for determining:

$$G_F = \tilde{G}_F / \mathscr{F}$$

(in the semi-simple case, for simplicity) is similar to the F-theory argument in *e.g.* [Aspinwall, 1998; Mayrhofer, Morrison, Till, Weigand, 2014; Cvetic, Lin, 2017].

For any state, we have:

$$\sum_{l=1}^{\mathrm{rank}(F_\infty)} \lambda_{\infty,l}^{(P_{\mathrm{tor}})} w_i^{(F_\infty)} + \sum_{i=1}^{\mathrm{rank}(\mathfrak{g}_F^{\mathrm{NA}})} \lambda_{v,i}^{(P_{\mathrm{tor}})} w_i^{(\mathfrak{g}_F^{\mathrm{NA}})} \in \mathbb{Z} \; .$$

For the pure flavour states that satisfy the physical state condition:

$$\sum_{i=1}^{\mathrm{rank}(\mathfrak{g}_F^{\mathrm{NA}})} \lambda_{v,i}^{(P_{\mathrm{tor}})} w_i^{(\mathfrak{g}_F^{\mathrm{NA}})} \in \mathbb{Z} \; , \qquad \forall \, P_{\mathrm{tor}} \in \mathscr{F} \; .$$

This directly implies the advertised result. To determine the precise action of $\mathscr{F}$, we compute the intersection of the sections with the fibers explicitly.
(Further arguments confirm our general results [CC, Magureanu, 2021].)

The U-plane of the E_n 5d SCFTs

The fiber at infinity

Consider the E_n theory. One can determine the large volume monodromy from the semi-classical periods.

Let us give a more "5d QFT" derivation: Take a limit where the 5d $SU(2)$, $N_f = n-1$ gauge-theory description is valid. At one-loop, the prepotential of the theory on $\mathbb{R}^4 \times S^1$ reads: [Nekrasov, 1998]

$$\mathcal{F} = \quad \mu_0 a^2 + \frac{2}{(2\pi i)^3}\mathrm{Li}_3\left(e^{4\pi i a}\right) - \frac{1}{(2\pi i)^3}\sum_{a=1}^{n-1}\sum_{\pm}\mathrm{Li}_3\left(e^{2\pi i(\pm a+\mu_a)}\right)$$

and $a_D = \frac{\partial \mathcal{F}}{\partial a}$. The large volume monodromy is:

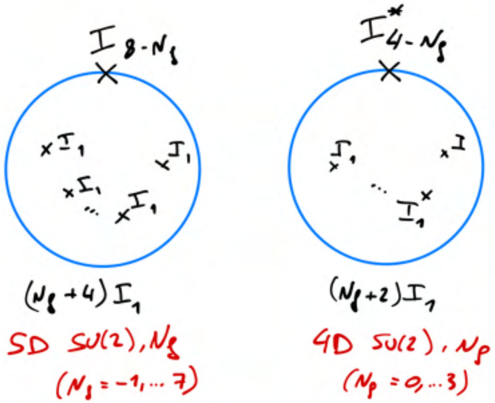
$$a_D \to a_D + (9-n)a + \mu_0 - \sum_{a=1}^{n-1}\mu_a \ , \qquad a \to a+1$$

We thus have:

$$\boxed{\mathbb{M}_\infty = T^{9-n} = \begin{pmatrix} 1 & 9-n \\ 0 & 1 \end{pmatrix}}$$

This determines **the fiber at infinity,** $F_\infty = I_{9-n}$**,** as anticipated.

Rational elliptic surfaces and *generic masses*:



The I_k fiber has monodromy conjugate to T^k. The bulk I_1 corresponds to a single BPS particle becoming massless:

$$M_*^{(m,q)} = B^{-1} T B = \begin{pmatrix} 1 + mq & q^2 \\ -m^2 & 1 - mq \end{pmatrix} \ .$$

The massless curves

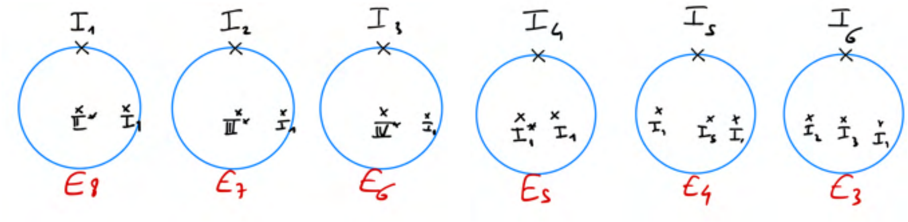
Consider now $M_I = 1$. One finds:

$$
\begin{array}{lcl}
E_8 & : & II^* \oplus I_1 \\
E_7 & : & III^* \oplus I_1 \\
E_6 & : & IV^* \oplus I_1 \\
\hline
E_5 & : & I_1^* \oplus I_1 \\
E_4 & : & I_5 \oplus I_1 \oplus I_1 \\
E_3 & : & I_3 \oplus I_2 \oplus I_1 \\
E_2 & : & I_2 \oplus I_1 \oplus I_1 \oplus I_1 \\
E_1 & : & I_2 \oplus I_1 \oplus I_1 \\
\tilde{E}_1 & : & I_1 \oplus I_1 \oplus I_1 \oplus I_1 \\
E_0 & : & I_1 \oplus I_1 \oplus I_1
\end{array}
$$

in agreement with old 'classic' results. [Ganor, Morrison, Seiberg, 1996]

- ◇ This reproduce the E_n flavor symmetry, including abelian factors.
- ◇ The 4d LEEFT is IR free for $n < 6$

The massless curves

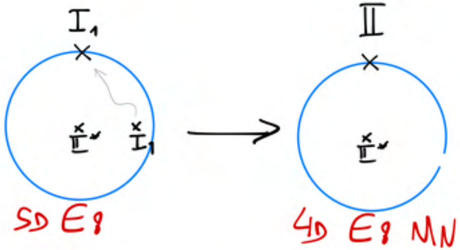


- It reproduces the 5d Higgs branches (one E_n-instanton moduli spaces).
- Note the case of E_5: 4d $SU(2)$ with $N_f = 5$ in the IR.
- 5d RG flows $E_n \to E_{n-1}$ reproduced.

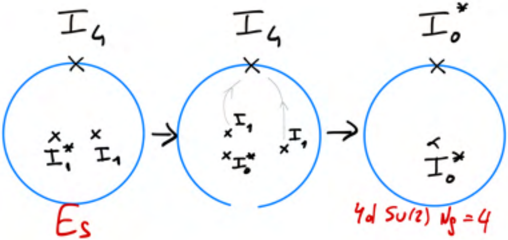
RG flows to 4d

Two types of flows:

- **"zooming in":**
  Here we just decouple the KK scale.



- **"geometric engineering limit":**
  We decouple the KK scale *and* the instanton particles.



Extremal rational elliptic surfaces

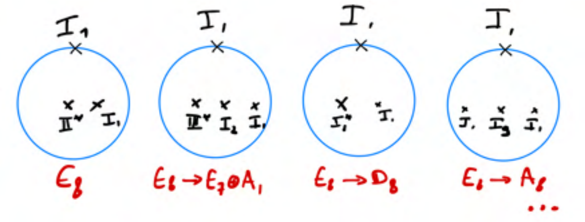
Subclass of rational elliptic surfaces with a section and with $\mathrm{rk(MW)} = 0$. Classification by [Miranda, Persson, 1986]. Small list of 16 surfaces.

- Four of them have only 2 singular fibers:

$$(II, II^*)\ , \qquad (III, III^*)\ , \qquad (IV, IV^*)\ , \qquad (I_0^*, I_0^*)\ .$$

  They describe the 7 "classic" 4d SCFTs of rank one we just reviewed.

- The other 12 extremal surfaces all describe points on the extended Coulomb branch of the 5d E_n SCFTs. They are all the possible maximal Dynkin subalgebras of E_n ($n \neq \tilde{1}, 2$). For instance:



- Recall that the same surface can describe different theories, by choosing 'the point at infinity':

$\{F_v\}$	*Notation*	Φ_{tor}	*Field theory*	$\mathfrak{g}_F$	*Modularity*
II^*, I_1, I_1	X_{211}	$-$	$D_{S^1}E_8$	E_8	$-$
			AD H_0	$-$	
III^*, I_2, I_1	X_{321}	$\mathbb{Z}_2$	$D_{S^1}E_8$	$E_7 \oplus A_1$	$\Gamma_0(2)$
			$D_{S^1}E_7$	E_7	
			AD H_1	A_1	
IV^*, I_3, I_1	X_{431}	$\mathbb{Z}_3$	$D_{S^1}E_8$	$E_6 \oplus A_2$	$\Gamma_0(3)$
			$D_{S^1}E_6$	E_6	
			AD H_2	A_2	
I_4^*, I_1, I_1	X_{411}	$\mathbb{Z}_2$	$D_{S^1}E_8$	D_8	$\Gamma_0(4)$
			4d pure $SU(2)$	$-$	
I_1^*, I_4, I_1	X_{141}	$\mathbb{Z}_4$	$D_{S^1}E_8$	$D_5 \oplus A_3$	$\Gamma_0(4)$
			$D_{S^1}E_5$	D_5	
			4d $SU(2)$ $N_f = 3$	A_3	
I_2^*, I_2, I_2	X_{222}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_{S^1}E_7$	$D_6 \oplus A_1$	$\Gamma(2)$
			4d $SU(2)$ $N_f = 2$	$A_1 \oplus A_1$	
I_9, I_1, I_1, I_1	X_{9111}	$\mathbb{Z}_3$	$D_{S^1}E_8$	A_8	$\Gamma_0(9)$
			$D_{S^1}E_0$	$-$	
I_8, I_2, I_1, I_1	X_{8211}	$\mathbb{Z}_4$	$D_{S^1}E_8$	$A_7 \oplus A_1$	$\Gamma_0(8)$
			$D_{S^1}E_7$	A_7	
			$D_{S^1}E_1$	A_1	
I_5, I_5, I_1, I_1	X_{5511}	$\mathbb{Z}_5$	$D_{S^1}E_8$	$A_4 \oplus A_4$	$\Gamma_1(5)$
			$D_{S^1}E_4$	A_4	
I_6, I_3, I_2, I_1	X_{6321}	$\mathbb{Z}_6$	$D_{S^1}E_8$	$A_5 \oplus A_2 \oplus A_1$	$\Gamma_0(6)$
			$D_{S^1}E_7$	$A_5 \oplus A_2$	
			$D_{S^1}E_6$	$A_5 \oplus A_1$	
			$D_{S^1}E_3$	$A_2 \oplus A_1$	
I_4, I_4, I_2, I_2	X_{4422}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$D_{S^1}E_7$	$A_3 \oplus A_3 \oplus A_1$	$\Gamma_0(4) \cap \Gamma(2)$
			$D_{S^1}E_5$	$A_3 \oplus A_1 \oplus A_1$	
I_3, I_3, I_3, I_3	X_{3333}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$D_{S^1}E_6$	$A_2 \oplus A_2 \oplus A_2$	$\Gamma(3)$

New $5d \to 4d$ limits

- Other interesting points on the 4d CB of each E_n KK theory?
- Most of these points are at $|M_I| = 1$. In field theory, that corresponds to turning on Wilson line along the S^1 in an otherwise massless theory. In IIA or M-theory, this corresponds to 'quantum periods' on vanishing cycles:

$$M_I \sim e^{i \int_{\mathcal{C}_I} B} = e^{i \int_{\mathcal{C}_I \times S^1} C_3} \quad \text{with} \qquad \mathrm{vol}(\mathcal{C}_I) = 0$$

- The most interesting points are Argyres-Douglas points H_0, H_1, H_2 which are **not** the ones we would obtain by tuning the masses in 4d $SU(2)$ with $N_f = 1, 2, 3$ flavors.
- The existence of such non-trivial limits was recently argued by [Bonelli, del Monte, Tanzini, 2020] using the 5d BPS quiver, as well as from a correspondence between these SCFTs and (difference) Painlevé equations [Bonelli, Lisovyy, Maruyoshi, Sciarappa, Tanzini, 2016]. We construct these limits explicitly.

Example: E_3 and AD points

Consider the E_3 SCFT. Its SW curve can be written as:

$$E_3 \;:\qquad \frac{\sqrt{\lambda}}{t}\left(1+\frac{M_2}{w}\right)+t\sqrt{\lambda}\,(1+M_1 w)+\frac{1}{w}+w-2U=0\ .$$

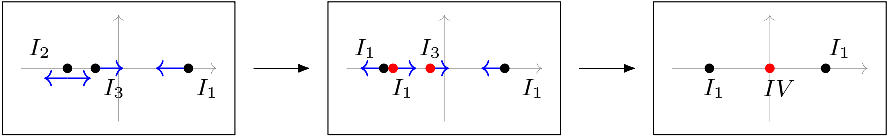
- Recall that this is "5d $SU(2)$, $N_f=2$".
- We have $M_0=\lambda\sim e^{-\frac{\beta}{g_{5\text{d}}^2}}$ and two 'hypermultiplet masses' M_1, M_2.
- On the other hand, the AD theories are known to exist on the CB of 4d $SU(2)$ with N_f flavors:
  - $H_0 \subset 4\text{d}\ SU(2), N_f=1$
  - $H_1 \subset 4\text{d}\ SU(2), N_f=2$
  - $H_2 \subset 4\text{d}\ SU(2), N_f=3$

  So, we may expect H_0 and H_1 to appear here too. We get 'more'.

Example: E_3 and AD points

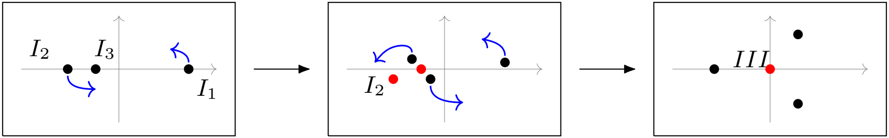
Starting from the massless curve $\lambda = M_1 = M_2 = 1$, we can get the AD fixed point H_2 at:

$$IV \ : \ (\lambda, M_1, M_2) = \left(1, e^{\frac{i\pi t}{2}}, e^{-\frac{i\pi t}{2}}\right)$$



We can then "zoom in" to the 4d SCFT. Similarly, we find H_1:

$$III \ : \ (\lambda, M_1, M_2) = \left(e^{\frac{4i\pi t}{3}}, e^{\frac{2i\pi t}{3}}, e^{\frac{2i\pi t}{3}}\right)$$



Modularity of the U-plane

In many interesting special limits, **the U-plane is a modular curve:**

$$\overline{\mathcal{B}} \cong \mathbb{H}/\Gamma \ , \qquad\qquad \Gamma \subset SL(2,\mathbb{Z})$$

This means, in particular, that the mirror map is a modular function:

$$a = a(U) \qquad \leftrightarrow \qquad U = U(\tau)$$

Example: the massless curves:

$$\begin{array}{lcccl} E_7 & : & III^* \oplus I_1 & : & \Gamma^0(2) \\ E_6 & : & IV^* \oplus I_1 & : & \Gamma^0(3) \\ E_5 & : & I_1^* \oplus I_1 & : & \Gamma^0(4) \\ E_4 & : & I_5 \oplus I_1 \oplus I_1 & : & \Gamma^1(5) \\ E_3 & : & I_3 \oplus I_2 \oplus I_1 & : & \Gamma^0(6) \\ E_1 & : & I_2 \oplus I_1 \oplus I_1 & : & \Gamma^0(8) \\ E_0 & : & I_1 \oplus I_1 \oplus I_1 & : & \Gamma^0(9) \end{array}$$

The massless E_8, E_2 and $\tilde{E}_1$ are not modular.

MW group and global symmetry

The general prescription for the global symmetry works here too. We find:

$$G_F = \mathrm{E}_n/Z(\mathrm{E}_n)$$

for the massless theories with semi-simple symmetry group.

- This agrees with the 5d result of [Apruzzi, Bhardwaj, Oh, Schafer-Nameki, 2021], which found G_F centerless using directly the M-theory geometry.
- The fiber $F_\infty = I_8$ does not determine the SQFT uniquely.
  Two distinct choices for $\mathcal{Z}^{[1]}$, either $\mathbb{Z}_2$ or trivial. This gives E_1 or $\tilde{E}_1$.
- The case E_1 is special, with $\Phi = \mathbb{Z}_4$ and $\mathcal{Z}^{[1]} = \mathbb{Z}_2$, with:

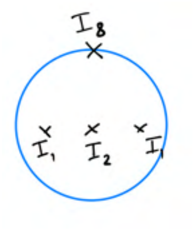
  $$\mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathscr{F} = \mathbb{Z}_2$$

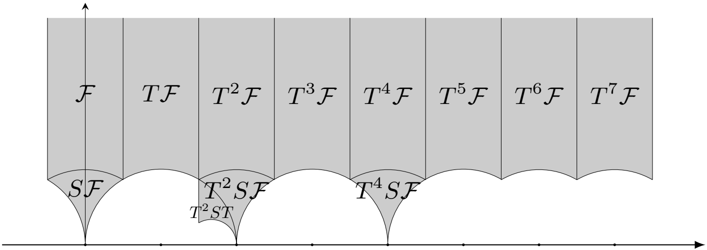
  so that $G_F = SO(3)$.
- All other theories have $\mathcal{Z}^{[1]} = 0$.

Example: the massless E_1 theory

This is "5d pure $SU(2)_0$ at infinite coupling."

The CB of the massless is a modular curve for the congruence subgroup $\Gamma^0(8)$:





Singularities and monodromies:

$$M_{(-2)} = STS^{-1}\ , \qquad M_{(0)} = (T^2S)T^2(T^2S)^{-1}\ , \quad M_{(-2)} = (T^4S)T(T^4S)^{-1}\ .$$

- At $U = -2$, the monopole $(1, 0)$ is massless, $a_D \to 0$. "Conifold point."
- At $U = 0$, two dyons $(-1, 2)$ are massless.
- At $U = 2$, the dyon $(1, -4)$ is massless.

Modular curves and quiver points

We can classify all modular CB configurations for any of the rank-one theories.

For instance, for $D_{S^1}E_8$ and restricting to congruence subgroups (for simplicity):

$\{F_v\}$	rk(Φ)	Φ_{tor}	$\mathfrak{g}_F$	$\Gamma \in \text{PSL}(2,\mathbb{Z})$
I_1, I_2, III^*	0	$\mathbb{Z}_2$	$E_7 \oplus A_1$	$\Gamma_0(2)$
I_1, I_3, IV^*	0	$\mathbb{Z}_3$	$E_6 \oplus A_2$	$\Gamma_0(3)$
$2I_1, I_4^*$	0	$\mathbb{Z}_2$	D_8	$\Gamma_0(4)$
I_1, I_4, I_1^*	0	$\mathbb{Z}_4$	$D_5 \oplus A_3$	$\Gamma_0(4)$
$2I_1, 2I_5$	0	$\mathbb{Z}_5$	$A_4 \oplus A_4$	$\Gamma_1(5)$
I_1, I_6, I_3, I_2	0	$\mathbb{Z}_6$	$A_5 \oplus A_2 \oplus A_1$	$\Gamma_0(6)$
$2I_1, I_8, I_2$	0	$\mathbb{Z}_4$	$A_7 \oplus A_1$	$\Gamma_0(8)$
$3I_1, I_9$	0	$\mathbb{Z}_3$	A_8	$\Gamma_0(9)$
I_1, III^*, II	1	–	E_7	$PLS(2,\mathbb{Z})$
I_1, III, IV^*	1	–	$E_6 \oplus A_1$	$PLS(2,\mathbb{Z})$
I_1, I_2^*, III	1	$\mathbb{Z}_2$	$D_6 \oplus A_1$	$\Gamma_0(2)$
I_1, I_3^*, II	1	–	D_7	$\Gamma_0(3)$
$I_1, I_5, 2III$	2	–	$A_4 \oplus 2A_1$	$\Gamma_0(5)$
$I_1, I_7, 2II$	2	–	A_6	$\Gamma_0(7)$

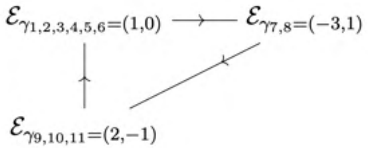
Modular curves and quiver points

One can then identify the light particles and, in favourable cases, the 5d BPS quiver.

[Alim, Cecotti, Cordova, Espahbodi, Rastogi, Vafa, 2011; CC, Del Zotto, 2019]

Example: The $D_{S^1}E_8$ CB configuration $\mathcal{S} = (I_1, I_6, I_3, I_2)$, with:

$$\mathscr{S}\ : \qquad I_6 :\ 6(1,0)\ , \qquad I_2 :\ 2(-3,1)\ , \qquad I_3 :\ 3(2,-1)\ ,$$



This is a correct 3-blocks quiver for dP_8 [Wijnholt, 2002].

By removing γ_1, we get a BPS quiver for the 4d E_8 MN theory.

In favourable cases, we can prove that the quiver point exists, by computing the periods exactly.

Summary and outlook

Summary:

- We revisited a general approach to rank-one 4d $\mathcal{N} = 2$ SQFT in terms of rational elliptic surfaces.
- We pointed out that the Persson classification of RES gives classification of CB configurations.
- We determined the flavour symmetry group directly from the SW geometry.
- We discussed **the Coulomb branch physics of 5d SCFTs** on $\mathbb{R}^4 \times S^1$.
- We observed some interesting new relations between 5d E_n SCFTs and 4d Argyres-Douglas theories.
- We studied global properties of the U-plane, such as modularity.

Outlook:

- We initiated a study of quiver points on the U-plane. More systematic analysis needed.
- These elementary considerations are fundamental to a better understanding of partition functions of 5d SCFTs on five-manifolds. *Work in progress.*