Multi trace correlator in the SYK model and their gravitational interpretation

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Based on work in with N. Brukner, V. Narovlansky and A. Raz, arXiv:2104.03336;

Gravity Perspective:

The SYK model is dual to 2D theory which is not really understood. It's much more than JT gravity. We would like to see new effects in it that go beyond the latter (necessary for factorization).

Since higher dimensional BH sometimes flow to $AdS_2 * M$, then maybe all BH include similar unfamiliar stuff.

We will discuss here new effects that have to do with multiboundary universes - disconnected boundaries "talk" to each other much more strongly then wormholes in JT gravity. But even in a single boundary space, they imply new light fields and boundary couplings.

Maybe: in higher dimensions this means new degrees of freedom in the near horizon (which the outside observer does not couple to), and new direct interactions between the stretched horizons on both sides of the BH.

To show this: we will compute the dominant multi-trace correlators in the full SYK model (larger then the topological recursion of ordinary RMT, non-asymtotic time scales). This is rigorous.

Then ask what do we need to do in the gravitational dual to reproduce these computation. This is tentative.

Quantum chaos/Math perspective:

The SYK Hamitonian is a k-local Hamiltonian. Matrix elements in the Hamiltonian are sparse and strongly correlated (very different from β-ensembles).

This brings about fluctuations of "global modes" in which remote parts of the spectrum are correlated.

Any physical system will be dominated by these before it reaches the ramp+plateau/Bohigas-Giannoni-Schmit regime. Little is known about the systematics of these modes.

Outline

- Setup and framework,
- Summary of results,
- $\langle tr(H^{k_1})tr(H^{k_2})\rangle$ (leading order - field theory and combinatorics),
- Arbitrary number of traces (leading order),
- New fluctuation parameters and new fields in the dual,
- Summary

Setup: Multi-trace correlators and mult-boundary spacetimes

The Sachdev-Ye-Kitaev (SYK) model

Conventions: N Majorana fermions ψ_i, $i = 1, \cdots, N.$ satisfying

$$\{\psi_i, \psi_j\} = 2\delta_{ij}$$

with random (disordered) all-to-all p-local interaction

$$H = i^{p/2} \sum_{1 \le i_1 < \cdots < i_p \le N} J_{i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p}$$

J are random gaussian couplings, $\langle J^2_{i_1 \cdots i_p} \rangle_J = \binom{N}{p}^{-1} \mathcal{J}^2$. The usual large N limit is $N \to \infty, \ p \ fixed.$

After ensemble average[1], it was shown to be almost conformal have the maximal chaos exponent (Sachdev, Ye; Georges, Pacollet, Sachdev; Kitaev; Polchinksi, Rosenhaus; Maldacena, Stanford), and it is dual to AdS_2 (and 2D JT gravity at low energues) (Sachdev; Jensen; Maldacena, Stanford; Almheiri, Polchinksi;...).

[1] Although self averaging

We are interested in the limit $N \to \infty$. We can either take p $fixed$ or $p = \lambda\sqrt{N}$, λ $fixed$ - the double scaled limit, and the basic formulas still apply.

The observables that we will be interested in are the leading order (in $1/N^{\kappa}$) contribution to the connected part of

$$M(k_1, k_2, \ldots, k_n) \equiv \langle \mathrm{tr}\left[H^{k_1}\right] \mathrm{tr}\left[H^{k_2}\right] \ldots \mathrm{tr}\left[H^{k_n}\right] \rangle_J .$$

where $\langle\rangle_J$ denotes the ensemble average. I.e.,

$$M_c(k_1) = M(k_1), \quad M_c(k_1, k_2) = M(k_1, k_2) - M_c(k_1)M_c(k_2),$$

$$M_c(k_1, k_2, \ldots, k_n) = M(k_1, k_2, \ldots, k_n) \; - \sum_{\substack{\text{partitions } p \\ \text{of } \{1,\ldots,n\}}} \; \prod_{\{i_1,\ldots i_m\}\in p} M_c(k_{i_1}, \ldots, k_{i_m}).$$

$$\Big(M_c(k_1, \ldots, k_n) \to Z_c(\beta_1, \ldots, \beta_n) \Big)$$

We will write $M_c(k_1, ..k_n)$ as

- Some "transformations" of each $M(k)$ depending on parameters (which will have a gravity interpretation).
- A measure on these parameters which links the different $M(k)$ (like α-parameters but a much larger contribution).

Example, at leading order

$$M_c(k_1, k_2)|_{\text{leading}} = \frac{k_1 k_2}{2} \binom{N}{p}^{-1} M(k_1) M(k_2).$$

The expansion parameter is $\binom{N}{p}^{-1}$. Each $M(k)$ receives corrections (in $1/N$) which are much larger.

From this it follows

$$\rho_c(E, E') = \frac{1}{2} \binom{N}{p}^{-1} \frac{d}{dE} \left(E \rho_0(E)\right) \frac{d}{dE'} \left(E' \rho_0(E')\right),$$

We will write it as

$$\rho(E, h) = \exp\left(\binom{N}{p}^{-1/2} h \partial_E E \right) \rho_0(E)$$

$$\rho_c(E_1, E_2) = \int dh\, e^{-h^2/2} \rho(E_1, h) \rho(E_2, h) \ - \ \text{disconnected},$$

This is a "global fluctuation mode" which acts throughout the entire spectrum. In fact

$$\rho_c(E_1, \cdots, E_n) = \int dh\, P(h) \rho(E_1, h) \cdots \rho(E_n, h) \ - \ \text{disconnected},$$

for an appropriate $P(h)$, captures the leading order connected multi-trace correlator.

In fact

$$\rho_c(E_1, \cdots, E_n) = \int d\vec{h}\, P(\vec{h})\rho(E_1, \vec{h}) \cdots \rho(E_n, \vec{h}) \; - \; \text{disconnected},$$

where

$$\rho(E) = \rho_0(E) + N^{-\kappa_1}\delta\rho_1(E, h_1) + N^{-\kappa_2}\delta\rho_2(E, h_1, h_2) + ... \qquad (1)$$

We can in fact extend this to a whole series.

Each h_i will be associated with some new light fields that we need to introduce into the bulk, and some new random couplings (α-parameters a la Coleman).

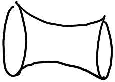
$$h_2 = \sum_I J_I^2; \quad h_3 = \sum_{I_1 \oplus I_2 \oplus I_3 = 0} J_{I_1} J_{I_2} J_{I_3} \quad .... \; other\ SO(N)\ invariants \qquad (2)$$

The list of SO(N) invariants is the same as specifying J's (up to SO(N) tranformations). We will call these "fluctuation parameters".

Warning: lack of universality.

Early times and the issue of a single realization:

Two recent developments:



$\frac{1}{2^N} \ll \binom{N}{p}^{-1}$

- Very late time spectral form factor is given by connected multi-boundary spacetimes with non-trivial topology Cotler et al; Saad, Shenker, Stanford; Penington..; maldacena..; Yang..; ..Yau..; ...
- The Page curve can be understood by via the islands construction Almheiri..; Englehardt..; Maldacena..; Mahajan..; Marolf..; Penington...; Zhao..

But there are still persistent questions

- The dual of a single realization; or are there wormholes in higher dimensions for specific dualities?
- How exactly does information come out of the black hole? Does (how) the effective action break down?
- .... microstates.. firewall.... singularities...infalling observer....

A slightly different question

Suppose we gradually (in some parameter) provide more and more information about the J's. How do we take this into account in the bulk? (Even after we provide some information, there is still an ensemble average).

1. The information about J_I sits in the fluctuation parameters $h_2 = \sum_I J_I^2; \quad h_3 = \sum_{I_1 \oplus I_2, \quad I_3=0} J_{I_1} J_{I_2} J_{I_3}$ + other SO(N) invariants. They will correspond to the α parameters that "correlate between universes". Their effect is smaller and smaller in some expansion parameter (but larger then wormholes).

2. The h_2 is responsible for dilation of the spectrum discussed above. $H \to hH$ with $P(h)$. h is the fluctuating coupling in the dual description. We can transfer it to JT or any other effective Hamiltonian/Lagrangian that we use.

3. h_3 and above: They multiply (or appear) in specific couplings in the gravity. To realize their effect in gravity we need to add additional light fields in a specific arrangement. We will call these "fluctuation fields" (a single fluctuation parameter might require several fields + discrete fluxes).

Comments

Caveats:

1. By bulk we mean the effective Hamiltonian after doing the remaining ensemble average.

2. Very tentative.

3. Time frames - much before the ramp and slope. The fluctuations parameters affect the details of the ramp and slope. For example the h_2 fluctuation parameter shifts them (just a bit) in time.

$$M(k_1, k_2) = Tr(H^{k_1})Tr(H^{k_2})$$

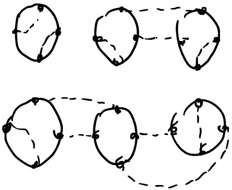
Basic diagramtics 1

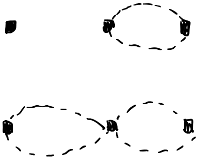
$$M_c(k_1,k_2) = \sum_{I_1,\cdots,I_k} \langle tr\left[J_{I_1}\Psi_{I_1}\cdots J_{I_{k_1}}\Psi_{I_{k_1}}\right] \mathrm{tr}\left[J_{I_{k_1+1}}\Psi_{I_{k_1+1}}\cdots J_{I_k}\Psi_{I_k}\right]\rangle_{J,c} \tag{3}$$

where $k = k_1 + k_2$. In step 1, we contract pairs of J's since they are Gaussian random variables.

To introduce some notation, let's go back to an arbitrary number of traces. We encode the contractions in the following diagrams.

We can simplify the notation by suppressing internal contractions and just keeping track of the number of contraction between the traces.

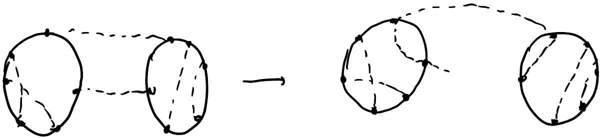




Going back to the leading order for two traces. The 2nd class of diagrams is convenient because the fermions also impose that the index sets are the same, which means that

$$\begin{aligned} &\binom{N}{p}^{-k/2} \sum_{I,J,K,\cdots} \mathrm{tr}\left(\Psi_I \Psi_J \cdots \Psi_I \cdots\right) \mathrm{tr}\left(\Psi_I \Psi_K \cdots \Psi_I \cdots\right) \\ &= \binom{N}{p}^{-k/2-1} \sum_{I,I',J,K,\cdots} \mathrm{tr}\left(\Psi_I \Psi_J \cdots \Psi_I \cdots\right) \mathrm{tr}\left(\Psi_{I'} \Psi_K \cdots \Psi_{I'} \cdots\right). \end{aligned} \tag{4}$$

This means that each trace just gives rise to $M(k)$, or diagrammatically



or

$$M_c(k_1, k_2)|_{\text{2 pairings}} = \frac{k_1 k_2}{2} \binom{N}{p}^{-1} M(k_1) M(k_2). \tag{5}$$

$$M_c(k_1,k_2)|_{2\text{ pairings}} = \frac{k_1 k_2}{2}\binom{N}{p}^{-1} M(k_1)M(k_2). \tag{6}$$

or

$$\mathcal{Z}_c(\beta_1,\beta_2) = \frac{\epsilon}{2}\beta_1\frac{\partial\mathcal{Z}(\beta_1)}{\partial\beta_1}\beta_2\frac{\partial\mathcal{Z}(\beta_2)}{\partial\beta_2} + \text{higher order terms.} \tag{7}$$

$$\rho_c(E,E') = \frac{\epsilon}{2}\ \frac{d}{dE}\left(E\rho_0(E)\right)\ \frac{d}{dE'}\left(E'\rho_0(E')\right), \tag{8}$$

with

$$\epsilon = \binom{N}{p}^{-1} \tag{9}$$

Note that this expression is correct irrespective of the details of M (whether leading order in N or subleading). We might as well use the exact M.

Note that in the simplest case we have two J_I in each trace. So the connected part is

$$\sum_I \left(\langle J_I^4\rangle - \langle J_I^2\rangle^2\right) \tag{10}$$

so it is a fluctuation of the $h_2 = \sum_I J_I^2$

Re-emergence of the couplings, and spacetime

Re-emergence of the couplings

Since we contracted J, or the multi-fermion operator, within each trace, a better way to think about it is as interaction between chords in the different traces. If we want to take into account the fluctuation of $h_2 = \sum J_I^2$, then we can write each $M(k)$ as

$$M(k, h_2) = M_1(k, h_2 = 1)h_2^{k/2} \tag{11}$$

and the expressions above become

$$M(k_1, \ldots, k_n) = C^{-1} \int d\phi_I \; e^{-\sum_I \phi_I^2/(2\epsilon)} \prod_{i=1}^{n} \left\{ M(k_i) \left(\sum_I \phi_I^2 \right)^{k_i/2} \right\} \tag{12}$$

(and we take the connected part, and used ϕ instead of J).

We can also write this as

$$\mathcal{Z}(\beta_1, \ldots, \beta_n) = \mathcal{E}_\phi \left(\prod_{i=1}^{n} \mathcal{Z} \left(\beta_i \sqrt{\sum_I \phi_I^2} \right) \right). \tag{13}$$

We can also turn it into a single integral on h_2.

Spacetime

$$\mathcal{Z}(\beta_1, \ldots, \beta_n) = \mathcal{E}_{h_2}\left(\prod_{i=1}^{n} \mathcal{Z}\left(\beta_i h_2\right)\right). \tag{14}$$

It is now straightforward (but not done in detail) to transfer this to a GR statement since we have a GR expression for $\mathcal{Z}(\beta)$ and in particular $\mathcal{Z}(\beta h_2)$. By this one means

- The Schwarzian action,
- The full all-scale partition function at finite p in whatever approximation we want (for example take also p large),
- The double scaled transfer matrix
- The JT action (with any additional structure to go to higher energies)

Single realization: Suppose we are only told that h_2 is slightly different then it's expectation value. We still have a large ensemble average, minus one constraint. A natural guess would be that the modified GR action in this case would just be $\mathcal{Z}\left(\beta h_2\right)$

This is the simplest case. The other fluctuation parameters are not this pretty, but they provide more interesting data about the dual.....

New fluctuations parameters - field theory

Computation in Quantum mechanics

Consider $\langle M(k_1)M(k_2)\rangle_J$. Non-zero contribution because of diagrams like

The core of the computation $M_c(3,3)$

$$= 3\binom{N}{p}^{-3} \sum_{\substack{|I_1|=|I_2|=|I_3|=p/2 \\ I_i\cap I_j=\emptyset,\ i\neq j}} \Big(\mathrm{tr}(\Psi_{I_1}\Psi_{I_2}\Psi_{I_2}\Psi_{I_3}\Psi_{I_3}\Psi_{I_1})\,\mathrm{tr}(\Psi_{I_1}\Psi_{I_2}\Psi_{I_2}\Psi_{I_3}\Psi_{I_3}\Psi_{I_1})$$

$$+\,\mathrm{tr}(\Psi_{I_1}\Psi_{I_2}\Psi_{I_2}\Psi_{I_3}\Psi_{I_3}\Psi_{I_1})\,\mathrm{tr}(\Psi_{I_1}\Psi_{I_2}\Psi_{I_3}\Psi_{I_1}\Psi_{I_2}\Psi_{I_3})\Big)$$

$$= \begin{cases} 6\binom{N}{p}^{-3}\binom{N}{3p/2}\binom{3p/2}{p}\binom{p}{p/2}, & 4 \mid p, \\ 0, & 4 \nmid p. \end{cases}$$

and the structure for $M_c(k_1, k_2)$ is similar

$$M_c(k_1,k_2) = \frac{k_1 k_2}{9} M_c(3,3) W_{k_1-3} W_{k_2-3} + \text{higher order terms}$$

$$\begin{aligned} W_k &\equiv \binom{N}{p/2}^{-3} \sum_{\substack{|I_1|=|I_2|=|I_3|=p/2 \\ k_1+k_2+k_3=k}} \langle tr\left(\Psi_{I_1}\Psi_{I_2}H^{k_1}\Psi_{I_2}\Psi_{I_3}H^{k_2}\Psi_{I_3}\Psi_{I_1}H^{k_3}\right)\rangle_J = \\ &= \binom{N}{p/2}^{-3} \sum_{\substack{|I_1|=|I_2|=|I_3|=p/2 \\ k_1+k_2+k_3=k}} \langle tr\left(\Psi_{I_2}H^{k_1}\Psi_{I_2}\Psi_{I_3}H^{k_2}\Psi_{I_3}\Psi_{I_1}H^{k_3}\Psi_{I_1}\right)\rangle_J . \end{aligned}$$

The main point is that $W(k)$ is an integrated 6-point function. It is made out of 3 insertions of random operators of length $p/2$.

In some cases we know, and can plug in, this 6-pt function (note that there are no lines intersecting) - for example, in the double scaled limit, or in the low energies.

But can we draw some conclusions just from the fact that it's a correlator, without resorting to the detailed expressions?

Graphically we want to describe diagrams of the form

It is a 6-pt function of pairs of particles, each corresponding to an operator with p/2 symbols - tachyons with dimension 1/2, which we have to include in the bulk, just in order to reproduce the fluctuations of the Hamitonian.

In each trace 3 Hamiltonians were converted into a pair of such particles - H mixes with : O^2 : where O is of dimension 1/2.

The strength of the effect:

$$\langle \sum_{I_1+I_2+I_3=0} J_{I_1}^2 J_{I_2}^2 J_{I_3}^2 \rangle - C \langle \sum_{I_1+I_2+I_3=0} J_{I_1} J_{I_2} J_{I_3} \rangle^2 \tag{15}$$

which is the fluctuation of $h_3 = \sum_{I_1+I_2+I_3=0} J_{I_1} J_{I_2} J_{I_3}$ + subleading.

Effective action

Before we had

$$\mathcal{Z}_c(\beta_1, \cdots, \beta_n) = \int_{conn} dh_2 P(h_2) Z(h_2\beta_1) \cdots Z(h_2\beta_n)$$

now we want (note W replaced by Z)

$$\mathcal{Z}_c(\beta_1, \cdots, \beta_n) \sim \int_{conn} dh_3 dh_2 P(h_3, h_2) Z(\beta_1, h_2, h_3) \cdots Z(\beta_n, h_2, h_3) \tag{16}$$

The partition functions are computed using

$$H_{\text{eff}} = h_2 H + h_3 \chi : O^2 : \ , Z_{\text{eff}}(\beta, h_2, h_3) = \langle e^{-\beta H_{\text{eff}}} \rangle_\chi .$$

$P(h_3, h_2)$ can be computed. The information between the universes is carried by h_2 and h_3, and χ is a new "discrete flux" which one needs to introduce to make this work.

A slightly different question

1. Adding more and more information about J means specifying more and more SO(N) invariant combinations of the J's (algebraic structure associated with the operators used). Examples incluse

$$h_2 = \sum_I J_I^2; \quad h_3 = \sum_{I_1 \oplus I_2 \oplus I_3 = 0} J_{I_1} J_{I_2} J_{I_3} \quad .... \ other \ SO(N) \ invariants \tag{17}$$

which proliferate at higher invariants.

2. These fluctuation parameters correlate between universes, with a smaller and smaller effect.

3. They multiply (or appear) in specific couplings in the gravitational description. To realize them in gravity we need to add additional light fields (fluctuation fields) in a specific arrangement (a single fluctuation parameter may require several fields + discrete fluxes).

4. In particular we get families of low conformal dimension fields, with specific (multi-trace?) couplings in order to generate the correct correlations.

Back to wormholes and black hole evaporation

1. Outside observer: consider $AdS_2 * M$ as a low energy limit. The outside observer is equipped with a small set of random operator probes. The near horizon is populated with a very large set of light fields, but it cannot couple to them.

2. If we have a specific J - what do all these fields do, for example in the wormhole background? Do they destabilize it? (we know nothing about what they do in the bulk).

3. For example: do these fluctuation fields change the plateau? Most likely yes - for example h_2 rescales energies and slightly shifts the time scale of the plateau.

Back to wormholes and black hole evaporation

- Multitrace (multi-universe) correlators in SYK are much larger then in RMT and wormholes. They have to do with global fluctuations of the spectrum. Under reasonable combinatorial control.
- If we want to interpret them in gravity, we need to introduce an infinite set of lighter and lighter fields.
- The coupling of these fields carry the information on the details of the realization.

Thank you