Asymptotic Flux Compactifications and the *Tameness* of the String Landscape

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Based on:

2010.15838

2105.02232 & 2108.11962 with Brice Bastian, Damian van de Heisteeg

2109.nnnn with Erik Plauschinn, Damian van de Heisteeg

21xx.nnnn with Ben Bakker, Christian Schnell, Jacob Tsimerman

Some words of motivation

Talk deals with aspects of Swampland Program:

Identify the general principles that have to be satisfied in an effective theory compatible with quantum gravity.

QFTs that can be coupled to quantum gravity

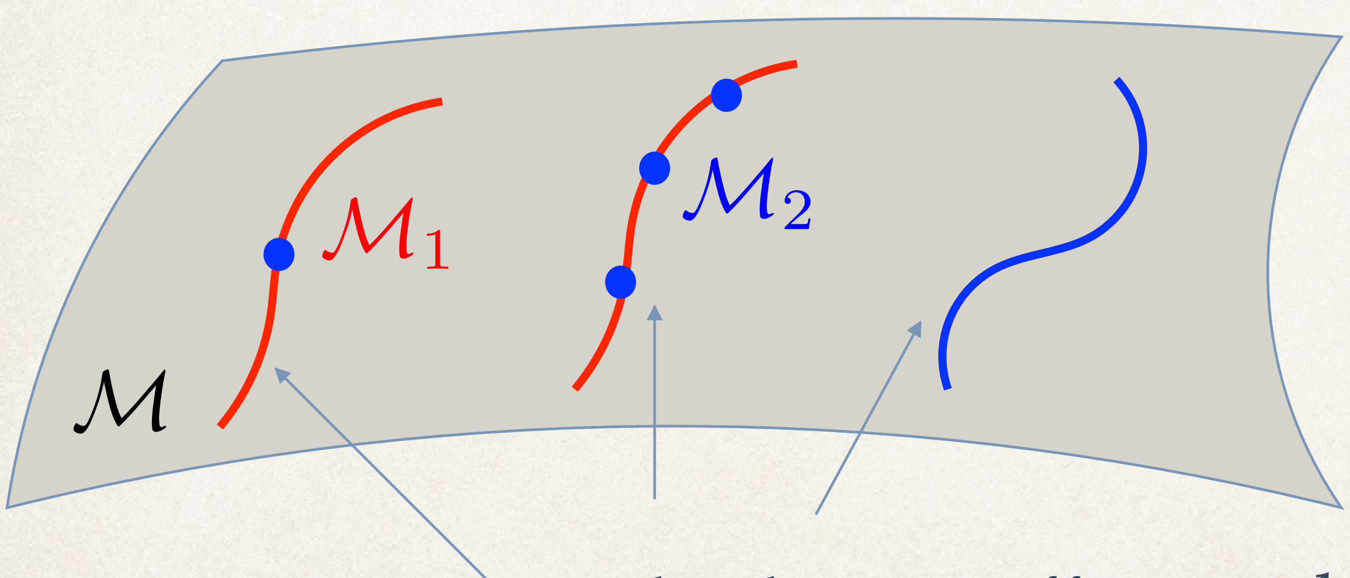
apparently consistent QFTs that **cannot** be coupled to quantum gravity

Field spaces and scalar potentials

- Consider effective theory with a cut-off Λ
  - scalar fields ϕ^i spanning field space $\mathcal{M}$
  - scalar potential $V(\phi)$: maps from $\mathcal{M}$ to real line

- Which field spaces $\mathcal{M}$ can appear in the landscape?
  Which scalar potentials $V(\phi)$ are in the landscape?

Related when lowering cut-off



Multiple new effective theories

Λ

integrate out heavy fields

Λ_1

integrate out heavy fields

Λ_2

Finiteness as a key principle of the landscape

- Is the number of distinct effective theories compatible with quantum gravity/string theory finite?
  - long part of the string phenomenology program e.g. [Douglas '03] [Acharya,Douglas '06]
  - much recent activity: finiteness of spectra, ranks of gauge groups

    [Adams,DeWolfe,Taylor] [Kim,Shiu,Vafa] [Kim,Tarazi,Vafa] [Cvetic,Dierigl,Lin,Zang] [Dierigl,Heckman] [Font,Fraiman,Grana,Nunez,DeFreitas] [Hamada,Vafa] [Taylor etal],[Kim,Shiu,Vafa],[Lee,Weigand],[Tarazi,Vafa]
- In this talk: indicate a new non-trivial finiteness proof and promote finiteness to a new universal principle to constrain effective theories (no susy, no holomorphicity…)

Outline

Part 1: Lessons about the complex structure moduli space and flux vacua

→ Scalar field spaces and scalar potentials in Type IIB flux compactifications are remarkably constrained.

Part 2: Finiteness of self-dual flux vacua and the structure of the flux vacuum landscape

→ Constraints are 'just enough' to ensure non-trivial finiteness property.

Part 3: Structure ensuring finiteness: a new principle

→ o-minimal structures and tame topology to describe the landscape

Flux compactifications: some lessons we learned

Type IIB / F-theory flux compactifications

- Type IIB flux compactifications review: [Graña] [Kachru,Douglas] …

background flux: $F_3, \; H_3 \in H^3(Y_3, \mathbb{Z})$ $\qquad \int_{Y_3} F_3 \wedge H_3 < K$

vacuum condition: $*G_3 = iG_3 \qquad G_3 = F_3 - \tau H_3$

- lift to F-theory flux compactifications

background flux: $G_4 \in H^4(Y_4, \mathbb{Z})$ $\qquad \int_{Y_4} G_4 \wedge G_4 < K$

vacuum condition: $*G_4 = G_4$

→ well studied set of N=0,1 vacua with (partially) fixed complex structure moduli, backreaction under control, higher-derivative corrections consistently included

[Becker,Becker]…[TG,Pugh,Weissenbacher]…[Cicoli,Quevedo,Savelli,Schachner,Valandro]

Using supersymmetry of the effective theory?

- Hodge star $*$ changes over complex structure moduli space → complicated

- an alternative picture: express theory in terms of N=1 data

$$K_{\text{cs}} = -\log \int_{Y_D} \Omega \wedge \bar{\Omega} \qquad\qquad W = \int_{Y_D} G_D \wedge \Omega$$

$\Omega \in H^{D,0}$: unique (D,0)-form on the Calabi-Yau D-fold $D = 3$, $D = 4$

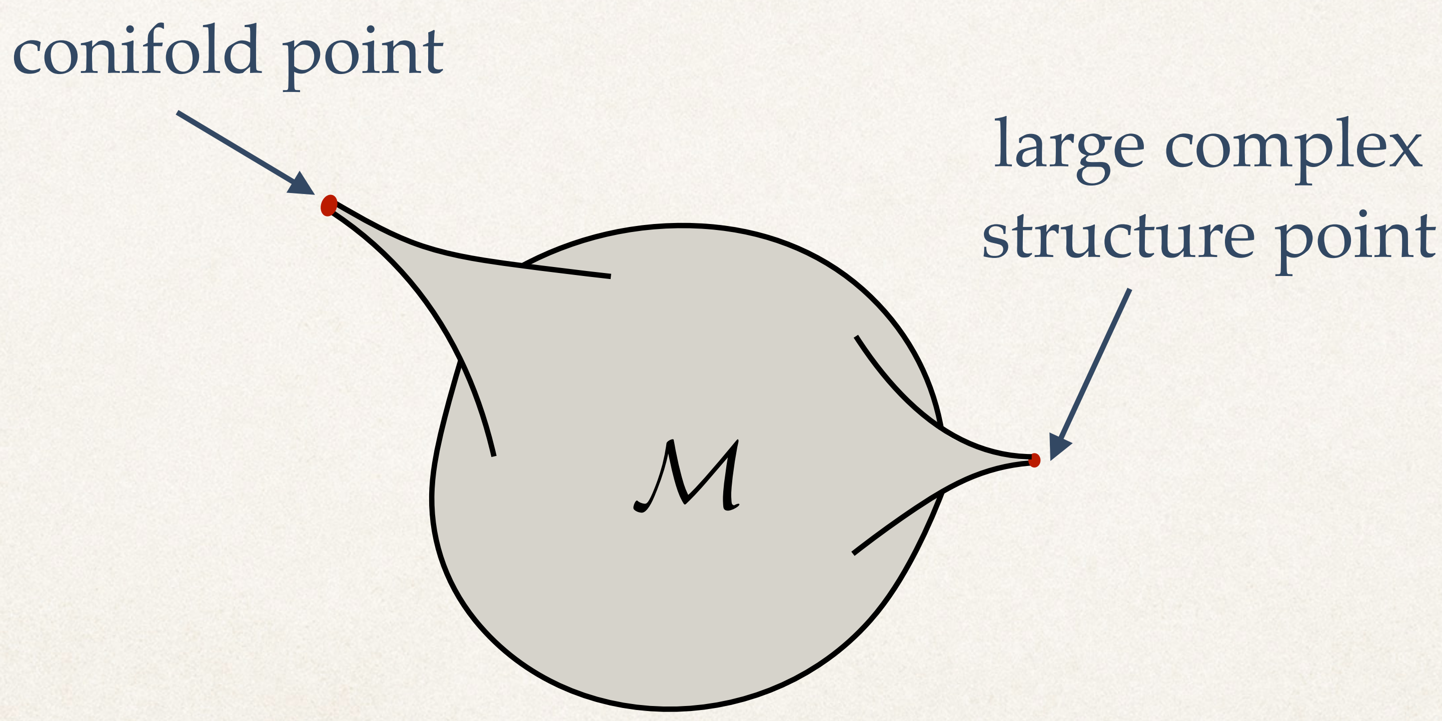
- vacuum condition: $D_{z^i} W = 0 \qquad (D_\tau W = 0)$ (self-dual fluxes)

- periods of Ω: $\Pi_i(z) = \int_{\mathcal{C}_i} \Omega$ → complicated transcendental functions → hard to compute or 'see' properties

Complex structure moduli space

- General conclusions in certain regions of the moduli space?
- Complex structure moduli space $\mathcal{M}$ has boundaries + asymptotic regions

Example: mirror quintic



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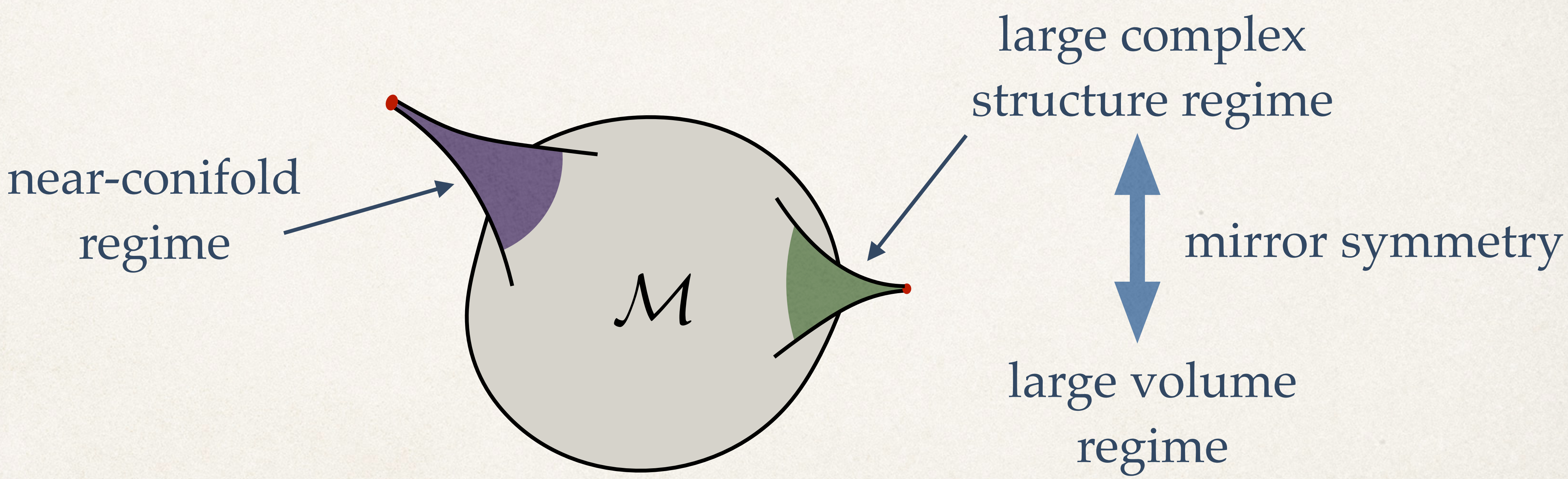
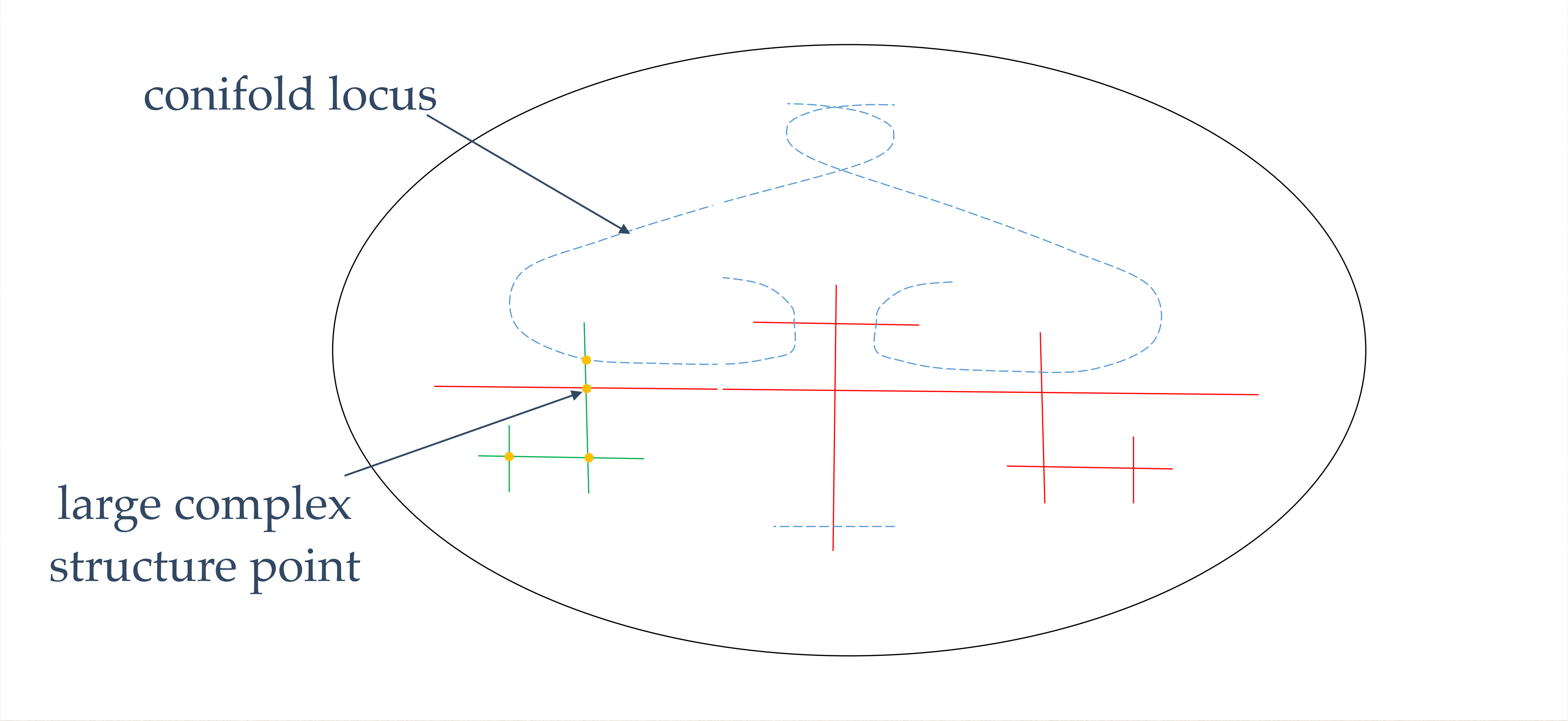


Complex structure moduli space

- Note: geometry of boundaries + asymptotic regions can be very involved for higher-dimensional moduli spaces

Example: mirror $\mathbb{P}^{1,1,1,6,9}[18]$

[Candelas,Font,Katz,Morrison]
[Candelas,De La Ossa,Font,Katz,Morrison]



Complex structure moduli space

- Systematic understanding of asymptotic moduli space without scanning through explicit examples?
- Identify states and flux vacua in the asymptotic regions of $\mathcal{M}$?
- Test conjectures: distance conjecture, WGC, axionic/emergent string conjecture, tadpole conjecture, finiteness conjectures

Complex structure moduli space

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$\rightarrow$ reviews: [Palti] [Valenzuela etal.] [Grana,Herraez]

see also [Lee,Lerche,Weigand],[Cecotti]

$\rightarrow$ develop tools in asymptotic Hodge theory and apply them to test conjectures

[TG,Palti,Valenzuela]
[TG,Li,Palti]
[TG,Li,Valenzuela]

full power starts to become apparent in our more recent works
[TG,Ruehle,vd Heisteeg],[TG],[TG,Monnee,vd Heisteeg][Bastian,TG,vd Heisteeg]

Hodge structures and their variation

- well-known Hodge decomposition of cohomology groups:

$$H^D(Y_D,\mathbb{C}) = H^{D,0} \oplus H^{D-1,1} \oplus ... \oplus H^{1,D-1} \oplus H^{0,D}$$

$\rightarrow$ (p,q)-forms in $H^{p,q}$

Hodge structures and their variation

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$$H^D(Y_D,\mathbb{C}) = H^{D,0} \oplus H^{D-1,1} \oplus \ldots \oplus H^{1,D-1} \oplus H^{0,D}$$

spanned by Ω

- (p,q)-splitting changes when moving in complex structure moduli space

$$\partial_{t^i} H^{D,0} \subset H^{D,0} \oplus H^{(D-1,1)} \qquad \rightarrow \text{ moduli dependence of } \Omega$$

- Hodge star on (p,q)-forms: $\quad *\omega = i^{p-q}\,\omega \qquad \omega \in H^{p,q}$

$\rightarrow$ (p,q)-splitting determines Hodge star and Hodge norm:

$$\|\omega\|^2 = \int_{Y_D} \bar{\omega} \wedge *\omega$$

Lesson 1: Classification of boundaries

- On each co-dimension n boundary in complex structure moduli space:

  Middle cohomology $H^D(Y_D,\mathbb{C})$ admits boundary (*p*,*q*)-decomposition and decomposition into representations of $\mathrm{sl}(2,\mathbb{C})^n$

  [Schmid][Cattani,Kaplan,Schmid]

Example: Y_3 sending one parameter to boundary: $\mathrm{sl}(2,\mathbb{C})$

$$H^3(Y_3,\mathbb{C}) = H^{3,0}_\infty \oplus H^{2,1}_\infty \oplus H^{2,1}_\infty \oplus H^{0,3}_\infty$$

$$H^{q,3-q}_\infty = \mathrm{span}_{\mathbb{C}}\Big\{|d,l\rangle,\ d=0,\dots,3;\ l=-3,\dots,3\Big\}$$

see, e.g., [TG '20] for details

sl(2)-highest-spin sl(2)-spin

In general: multiple sl(2)-spins

Lesson 1: Classification of boundaries

- On each co-dimension n boundary in complex structure moduli space:

  Middle cohomology $H^D(Y_D,\mathbb{C})$ admits boundary (p,q)-decomposition and decomposition into representations of $\mathrm{sl}(2,\mathbb{C})^n$

  [Schmid][Cattani,Kaplan,Schmid]

- Classification of boundaries using sl(2)-representations and positivity

works for all Kähler manifolds

ensure: $\int \alpha \wedge *_\infty \alpha > 0$

Calabi-Yau threefold examples

- All cases: $h^{2,1} = 1$ ('mild' degeneration I_0)

$$\begin{aligned} \mathrm{I}_1 &: \quad \text{conifold point}\,, \\ \mathrm{II}_0 &: \quad \text{Tyurin degeneration}\,, \\ \mathrm{IV}_1 &: \quad \text{large complex structure point} \end{aligned}$$

- All cases: $h^{2,1} = 2$ ('mild' degeneration+one-modulus)

[Kerr,Pearlstein,Robles]
[TG,Li]

$$\begin{aligned} \mathrm{I}_2 \text{ class} &: \quad \langle \mathrm{I}_1|\mathrm{I}_2|\mathrm{I}_1\rangle\,,\ \langle \mathrm{I}_2|\mathrm{I}_2|\mathrm{I}_1\rangle\,,\ \langle \mathrm{I}_2|\mathrm{I}_2|\mathrm{I}_2\rangle\,, \\ \text{Coni-LCS class} &: \quad \langle \mathrm{I}_1|\mathrm{IV}_2|\mathrm{IV}_1\rangle\,,\ \langle \mathrm{I}_1|\mathrm{IV}_2|\mathrm{IV}_2\rangle\,, \\ \mathrm{II}_1 \text{ class} &: \quad \langle \mathrm{II}_0|\mathrm{II}_1|\mathrm{I}_1\rangle\,,\ \langle \mathrm{II}_1|\mathrm{II}_1|\mathrm{I}_1\rangle\,,\ \langle \mathrm{II}_0|\mathrm{II}_1|\mathrm{II}_1\rangle\,,\ \langle \mathrm{II}_1|\mathrm{II}_1|\mathrm{II}_1\rangle\,, \\ \text{LCS class} &: \quad \langle \mathrm{II}_1|\mathrm{IV}_2|\mathrm{III}_0\rangle\,,\ \langle \mathrm{II}_1|\mathrm{IV}_2|\mathrm{IV}_2\rangle\,,\ \langle \mathrm{III}_0|\mathrm{IV}_2|\mathrm{III}_0\rangle\,,\ \langle \mathrm{III}_0|\mathrm{IV}_2|\mathrm{IV}_1\rangle\,, \\ &\quad \langle \mathrm{III}_0|\mathrm{IV}_2|\mathrm{IV}_2\rangle\,,\ \langle \mathrm{IV}_1|\mathrm{IV}_2|\mathrm{IV}_2\rangle\,,\ \langle \mathrm{IV}_2|\mathrm{IV}_2|\mathrm{IV}_2\rangle\,, \end{aligned}$$

[Alvarez-Garcia,Blumenhagen et al. '20]
[Demirtas et al. '20]

Seiberg-Witten theory

LCS: in [Kreuzer,Skarke]
(after mirror symmetry)

Lesson 2: Approximating the Hodge star

- In any asymptotic regime of the complex structure moduli space:
  Hodge star can be approximated using $\mathrm{sl}(2,\mathbb{C})^n$ - spins

Example: Y_D boundary at $t_i \equiv x_i + iy_i \to i\infty$, $\alpha \in H^D(Y_D,\mathbb{C})$

regime: $y_1 \gg y_2 \gg \ldots \gg y_n$ ('strict asymptotic')

$$\|\alpha\|^2 \sim \sum_{l_1,\ldots,l_n} (y^1)^{l_1-n}(y^2)^{l_2-l_1}\ldots(y^n)^{l_{n-1}-l_n}\|\left(e^{-x^i N_i}\alpha\right)_{l_1\ldots l_n}\|_\infty$$

sl(2)-spins

$\rightarrow$ **leading, most 'crude' approximation, but easy to handle**

Lesson 3: Reconstructing the moduli space

- In any asymptotic regime of the complex structure moduli space: Asymptotic moduli space geometry (Hodge star and periods) can be reconstructed from data associated to boundaries

boundary data:

- $\mathrm{sl}(2,\mathbb{C})^n$- data, boundary (p,q)-decomposition
  $\rightarrow$ classified + simple normal forms
- **Extra:** chain of phase operators $\delta_n, \ldots, \delta_1$
  $\rightarrow$ used: most general Ansatz compatible with other boundary data

- Holographic perspective for *n*=1: $\mathrm{sl}(2,\mathbb{C})$ boundary data can be used to reconstruct bulk Hodge star in a near boundary expansion

  $\rightarrow$ WZW model on the moduli space

[TG][TG,Monnee,vd Heisteeg]
+ to appear

Lesson 3: Reconstructing the moduli space

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  Asymptotic moduli space geometry (Hodge star and periods) can be reconstructed from data associated to boundaries

boundary data:

- $\mathrm{sl}(2,\mathbb{C})^n$- data, boundary (p,q)-decomposition
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- **Extra:** chain of phase operators $\delta_n, \ldots, \delta_1$
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- reconstruction of CY_3 periods - [Bastian,TG,vd Heisteeg]
  combine [Cattani,Kaplan,Schmid],[Fernandez,Cattani],[Brosnan,Pearlstein,Robles]

Lesson 3: Reconstructing the moduli space

- In any asymptotic regime of the complex structure moduli space:
  Asymptotic moduli space geometry (Hodge star and periods) can be reconstructed from data associated to boundaries

- reconstruct periods with polynomial and essential exponential 'instanton' corrections

essential instantons
at almost all boundaries in
accordance with conjecture
[Palti,Weigand,Vafa]

$$\Pi(t) = \Pi_{\mathrm{pol}}(t) + \Pi_{\mathrm{ess}}(t) = e^{t^i N_i} a_0 + \mathcal{O}(e^{2\pi i t})$$

cannot be dropped: required e.g. to ensure non-degeneracy of moduli metric

Modeling one-parameter periods

- Results for one parameter example: $z = e^{2\pi i t}$

I_1 $\rightarrow$ finite distance
$$\Pi = \begin{pmatrix} 1 + \frac{a^2}{8\pi} z^2 \\ az \\ i - \frac{ia^2}{8\pi} z^2 \\ \frac{ia}{2\pi} z \log[z] \end{pmatrix}$$

II_0 $\rightarrow$ infinite distance
$$\Pi = \begin{pmatrix} 1 + az \\ i - iaz \\ \frac{\log[z]}{2\pi i} + \frac{az}{2\pi i}(\log[z] - 2) \\ \frac{\log[z]}{2\pi} - \frac{az}{2\pi}(\log[z] - 2) \end{pmatrix}$$

Modeling two-parameter periods

- Results for two-cubes: II_1 class : $\langle \mathrm{II}_0|\mathrm{II}_1|\mathrm{I}_1\rangle\,,\ \langle \mathrm{II}_1|\mathrm{II}_1|\mathrm{I}_1\rangle\,,\ \langle \mathrm{II}_0|\mathrm{II}_1|\mathrm{II}_1\rangle\,,\ \langle \mathrm{II}_1|\mathrm{II}_1|\mathrm{II}_1\rangle\,,$

$$\mathbf{\Pi} = \begin{pmatrix} 1+cz_1+\frac{1}{4}\big(a^2n_1z_2^2+2abz_1z_2\frac{1-n_1n_2^2}{1-n_2}+b^2n_2z_1^2\big) \\ i-icz_1-\frac{i}{4}\big(a^2n_1z_2^2+2abz_1z_2\frac{1-n_1n_2^2}{1-n_2}+b^2n_2z_1^2\big) \\ bn_2z_1+az_2 \\ \frac{\log[z_1]+n_2\log[z_2]}{2\pi i}\Big(1+cz_1+\frac{1}{4}\big(a^2n_1z_2^2+2abz_1z_2\frac{1-n_1n_2^2}{1-n_2}+b^2n_2z_1^2\big)\Big)+f(z) \\ \frac{\log[z_1]+n_2\log[z_2]}{2\pi}\Big(1-cz_1-\frac{1}{4}\big(a^2n_1z_2^2+2abz_1z_2\frac{1-n_1n_2^2}{1-n_2}+b^2n_2z_1^2\big)\Big)-if(z) \\ i(bn_2z_1+az_2)\frac{n_1\log[z_1]+\log[z_2]}{2\pi}-\frac{1-n_1n_2}{2\pi i}(bz_1-az_2) \end{pmatrix}$$

$$f(z)=\frac{ic\,z_1}{\pi}+\frac{1-n_1n_2}{8\pi i}\left(a^2z_2^2+2ab\,z_1z_2\frac{1+n_2^2}{1-n_2}+b^2n_2z_1^2\right)$$

parameters	$\langle \mathrm{II}_0\vert\mathrm{II}_1\vert\mathrm{I}_1\rangle$	$\langle \mathrm{II}_1\vert\mathrm{II}_1\vert\mathrm{I}_1\rangle$	$\langle \mathrm{II}_0\vert\mathrm{II}_1\vert\mathrm{II}_1\rangle$	$\langle \mathrm{II}_1\vert\mathrm{II}_1\vert\mathrm{II}_1\rangle$
log-mon.	$n_1=n_2=0$	$n_1\in\mathbb{Q}_{>0},\ n_2=0$	$n_1=0, n_2\in\mathbb{Q}_{>0}$	$n_1,n_2\in\mathbb{Q}_{>0},\ n_1n_2\neq 1$
inst. coeff.	$a,b\in\mathbb{R}_{\neq 0}, c\in\mathbb{C}\ \Vert\ a,c\in\mathbb{R}_{\neq 0}, b\in\mathbb{C}$		$a,b\in\mathbb{R}-\{0\},\ c=0$	

Lesson 4: Systematic moduli stabilization

- New procedure to find flux vacua: [TG,Plauschinn,vd Heisteeg]

Example: (imaginary) self-dual flux $*G_3 = iG_3$

sl(2)-approximation → asymptotic periods w. essential expon. corrections (nilpotent orbit) → full periods with all exponential corrections (hard to find in practice)

polynomial, vacua easy to find

[TG,Li,Valenzuela]

'refinement' of of sl(2)-vacua

exact vacua

Note: flat directions in sl(2)-approx. might be stabilized in successive steps

e.g. linear scenario [Palti,Tasinato,Ward] [Marchesano,Prieto,Wiesner]

Lesson 4: Systematic moduli stabilization

- New procedure to find flux vacua: [TG,Plauschinn,vd Heisteeg]

sl(2)-approximation	→	asymptotic periods w. essential expon. corrections (nilpotent orbit)	→	full periods with all exponential corrections

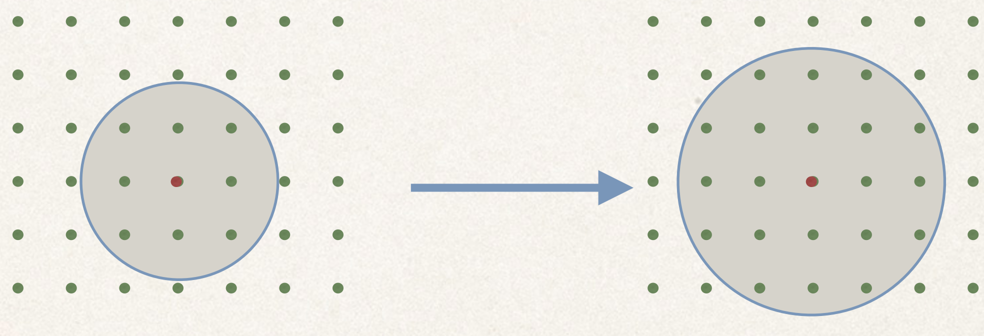
- Remarks:
  - algorithmic approach to stabilize moduli (also away from large complex structure) → abstract results, favorable numeric
  - naturally implement hierarchies (e.g. moduli masses, small W_0) linked to classification of boundaries [Bastian,TG,vd Heisteeg]
  - possible for large number of moduli + fluxes [Graña,TG,Herraez, Plauschinn, vd Heisteeg] → in progress
    → tadpole conjecture? [Bena,Blåbäck,Graña,Lüst]

Finiteness of self-dual flux vacua

Self-dual flux vacua

- Recall: integral $G_4 \in H^4(Y_4, \mathbb{Z})$, self-dual $G_4 = *G_4$, tadpole $\int_{Y_4} G_4 \wedge G_4 < K$
- Important note: fix Y_4 in this discussion (finitely many CY)
- Evidence for finiteness of flux choices:
  [Ashoke,Douglas],[Douglas,Shiffman,Zelditch],[Douglas,Lu] using vacuum density
- This is a **very** hard math problem!

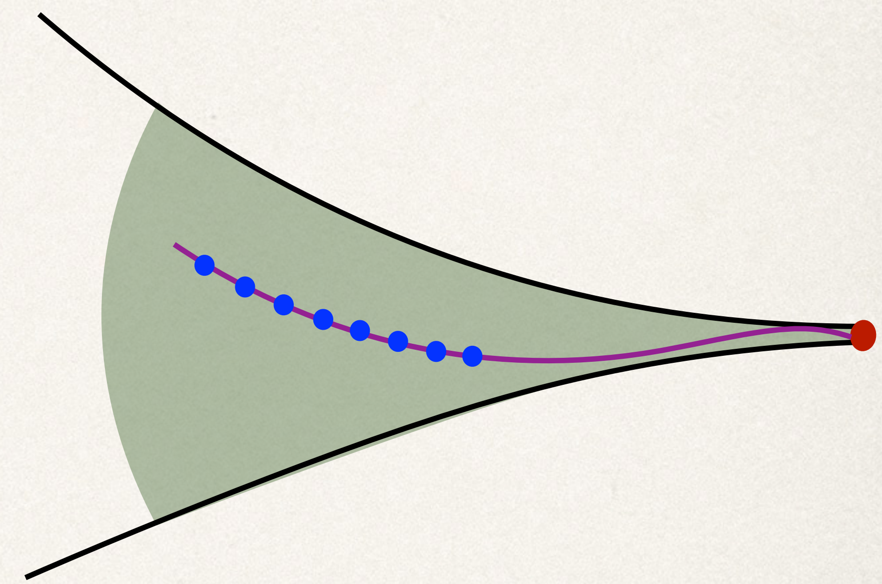
$$\int G_4 \wedge G_4 = \int G_4 \wedge *G_4 < K$$



$\rightarrow$ key challenge: cut off infinite tails at asymptotic regimes of $\mathcal{M}$ when Hodge star degenerates $\Rightarrow$ control near boundary regions

Direct approach to finiteness proof

- first idea: show that on every path to every boundary only finitely many vacua can arise



can be done for co-dimension one boundary
i.e. one coordinate send to limit
$\Rightarrow$ use full power of sl(2) asymptotic techniques [Schnell][TG]

- general story is orders of magnitude more complicated

- susy (2,2)-fluxes (W=0) $\rightarrow$ Hodge classes $H^4(Y,\mathbb{Z}) \cap H^{2,2}(Y,\mathbb{C})$
  apply theorem by [Cattani,Deligne,Kaplan] (paper is strongest evidence for Hodge conjecture)

use sl(2)-techniques to control every path to every boundary
however: they use holomorphicity ('Susy vaccum')

General proof - What is behind this?

- self-dual fluxes: more general questions [Bakker,TG,Schnell,Tsimerman]

→ use recent breakthrough by [Bakker,Klingler,Tsimerman]

Hodge theory —— tame topology (build-in finiteness)

→ period map: $\Phi: \quad \mathcal{M} \longrightarrow \Gamma\backslash G/K$

[BKT] show:

(1) arithmetic quotients have certain 'tame topology'

(2) period map is special map that is 'tame' (e.g. near boundaries)

shown by using sl(2)-techniques

(3) alternative proof to the theorem of [Cattani,Deligne,Kaplan]

General proof - What is behind this?

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  Hodge theory —— tame topology (build-in finiteness)

  → period map: $\Phi: \quad \mathcal{M} \longrightarrow \Gamma\backslash G/K$

  Very roughly:

  (1) arithmetic quotients can be covered by 'finitely many patches'

  (2) period map maps 'finitely many sets to finitely many sets'

General proof - What is behind this?

- self-dual fluxes: more general questions [Bakker,TG,Schnell,Tsimerman]

  $\rightarrow$ use recent breakthrough by [Bakker,Klingler,Tsimerman]

  Hodge theory — tame topology (build-in finiteness)

- What about the self-dual vacua?

vacuum condition: $V(t_*, G_4) = C\| * G_4 - G_4\|^2 \overset{!}{=} 0$ finitely many zero-sets?

$$V\,:\;\; \mathcal{M} \times (\text{flux lattice}) \;\to\; \mathbb{R}$$

infinite discrete set

$\rightarrow$ we show that 'tameness' is obtained when imposing tadpole bound

General proof - What is behind this?

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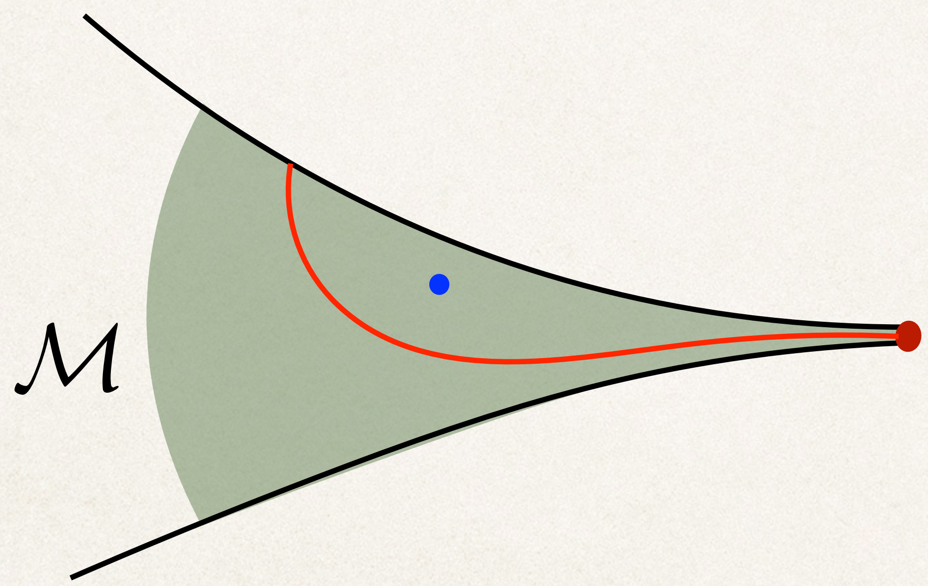
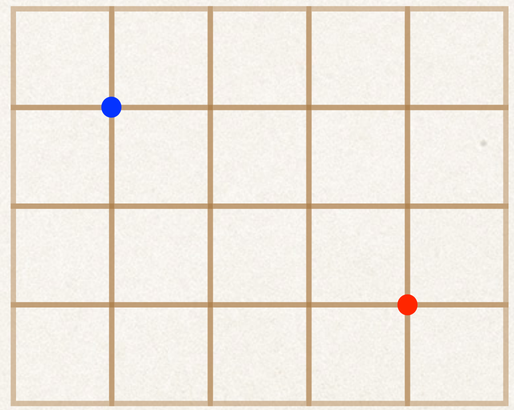
Hodge theory — tame topology (build-in finiteness)





$\Rightarrow$ pairs form finitely many subsets of $\mathcal{M} \times$ (flux lattice)

Minimal structure for the landscape

A mathematical structure with finiteness

- develop a mathematical framework that respects finiteness:
  - remove pathologies that can occur in ‘ordinary topology’
  - Grothendieck’s dream of a tame topology [Esquisse d’un programme]

- theory of o-minimal structures gives a generalization of algebraic geometry and provides a tame topology

intro book [van den Dries]

- recall: $V(t_*, G_4) \overset{!}{=} 0$ with tadpole to have finitely many solutions V has to be special

many functions do not work:

discussed by [Acharya,Douglas]

$$V(\phi) = \sin(\phi^{-1}) \qquad V(\phi) = \phi^8 \sin(\phi^{-1})$$

$\rightarrow$ infinitely many zeros near $\phi = 0$ ↯

A mathematical structure with finiteness

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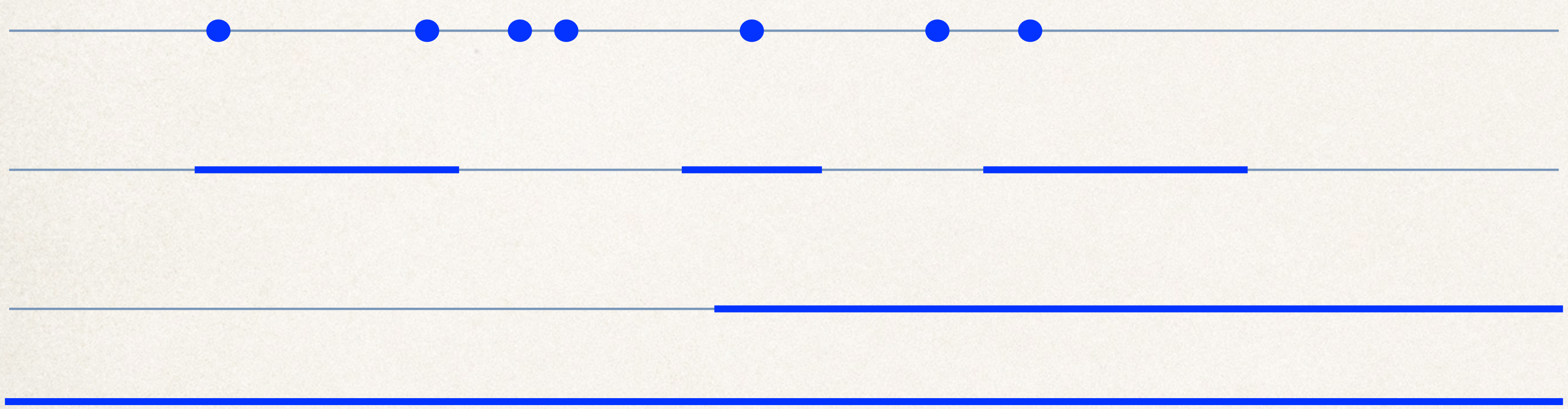
- theory of o-minimal structures gives a generalization of algebraic geometry and provides a tame topology

intro book [van den Dries]

- theory of o-minimal structures gives a precise answer to what this special property for the flux scalar potential is

Finite subsets on the real line

- simplest situation: finite subsets of $\mathbb{R}$



finitely many point

finitely many intervals

open intervals
(infinitely long)

- much harder to extend this to $\mathbb{R}^n$. Some intuitive requirements:
  - projections to $\mathbb{R}$ should give the above sets
  - finite unions, intersections, and products should be allowed

O-minimal structure as a tame topology

- Definition: An o-minimal structure $\mathcal{S}$ of sets $\{S_n\}_{n=0,1,..}$:
  - S_n are subsets of $\mathbb{R}^n$
  - S_n is closed under finite intersections, finite unions and complements
  - collection $\{S_n\}$ closed under finite Cartesian products & coordinate projections
  - S_n contain zero set of every polynomial in *n* variables is in $\mathbb{R}^n$
  - S_1 is the finite union of intervals and points

O-minimal structure as a tame topology

Now there is a clear definition of 'tame' functions:

- $\mathcal{S}$-definable functions among the S_n's are those whose graph is part of the o-minimal structure

Remarkable consequence: definable $f : \mathbb{R} \to \mathbb{R}$

split $\mathbb{R}$ into finite number of intervals f is either constant, or monotonic and continuous in each open interval

Another consequence: definable + holomorphic $f : \mathbb{C} \to \mathbb{C}$ is algebraic

Examples of o-minimal structures

- there is no unique choice of o-minimal structure on $\mathbb{R}^n$:
  - examples are obtained by stating which functions are allowed to generate some of the sets

- Some remarkable examples:
  - structure generated by graphs of real polynomials: $\mathbb{R}_{\mathrm{alg}}$
  - $\mathbb{R}_{\mathrm{alg}}$ plus graphs of restricted real analytic functions: $\mathbb{R}_{\mathrm{an}}$
  - $\mathbb{R}_{\mathrm{alg}}$ plus graph of exponential function: $\mathbb{R}_{\mathrm{exp}}$ [Wilikie '96]
  - combination of $\mathbb{R}_{\mathrm{an}}$ and $\mathbb{R}_{\mathrm{exp}}$: $\boxed{\mathbb{R}_{\mathrm{an,exp}}}$ [vd Dries, Miller '94]

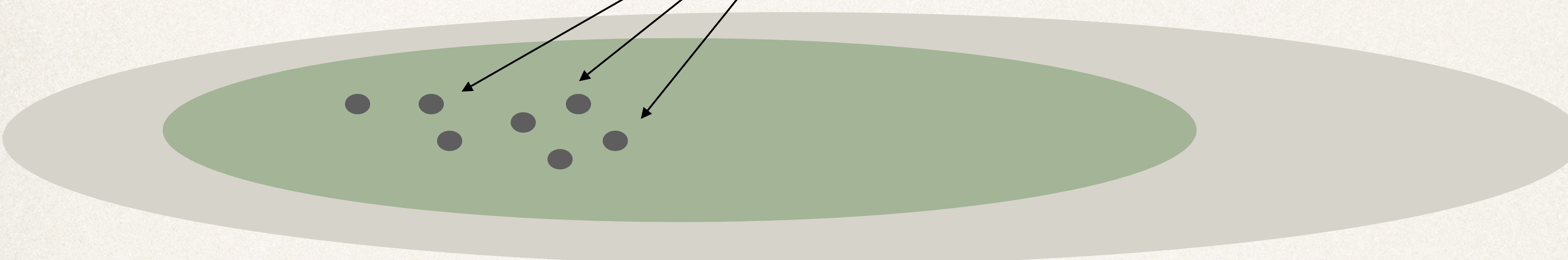
- Note: period map is a $\mathbb{R}_{\mathrm{an,exp}}$ - definable function [BKT '18]

A conjecture

Set of effective theories
arising from string theory

collect vectors:

$$\begin{pmatrix}\text{moduli space}\\ \text{rank gauge group}\\ \text{matter spectrum}\end{pmatrix}$$



Conjecture : The string landscape is definable in an $\mathbb{R}_{\mathrm{an,exp}}$ o-minimal structure and all coupling functions in the effective theory are definable.

Conclusions

- Uncover the structure of complex structure moduli space using asymptotic Hodge theory

  - sl(2) structure allows for a classification of boundaries
  - reconstruction of the near boundary periods

  $\Rightarrow$ no need to scan through CY-examples
  $\Rightarrow$ ready to make general proofs of recent conjectures

- Highly non-trivial finiteness result for the number of self-dual flux vacua

- Suggested to use o-minimal structure to describe the string theory vacuum landscape

  $\Rightarrow$ build-in finiteness properties
  $\Rightarrow$ general enough for also non-supersymmetric situations

Thanks for listening!

An application: vacua with small vacuum W

- construction of vacua with exponentially small vacuum superpotential near LCS + conifold
  [Demirtas,Kim,McAllister,Moritz] (LCS '19) (coni '20)
  [Alvarez-Garcia, Blumenhagen,Brinkmann,Schlechter][Honma,Otsuka][Broekel etal]

- constructions can be conceptually understood using asymptotic Hodge theory near every boundary + new vacua using essential instantons
  [Bastian,TG,vd Heisteeg]

recall: $\Pi(t) = \Pi_{\rm pol}(t) + \Pi_{\rm ess}(t)$

$$K_{\rm cs} = -\log(\mathcal{K}_{\rm pol} + \mathcal{K}_{\rm inst}) \qquad W = W_{\rm pol} + W_{\rm inst}$$

vacua?: $W_{\rm pol}\big|_* = 0 \qquad \partial_a W_{\rm pol}\big|_* = 0 \qquad \partial_\tau W_{\rm pol}\big|_* = 0$

Inconsistent near many boundaries (only approximation near LCS): restriction to polynomial behavior does not exchange with derivative

Small W from essential instantons

- **Example:** boundaries with "Type II points" (Seiberg-Witten points)

$$K_{\text{cs}} = -\log(y_1 + n y_2 + \mathcal{K}_{\text{inst}})$$

needs essential instantons for non-degeneracy of metric constructed

$\rightarrow$ metric has exponentially small eigenvalue

metric-essential instantons

- flux superpotential has polynomial and exponentially suppressed terms

solve: $W_{\text{pol}} = 0 \quad D_\tau W = 0 \quad D_{t^i} W = 0$ become polynomial

- moduli stabilized by metric-essential instantons in *W*: polynomial masses

$\rightarrow$ leading polynomial scalar potential after cancellation of exponentials

$\rightarrow$ matches the result for the Hodge norm in asymptotic Hodge theory