On Colour-Kinematics Duality and Double Copy

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Rencontres théoriciennes
Institut Henri Poincaré, 21 October 2021

Based on joint work 2007.13803, 2102.11390 and 2108.03030 with Branislav Jurčo, Hyungrok Kim, Tommaso Macrelli, Christian Saemann and Martin Wolf

Gravity and gauge theory

- Gravity as a gauge theory:
  - Gauge theory of Lorentz, (super) Poincaré or de Sitter symmetries
    [Utiyama '56; Kibble '61; MacDowell-Mansouri '77; Chamseddine-West '77; Stelle-West 79]
  - Holographic principle - AdS/CFT correspondence
    ['t Hooft '93; Susskind '94; Maldacena '97]

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  - Holographic principle - AdS/CFT correspondence ['t Hooft '93; Susskind '94; Maldacena '97]
- Here, we appeal to a third and (superficially) independent perspective:

$$\text{Gravity} = \text{Gauge} \times \text{Gauge}$$

- The theme of gravity as the "square" of Yang-Mills has appeared in a variety of guises going back to the KLT relations of string theory [Kawai-Lewellen-Tye '85] Cf. Field theory [Feynman–Morinigo–Wagner; Papini '65]
- Bern-Carrasco-Johansson colour-kinematic (CK) duality and double-copy of (super) Yang-Mills (plus matter) scattering amplitudes [Bern-Carrasco-Johansson '08, '10; Bern-Dennen-Huang-Kiermaier '10]

Gravity = Gauge × Gauge

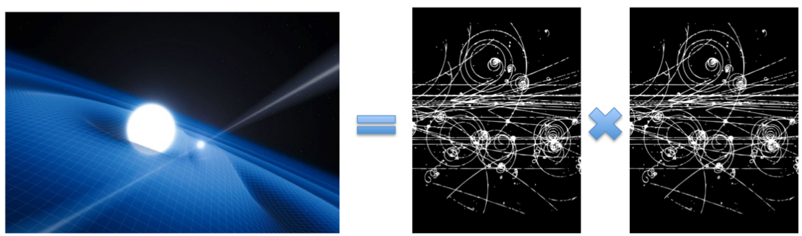
Longstanding open questions:

- Does CK duality, in some form, hold to all orders?
- Does the double copy hold: is Einstein really the square of Yang–Mills?

Gravity = Gauge × Gauge

Off-shell field theory approach:

- CK duality is property of the Yang–Mills Batalin–Vilkovisky action, up to counter terms [BJKMSW '21]
- Natural, but non-conventional notion of CK duality: counterterms required for unitarity break it
- Perturbative quantum Einstein–Hilbert gravity coupled to a Kalb–Ramond 2-form and dilaton *is* the square Yang–Mills theory [BJKMSW '20, '21]



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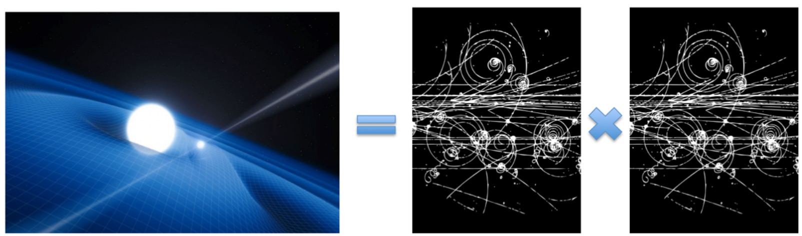


- Natural notion of CK duality $\leftrightarrow$ $\mathrm{BV}^{\square}_{\infty}$-algebra
- BV quantised Yang-Mills $\to$ L_∞-algebra that factorises:

Bi-adjoint ϕ^3 theory		YM theory		$\mathcal{N}=0$ supergravity
$\mathfrak{col}\otimes\mathfrak{col}\otimes\mathfrak{scal}$	$\longleftarrow$	$\mathfrak{col}\otimes\mathfrak{kin}\otimes\mathfrak{scal}$	$\longrightarrow$	$\mathfrak{kin}\otimes\mathfrak{kin}\otimes\mathfrak{scal}$

Order of Events

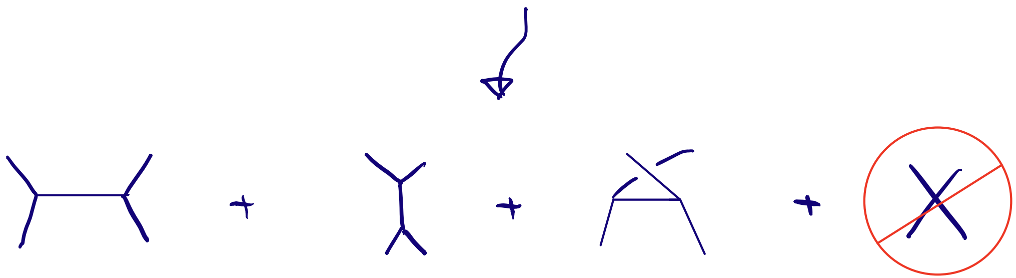
1. Review: BCJ CK Duality and Double-Copy
2. CK Duality Redux
3. BV Lagrangian Syngamy
4. Homotopy CK Duality and Double Copy
5. Generalisations: Supersymmetry

BCJ CK Duality and Double-Copy

Amplitudes and cubic diagrams

- Can write n-point L-loop gluon amplitude in terms of only cubic diagrams:

$$A_{\text{YM}}^{n,L} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$



- c_i: colour numerator, built from f^{abc}, read off diagram i
- n_i: kinematic numerator, built from p, ε ← Non-unique
- d_i: propagator, $\prod_{\text{int. lines}} p^2$, read off diagram i

Amplitudes and cubic diagrams

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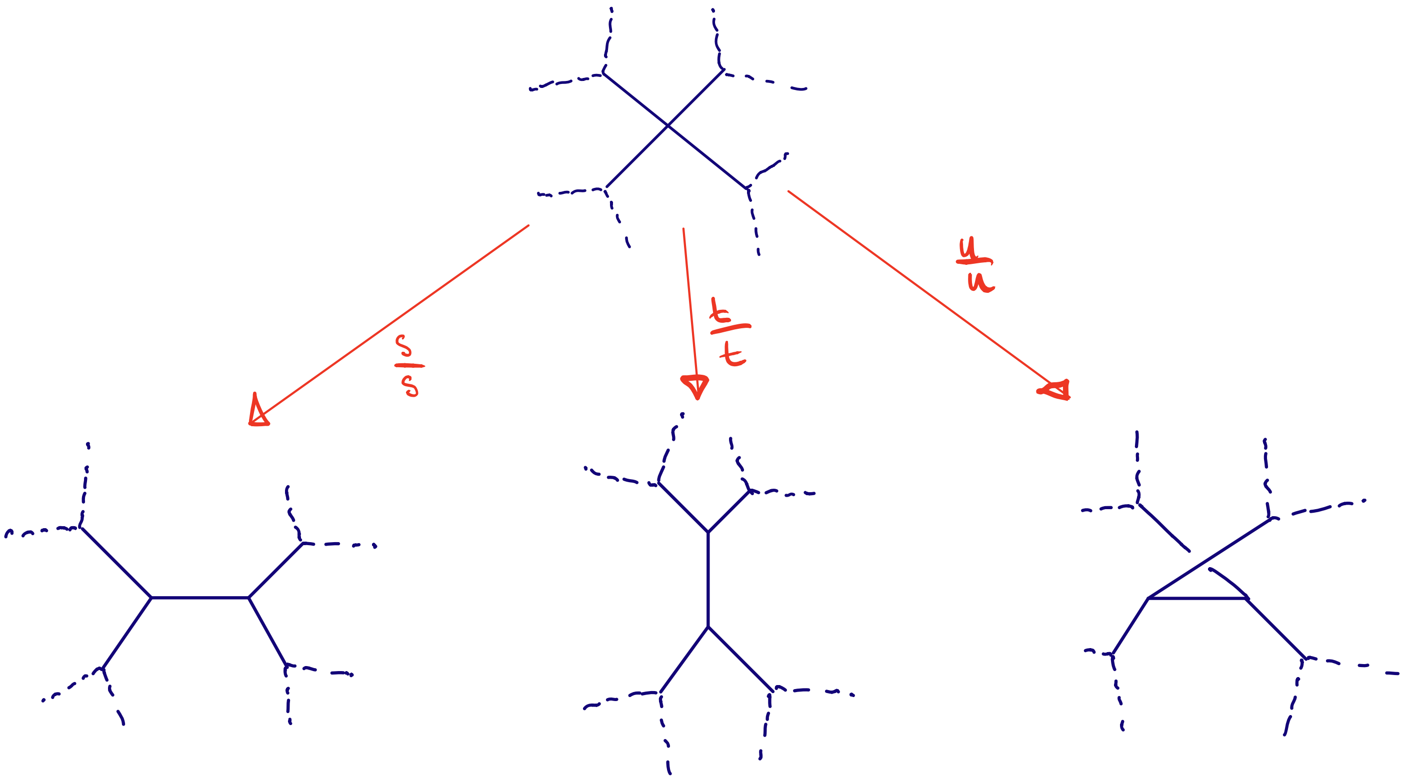


Amplitudes and cubic diagrams

- Can be realised in the YM Lagrangian through auxiliary fields:
  [Bern-Dennen-Huang-Kiermaier '10]

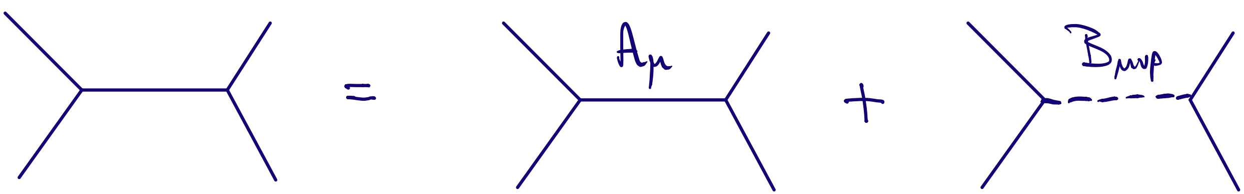
$$g^2[A_\mu, A_\nu][A^\mu, A^\nu] \;\rightarrow\; \tfrac{1}{2} B^{\mu\nu\kappa} \Box B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \frac{1}{\sqrt{2}} \partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu]$$

- Feynman diagrams give 'cubic' amplitudes directly:

$$A^{n,L}_{\text{YM}} = \sum_{\alpha \in \text{Feynman diag}} \int_L \frac{c_\alpha n_\alpha}{S_\alpha d_\alpha} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$

$$+ \sum_\varphi n_i^\varphi$$

- Example: 4-point s-channel diagram



$$n_s = n_s^A + n_s^B$$

BCJ colour-kinematic duality conjecture

- There is an organisation of the n-point L-loop gluon amplitude:

$$A_{\text{YM}}^{n,L} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$

such that

$$\boxed{\begin{array}{ccc} c_i + c_j + c_k = 0 & \Rightarrow & n_i + n_j + n_k = 0 \\ c_i \longrightarrow -c_i & \Rightarrow & n_i \longrightarrow -n_i \end{array}}$$

[Bern-Carrasco-Johansson '08]

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[Bern-Carrasco-Johansson '08]

- CK duality established at tree-level
  [Stieberger '09, Bjerrum-Bohr–Damgaard–Vanhove '09. . . Mizera '19; Reiterer '19]

- Significant evidence up to 4 loops in various (super)YM theories
  [Carrasco–Johansson '11; Bern–Davies–Dennen–Huang–Nohle '13; Bern-Davies-Dennen '14. . .]

- Quickly becomes difficult to check: remains conjectural at the loop level
  [Bern–Carrasco–Chen–Edison–Johansson–Parra-Martinez–Roiban–Zeng '18]

BCJ double-copy prescription

- Given CK dual amplitude of pure Yang-Mills

$$A_{\text{YM}}^{n,L} = \int_L \sum_{i \in \text{cubic diag}} \frac{c_i n_i}{S_i d_i}$$

$$S_{\text{YM}} = \frac{1}{2g^2} \int \text{tr} F \wedge \star F$$

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- Gives an amplitude of $\mathcal{N} = 0$ supergravity

$$A^{n,L}_{\mathcal{N}=0} = \sum_{i \in \text{cubic diag}} \int_L \frac{n_i n_i}{S_i d_i}$$

$$S_{\mathcal{N}=0} = \frac{1}{2\kappa^2} \int \star R - \frac{1}{d-2} d\varphi \wedge \star d\varphi - \frac{1}{2} \mathrm{e}^{-\frac{4}{d-2}\varphi} dB \wedge \star dB$$

where B is the Kalb-Ramond 2-form, φ is the dilaton
[Bern-Carrasco-Johansson '08, '10; Bern-Dennen-Huang-Kiermaier '10]

Implications and applications

- Conceptually compelling and computationally powerful: $\mathcal{N} = 8$ supergravity four-point to 5 loops! (finite)
  [Bern–Carrasco–Chen–Edison–Johansson–Parra-Martinez–Roiban–Zeng '18]

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- Can be explained by supersymmetry and $E_{7(7)}$ U-duality [Bjornsson–Green '10, Bossard–Howe–Stelle '11; Elvang–Freedman–Kiermaier '11; Bossard–Howe–Stelle–Vanhove '11]

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- At 7 loops any would-be cancellations are "not consequences of supersymmetry in any conventional sense" [Bjornsson–Green '10]
- $D = 4, \mathcal{N} = 5$ supergravity finite to 4 loops, contrary to expectations:

  "Enhanced" cancellations

  [Bern-Davies-Dennen '14]
- Such cancellations not seen for $\mathcal{N} = 8$ at 5 loops: implications unclear

Implications and applications

- Classical solutions and gravity wave astronomy
  [Monteiro–O'Connell–White '14; Cardoso–Nagy–Nampuri '16; Luna–Monteiro–Nicholson–Ochirov–O'Connell–Westerberg–White '16; Berman–Chacón–Luna–White '18; Kosower–Maybee–O'Connell '18; Bern–Cheung–Roiban–Shen–Solon–Zeng '19; Bern–Luna–Roiban–Shen–Zeng '20; Chacón-Nagy-White '21. . .]

- Geometric/world-sheet picture: ambitwistor string theories theories and scattering equation formalism $\rightarrow$ non-trivial gluon and spacetime backgrounds
  [Cachazo–He–Yuan '13 '14; Mason–Skinner '13; Adamo–Casali–Skinner '13; Adamo–Casali–Mason–Nekovar '17 '18; Geyer–Monteiro '18; Geyer–Mason '19; Geyer-Monteiro-Stark–Muchão '21. . .]

Off-shell BRST-Lagrangian double-copy

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  [Bern–Dennen–Huang–Kiermaier '10; Anastasiou–LB-Duff–Hughes–Nagy '14; LB '17; Anastasiou–LB–Duff-Nagy–Zoccali '18; LB–Jubb–Makwana–Nagy '20; LB-Nagy '20]
- CK duality manifesting actions and kinematic algebras
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- Covariant Color-Kinematics Duality: a closed-form, analytic expression for all tree-level BCJ numerators in YM theory! [Cheung-Mangan '21]

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- Covariant Color-Kinematics Duality: a closed-form, analytic expression for all tree-level BCJ numerators in YM theory! [Cheung-Mangan '21]
- Today: the YM BV action admits a natural form of 'anomalous' CK duality that immediately implies the double copy to all orders

Lighting overview

Step 1. Cubic tree-level off-shell CK duality manifesting Yang-Mills BRST-action:

$$S^{\mathrm{YM}}_{\mathrm{BRST\text{-}CK}} = c_{ab} C_{ij} A^{ai} \square A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

Step 2. BRST-action double-copy:

$$S_{\mathrm{DC}} = C_{ij} C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \square A^{j\tilde{\jmath}} + F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

Step 3. Double-copy BRST operator:

$$(Q_{\mathrm{YM}}, \tilde{Q}_{\mathrm{YM}}) \longrightarrow Q_{\mathrm{DC}} = Q^{\mathrm{lin}}_{\mathrm{diffeo}} + Q^{\mathrm{lin}}_{\mathrm{2\text{-}form}} + \cdots$$

Step 4. Assuming tree-level physical CK duality, perturbative quantum equivalence:

$$Q_{\mathrm{DC}}{}^2 = Q_{\mathrm{DC}} S_{\mathrm{DC}} = 0 \quad \Rightarrow \quad S_{\mathrm{DC}} \cong S^{\mathcal{N}=0}_{\mathrm{BRST}}$$

Corollary: Loop amplitude (integrands) computed from Feynman diagrams of $S^{\mathrm{YM}}_{\mathrm{BRST\text{-}CK}}$ manifest CK duality, *up to counterterms needed for unitarity*, and double-copy correctly to give amplitudes of $\mathcal{N} = 0$ supegravity

§2.

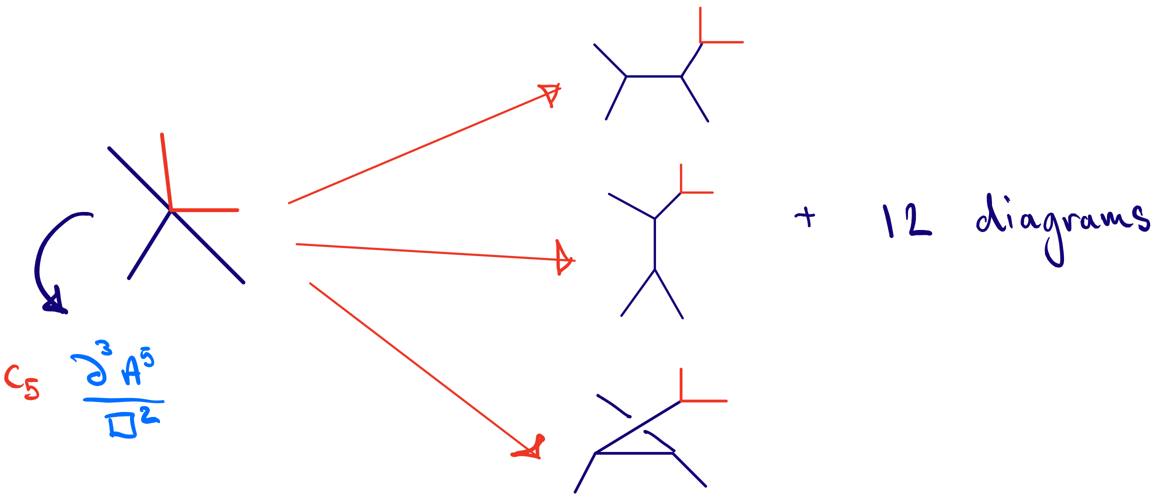
Colour-Kinematics Duality Redux

Colour-Kinematic Duality Redux

- There is a YM action such that the Feynman diagrams yield amplitudes manifesting CK duality for tree-level amplitudes:

$c_s = f^{abe} f_c^{\ de} f_e^{\ fg} + \cdots = 0$ by Jacobi

$$S^{\infty}_{\text{CK YM}} = \sum_{n=2}^{\infty} \int \mathcal{L}^{(n)}_{\text{YM}} \sim A\Box A + \partial AAA + \frac{\Box}{\Box} AAAA + \frac{\partial^3}{\Box^2} AAAAA + \cdots$$



[Bern–Dennen–Huang–Kiermaier 1004.0693; Tolotti–Weinzierl 1306.2975]

Colour-Kinematic Duality Redux

- This can be "strictified" to have only cubic interactions through infinite tower of auxiliaries [BJKMSW '21]

$$S^{\text{YM}}_{\text{on-shell CK}} = \int c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

  - i, j, k: kinematic indices over all fields including the auxiliaries:

$$A^{ai} = (A^a{}_\mu(x), B^a{}_{\mu\nu\rho}(x), \dots)$$

  - c_{ab}, f_{abc}: Lie gauge algebra Cartan-Killing form and structure constants
  - C_{ij}, F_{ijk}: Bi- and tri-linear differential operators

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- Example: 5-points

$$\begin{aligned}\mathcal{L}^{(5)}_{\text{YM}} &= C^{\mu\nu} \Box \bar{C}_{\mu\nu} + C^{\mu\nu\kappa} \Box \bar{C}_{\mu\nu\kappa} + C^{\mu\nu\kappa\lambda} \Box \bar{C}_{\mu\nu\kappa\lambda} + \\ &\quad + gC^{\mu\nu}[A_\mu, A_\nu] + g\partial_\mu C^{\mu\nu\kappa}[A_\nu, A_\kappa] - \tfrac{g}{2}\partial_\mu C^{\mu\nu\kappa\lambda}[\partial_{[\nu} A_{\kappa]}, A_\lambda] \\ &\quad + g\bar{C}^{\mu\nu}\left(\tfrac{1}{2}[\partial^\kappa \bar{C}_{\kappa\lambda\mu}, \partial^\lambda A_\nu] + [\partial^\kappa \bar{C}_{\kappa\lambda\nu\mu}, A^\lambda]\right)\end{aligned}$$

[Bern–Dennen–Huang–Kiermaier '10]

Colour-Kinematic Duality Redux

Tree-level gluon CK duality

- Cubic action manifesting tree-level CK duality for physical gluon states

$$S^{\text{YM}}_{\text{on-shell CK}} = \int c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

$$\text{Feynman diagrams} \longrightarrow A_n^{\text{tree}} = \sum_i \frac{c_i n_i}{d_i} \quad \text{s.t.} \quad c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0$$

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Generalise: Tree-level BRST CK duality

- Cubic action manifesting on-shell tree-level CK duality for physical gluons and unphysical longitudinal gluons and ghosts:

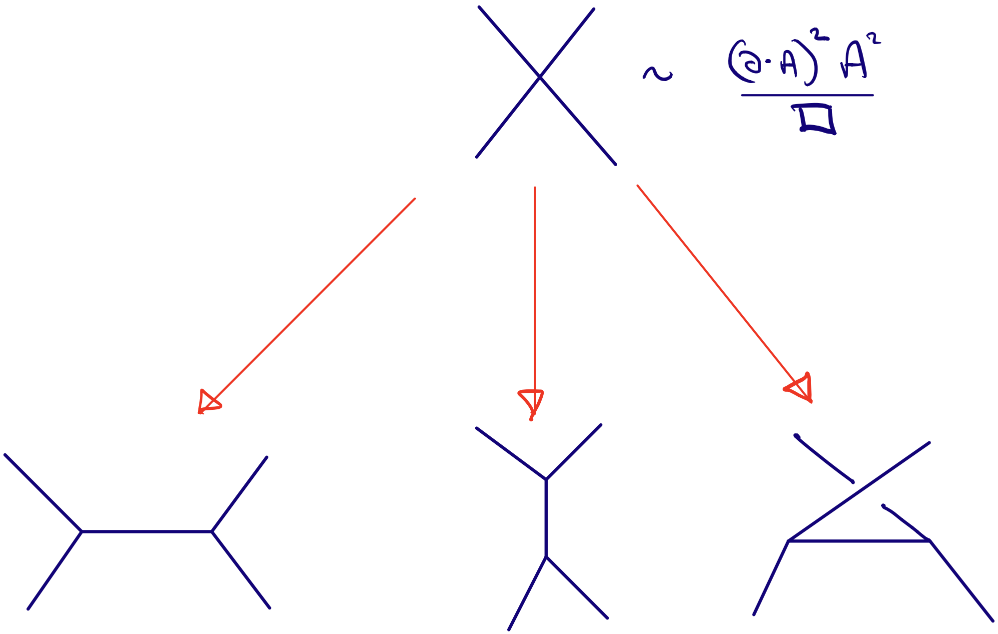
$$S_{\text{BRST-CK YM}} = \int c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

- Now i, j, k runs also over the BRST ghosts $c, \bar{c}$, the Nakanishi-Lautrup auxiliary b and auxiliary ghosts [BJKMSW '20]

Colour-Kinematic Duality Redux

Tree-level CK duality for longitudinal gluons

- Relax transversality $p_n \cdot \varepsilon_n \neq 0 \Rightarrow$ tree CK duality fails
- By analogy can compensate with new *non-zero* vertices [BJKMSW '20]:



- We can add them to the action without changing the physics [BJKMSW '20]

Colour-Kinematic Duality Redux

Tree-level CK duality for longitudinal gluons

- Using Lagrangian perspective, all CK failures can simultaneously be compensated by terms of the form

$$(\partial \cdot A) Y[A]$$

- Can add through the gauge-fixing functional

$$\begin{aligned} \text{Gauge-fixing func. } G[A]: \quad & \partial \cdot A && \mapsto && G'[A] && = && \partial \cdot A - 2\xi Y \\ \text{Nakanishi-Lautrup } b: \quad & b && \mapsto && b' && = && b + Y \end{aligned}$$

- Longitudinal CK duality $\Leftrightarrow$ gauge choice [BJKMSW '20, '21]

Colour-Kinematic Duality Redux

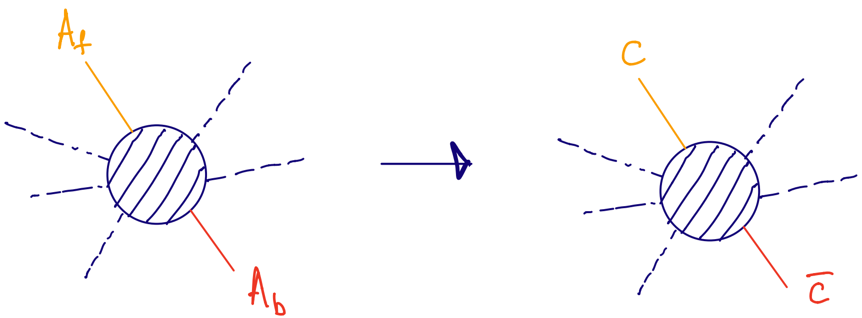
Tree-level CK duality for ghosts

- Use on-mass-shell BRST Ward identities

$$Q^{\rm lin}_{\rm YM} A_{\sf phys} = 0, \quad Q^{\rm lin}_{\rm YM} A_{\sf f} = c, \quad Q^{\rm lin}_{\rm YM} b = \bar{c}$$

- Analogous to global SUSY Ward identities

$$0 \;=\; \langle 0 | [Q^{\rm lin}_{\rm YM}, O_1 \cdots O_n] | 0 \rangle$$



- Transfers CK duality onto ghosts through

$$\mathcal{L}^{\sf YM}_{\sf ghost} = \bar{c} Q_{\rm YM} (\partial^\mu A_\mu - 2\xi Y)$$

Colour-Kinematic Duality Redux

On-shell tree-level CK manifesting BRST action

- Introduce new auxiliary gluons and ghosts:

$$S^{\text{YM}}_{\text{BRST CK-dual}} = c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

- i, j, k: run over all BRST fields including $b, c, \bar{c}$ and the tower of ghost auxiliaries

$$\begin{aligned}
\mathcal{L}^{\text{YM}}_{\text{BRST CK-dual}} = {} & \tfrac{1}{2} A_{a\mu} \Box A^{\mu a} - \bar{c}_a \Box c^a + \tfrac{1}{2} b_a \Box b^a + \xi\, b_a \sqrt{\Box}\, \partial_\mu A^{\mu a} \\
& - K^{\mu}_{1a} \Box \bar{K}^{1a}_{\mu} - K^{\mu}_{2a} \Box \bar{K}^{2a}_{\mu} - g f_{abc} \bar{c}^a \partial^\mu (A^b_\mu c^c) \\
& - \tfrac{1}{2} G^{\mu\nu\kappa}_a \Box G^a_{\mu\nu\kappa} + g f_{abc} \Big(\partial_\mu A^a_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa G^a_{\kappa\mu\nu} \Big) A^{\mu b} A^{\nu c} \\
& - g f_{abc} \Big\{ K^{a\mu}_1 (\partial^\nu A^b_\mu) A^c_\nu + [(\partial^\kappa A^a_\kappa) A^{b\mu} + \bar{c}^a \partial^\mu c^b] \bar{K}^{1c}_\mu \Big\} \\
& + g f_{abc} \Big\{ K^{a\mu}_2 \Big[(\partial^\nu \partial_\mu c^b) A^c_\nu + (\partial^\nu A^b_\mu) \partial_\nu c^c \Big] + \bar{c}^a A^{b\mu} \bar{K}^{2c}_\mu \Big\} + \cdots
\end{aligned}$$

- Feynman diagrams yield CK dual tree amplitudes for physical gluons and unphysical longitudinal modes and ghosts, but on-shell

Lifting to off-shell CK duality

- Relaxing on-shell momenta CK duality may be violated by terms $p_i^2 F_i$
- Can compensate with term $F_i \Box \Phi$ with non-local field redefinition

$$\Phi \mapsto \Phi + \sum_i \hat{F}_i$$

so that off-shell CK duality is manifest $\to$ loop CK duality [BJKMSW '21]

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- Price to pay: Jacobian determinants lead to counterterms in the renormalization ensuring unitarity
- In this sense, this manifest CK duality is anomalous on the physical Hilbert space (but is exact on the complete pre-Hilbert space)

Colour-Kinematic Duality Redux

Lifting to off-shell CK duality

- BV YM action with manifest *off-shell* CK duality

$$S^{\text{YM}}_{\text{BV CK-dual}} = \int c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck} + A^{+}_{ai} \left(Q^{i}{}_{j} A^{aj} + f^{a}_{bc} Q^{i}_{jk} A^{bj} A^{ck} \right)$$

- F^{ijk} satisfy the same identities as f^{abc} as *operators equations*,

$$\begin{aligned} c_{ab} &= c_{(ab)} & f_{abc} &= f_{[abc]} & c_{a(b} f^{a}_{c)d} &= 0 & f_{[ab|d} f^{d}_{c]e} &= 0 \\ C_{ij} &= C_{(ij)} & F_{ijk} &= F_{[ijk]} & C_{i(j} F^{i}_{k)l} &= 0 & F_{[ij|l} F^{l}_{|k]m} &= 0 \end{aligned}$$

- That is, the F_{ijk} are the structure constants of a *kinematic Lie algebra*, cf. [Monteiro–O'Connell '11, '13; Bjerrum–Bohr–Damgaard–Monteiro–O'Connell '12; Fu–Krasnov '16; Chen–Johansson–Teng–Wang 19; Campiglia-Nagy '21. . .]

- BV quantised Yang-Mills theory has manifest CK duality

- Anomalous due to Jacobian counterterms: standard Bern, Dennen, Huang, Kiemaier proof of loop double copy does not hold straightforwardly

§2.

BV Lagrangian Syngamy

BV Lagrangian Syngamy

- Parent theories in cubic factorised form:

$$S = c_{ab} C_{ij} \Phi^{ai} \Box \Phi^{aj} + f_{abc} F_{ijk} \Phi^{ai} \Phi^{bj} \Phi^{ck}$$

$$\tilde{S} = \tilde{c}_{\tilde{a}\tilde{b}} \tilde{C}_{\tilde{\imath}\tilde{\jmath}} \Phi^{\tilde{a}\tilde{\imath}} \Box \tilde{\Phi}^{\tilde{a}\tilde{\jmath}} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \tilde{\Phi}^{\tilde{a}\tilde{\imath}} \tilde{\Phi}^{\tilde{b}\tilde{\jmath}} \tilde{\Phi}^{\tilde{c}\tilde{k}}$$

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- Syngamy: meiotic reproduction of diploid theories, e.g.

Double-copy	$c_{ab} \to \tilde{C}_{\tilde{\imath}\tilde{\jmath}}$	$f_{abc} \to \tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}$	$\Phi^{ai} \to \Phi^{i\tilde{\imath}}$
Zeroth-copy	$C_{ij} \to \tilde{c}_{\tilde{a}\tilde{b}}$	$F_{ijk} \to \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}$	$\Phi^{ai} \to \Phi^{a\tilde{a}}$

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- Double/zeroth copy Lagrangians:

$$S \otimes \tilde{S} \to \begin{cases} S_{\text{DC}} = C_{ij} \tilde{C}_{\tilde{\imath}\tilde{\jmath}} \Phi^{i\tilde{\imath}} \Box \Phi^{j\tilde{\jmath}} + F_{ijk} \tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \Phi^{i\tilde{\imath}} \Phi^{j\tilde{\jmath}} \Phi^{k\tilde{k}} \\ S_{\text{ZC}} = c_{ab} \tilde{c}_{\tilde{a}\tilde{b}} \Phi^{a\tilde{a}} \Box \Phi^{a\tilde{b}} + f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \Phi^{a\tilde{a}} \Phi^{b\tilde{b}} \Phi^{c\tilde{c}} \end{cases}$$

BV Lagrangian Syngamy

- $S^{\mathrm{YM}}_{\mathrm{BRST\text{-}CK}} \otimes \tilde{S}^{\mathrm{YM}}_{\mathrm{BRST\text{-}CK}} \to \mathcal{N}=0$ supergravity

$$\begin{aligned} A^{ai} &\to A^{i\tilde{\imath}} &= h_{\mu\nu} \oplus B_{\mu\nu} \oplus \varphi \oplus \text{ghosts} \oplus \text{auxiliaries} \\ S^{\mathrm{YM}}_{\mathrm{BRST\text{-}CK}} &\to S^{\mathcal{N}=0}_{\mathrm{DC}} &= C_{ij} C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}} \end{aligned}$$

- $G \times \tilde{G}$ bi-adjoint scalar theory,

$$S^{\text{bi-adj}}_{\mathrm{DC}} = c_{ab} \tilde{c}_{\tilde{a}\tilde{b}} \Phi^{a\tilde{a}} \Box \Phi^{a\tilde{b}} + f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \Phi^{a\tilde{a}} \Phi^{b\tilde{b}} \Phi^{c\tilde{c}}$$

- Cf. scattering equation formalism [Hodges '11; Cachazo–He–Yuan '13 '14]

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$$\sum \frac{cn}{d} \qquad \to \qquad \sum \frac{nn}{d}$$

- Implies by construction the physical (h, B, φ) tree-level amplitudes are those of $\mathcal{N} = 0$ supergravity
- Cf. [Bern-Dennen-Huang-Kiermaier 1004.0693] for gravitons up to 6 points

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- What about quantum consistency $\Rightarrow$ double-copy BRST operator

BV Lagrangian Syngamy

- How do we we know that there exists some BRST Q such that:

$$QS_{\mathrm{DC}} = 0, \qquad Q^2 = 0$$

BV Lagrangian Syngamy

- How do we we know that there exists some BRST Q such that:

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- Double-copy of BV action implies double copy BRST operator Q_{DC}

$$S^{\mathrm{YM}}_{\mathrm{BV\;CK\text{-}dual}} = \int c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck} + A^{+}_{ai} \left(Q^{i}{}_{j} A^{aj} + f^{a}_{bc} Q^{i}_{jk} A^{bj} A^{ck} \right)$$

$$QA^{ai} = Q^{i}{}_{j} A^{bj} + f^{a}{}_{bc} Q^{i}{}_{jk} A^{bj} A^{ck} \qquad \tilde{Q}\tilde{A}^{\tilde{a}\tilde{\imath}} = Q^{\tilde{\imath}}{}_{\tilde{\jmath}} \tilde{A}^{\tilde{b}\tilde{\jmath}} + \tilde{f}^{\tilde{a}}{}_{\tilde{b}\tilde{c}} \tilde{Q}^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} \tilde{A}^{\tilde{b}\tilde{\jmath}} \tilde{A}^{\tilde{c}\tilde{k}}$$

$$f \to \tilde{F} \qquad \tilde{f} \to F$$

$$Q_{\mathrm{DC}} = Q_{\mathrm{L}} + Q_{\mathrm{R}}$$

$$Q_L = Q^{i}{}_{j} A^{j\tilde{\imath}} + Q^{i}{}_{jk} F^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} A^{j\tilde{\jmath}} A^{k\tilde{k}} \qquad Q_R = Q^{\tilde{\imath}}{}_{\tilde{\jmath}} A^{i\tilde{\jmath}} + F^{i}{}_{jk} Q^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

BV Lagrangian Syngamy

- For Yang-Mills we find linear diffeomorphisms and 2-form gauge (and gauge-for-gauge) symmetry:

$$Q^{\text{lin}}_{\mathsf{DC}} = Q^{\text{lin}}_{\mathsf{diffeo}} + Q^{\text{lin}}_{\mathsf{2\text{-}form}}$$

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  [Anastasiou-LB-Duff-Hughes-Nagy '14]

- $Q_{\rm DC}S_{\rm DC} = 0$, $Q^2_{\rm DC} = 0$ follows since F^{ijk} satisfy the same identities as f^{abc}

- Semi-classical equivalence with well-defined BRST operator
  $\longrightarrow$ quantum equivalence

- Einstein is the square of Yang–Mills (at least perturbatively)

- Straightforward supersymmetric completion

§4.

Homotopy CK Duality and Double Copy

Homotopy Algebras and BV Lagrangian Field Theories

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- Lie algebras $\to$ L_∞-algebras, first arose in string field theory:

Vector space $\mathfrak{g} = V_0$	Graded vector space $\mathfrak{L} = \bigoplus_n V_n$
Bracket $\mu_2 = [-,-]$	Higher brackets $\mu_1 = [-],\ \mu_2 = [-,-],\ \mu_3 = [-,-,-],\dots$
Relations $\mathit{Antisymmetry} + \mathit{Jacobi}$	Relations $\mathit{Antisymmetry} + \mathit{homotopyJacobi}$

[Zwiebach '93; Hinich–Schechtman '93]

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Relations *Antisymmetry + Jacobi*	Relations *Antisymmetry + homotopyJacobi*

[Zwiebach '93; Hinich–Schechtman '93]

- Associative algebras $\to$ A_∞-algebras [Stasheff '63]

- Commutative algebras $\to$ C_∞-algebras [Kadeishvili '88]

- Chevalley–Eilenberg formulation of Lie algebra $\mathfrak{g}$ with basis t_a:

$$\mathsf{CE}(\mathfrak{g}) = \bar{T}(\mathfrak{g}[1]^*), Q$$

$$Qt^a = -\tfrac{1}{2} f^a{}_{bc} t^b t^c, \qquad Q^2 = 0 \Leftrightarrow \text{Jacobi}$$

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- Chevalley–Eilenberg formulation of L_∞-algebra $\mathfrak{L}$ with basis t_a:

$$\mathrm{CE}(\mathfrak{L}) = \bar{T}(\mathfrak{L}[1]^*), Q$$

$$Qt^a = -\sum_n \tfrac{1}{n!} \mu_n{}^a{}_{a_1 \cdots a_n} t^{a_1} \cdots t^{a_n}, \qquad Q^2 = 0 \Leftrightarrow \text{homotopy Jacobi}$$

- Any BV field theory with operator Q_{BV} corresponds to an L_∞-algebra in the CE picture, see e.g. [Jurco-Raspollini-Saemann-Wolf '18]

Homotopy Algebras and BV Lagrangian Field Theories

- Yang-Mills theory $\mathfrak{L}^{\mathrm{YM}}$

$$
\begin{array}{ccccccc}
\mathfrak{L}^{\mathrm{YM}}_0 & \oplus & \mathfrak{L}^{\mathrm{YM}}_1 & \oplus & \mathfrak{L}^{\mathrm{YM}}_2 & \oplus & \mathfrak{L}^{\mathrm{YM}}_3 \\
c & \xrightarrow{d} & A & \xrightarrow{d^\dagger d} & A^+ & \xrightarrow{d^\dagger} & c^+ \\
 & & b & \xrightarrow{\mathrm{Id}} & \bar{c} & & \\
 & & \bar{c}^+ & \xrightarrow{-\mathrm{Id}} & b^+ & &
\end{array}
$$

- Homotopy Maurer-Cartan theory $\longrightarrow$ field strengths + gauge trans.
- Cartan-Killing form $\langle -,-\rangle_{\mathfrak{g}} \to$ cyclic structure $\langle -,-\rangle_{\mathrm{YM}}$ on $\mathfrak{L}^{\mathrm{YM}}$
- BV action $\sim \sum \frac{1}{(i+1)!}\langle a, \mu_i(a,\ldots,a)\rangle$
- L_∞ quasi-isomorphisms $\longrightarrow$ physical equivalence (field redefinitions etc)

Colour-Kinematic-Scalar Factorisation of Yang-Mills

- $\mathfrak{L}^{\mathsf{YM}}$ factorises into $\mathfrak{colour} \otimes \mathfrak{kinematics} \otimes \mathfrak{scalar}$

$$\mathfrak{L}^{\mathsf{YM}} = \underbrace{\underbrace{\mathfrak{colour}}_{L_\infty} \otimes \underbrace{\mathfrak{kinematics} \otimes \underbrace{\mathfrak{scalar}}_{A_\infty}}_{C_\infty}}_{L_\infty}$$

[BLKMSW '21]

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[BLKMSW '21]

- $\mathfrak{colour}$: gauge group Lie algebra

- $\mathfrak{kinematics}$: graded vector space of Poincaré representations of fields

$$\begin{array}{ccccccc} \mathbb{R}[-1] & \oplus & (\mathbb{R}^d \oplus \mathbb{R}) & \oplus & \mathbb{R}[1] & \oplus & \text{Auxiliaries} \\ c & & (A_\mu, b) & & \bar{c} & & A_{\mu\nu\rho} \cdots \end{array}$$

- $\mathfrak{scalar}$: A_∞-algebra of a scalar field theory

$$\langle -, - \rangle_{\mathsf{YM}} = \langle -, - \rangle_{\mathfrak{colour}} \langle -, - \rangle_{\mathfrak{kinematics}} \langle -, - \rangle_{\mathfrak{scalar}}$$

Homotopy Double Copy: Fields and Action

- Homotopy double-copy:

$$\mathfrak{L}^{\mathsf{YM}} = \mathfrak{g} \otimes \mathfrak{V} \otimes_\tau \mathfrak{S} \longrightarrow \mathfrak{V} \otimes_\tau \mathfrak{V} \otimes_\tau \mathfrak{S} = \mathfrak{L}^{\mathsf{DC}}$$

- Given $\mathfrak{kinematics}$ $\otimes_\tau$ $\mathfrak{scalar}$ double-copy completely determined:

fields				antifields		
factorisation	role	L_∞ deg	dim	factorisation	L_∞ deg	dim
$\lambda = [\mathrm{g},\mathrm{g}]\mathsf{s}_x \frac{1}{2}\lambda(x)$	ghost for ghost	-1	$\frac{d}{2}-3$	$\lambda^+ = [\mathrm{a},\mathrm{a}]\mathsf{s}_x^+ \frac{1}{2}\lambda^+(x)$	4	$\frac{d}{2}+3$
$\Lambda = [\mathrm{g},\mathrm{v}^\mu]\mathsf{s}_x \Lambda_\mu(x)$	ghost	0	$\frac{d}{2}-2$	$\Lambda^+ = [\mathrm{a},\mathrm{v}^\mu]\mathsf{s}_x^+ \Lambda_\mu^+$	3	$\frac{d}{2}+2$
$\gamma = [\mathrm{g},\mathrm{n}]\mathsf{s}_x \gamma(x)$	NL field of Λ_μ	0	$\frac{d}{2}-2$	$\gamma^+ = [\mathrm{a},\mathrm{n}]\mathsf{s}_x^+ \gamma^+(x)$	3	$\frac{d}{2}+2$
$B = [\mathrm{v}^\mu,\mathrm{v}^\nu]\mathsf{s}_x \frac{1}{2}B_{\mu\nu}(x)$	physical field	1	$\frac{d}{2}-1$	$B^+ = [\mathrm{v}^\mu,\mathrm{v}^\nu]\mathsf{s}_x^+ \frac{1}{2}B_{\mu\nu}^+(x)$	2	$\frac{d}{2}+1$
$\alpha = [\mathrm{n},\mathrm{v}^\mu]\mathsf{s}_x \alpha_\mu(x)$	NL field	1	$\frac{d}{2}-1$	$\alpha^+ = [\mathrm{n},\mathrm{v}^\mu]\mathsf{s}_x^+ \alpha_\mu^+(x)$	2	$\frac{d}{2}+1$
$\varepsilon = [\mathrm{g},\mathrm{a}]\mathsf{s}_x \varepsilon(x)$	anti-ghost of Λ_μ	1	$\frac{d}{2}-1$	$\varepsilon^+ = [\mathrm{g},\mathrm{a}]\mathsf{s}_x^+ \varepsilon^+(x)$	2	$\frac{d}{2}+1$
$\bar\Lambda = [\mathrm{a},\mathrm{v}^\mu]\mathsf{s}_x \bar\Lambda_\mu(x)$	anti-ghost	2	$\frac{d}{2}$	$\bar\Lambda^+ = [\mathrm{g},\mathrm{v}^\mu]\mathsf{s}_x^+ \bar\Lambda_\mu^+(x)$	1	$\frac{d}{2}$
$\bar\gamma = [\mathrm{a},\mathrm{n}]\mathsf{s}_x \bar\gamma(x)$	NL field of $\bar\Lambda_\mu$	2	$\frac{d}{2}$	$\bar\gamma^+ = [\mathrm{g},\mathrm{n}]\mathsf{s}_x^+ \bar\gamma^+(x)$	1	$\frac{d}{2}$
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$X = (\mathrm{g},\mathrm{v}^\mu)\mathsf{s}_x X_\mu(x)$	ghost	0	$\frac{d}{2}-2$	$X^+ = (\mathrm{a},\mathrm{v}^\mu)\mathsf{s}_x^+ X_\mu^+(x)$	3	$\frac{d}{2}+2$
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$\bar X = (\mathrm{a},\mathrm{v}^\mu)\mathsf{s}_x \bar X_\mu(x)$	anti-ghost	2	$\frac{d}{2}$	$\bar X^+ = (\mathrm{g},\mathrm{v}^\mu)\mathsf{s}_x^+ \bar X_\mu(x)$	1	$\frac{d}{2}$
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$h = (\mathrm{v}^\mu,\mathrm{v}^\nu)\mathsf{s}_x \frac{1}{2} h_{\mu\nu}(x)$	physical field	1	$\frac{d}{2}-1$	$h^+ = (\mathrm{v}^\mu,\mathrm{v}^\nu)\mathsf{s}_x^+ \frac{1}{2} h_{\mu\nu}^+(x)$	2	$\frac{d}{2}+1$
$\varpi = (\mathrm{n},\mathrm{v}^\mu)\mathsf{s}_x \varpi_\mu(x)$	NL field	1	$\frac{d}{2}-1$	$\varpi^+ = (\mathrm{n},\mathrm{v}^\mu)\mathsf{s}_x^+ \varpi_\mu^+(x)$	2	$\frac{d}{2}+1$
$\pi = (\mathrm{n},\mathrm{n})\mathsf{s}_x \frac{1}{2}\pi(x)$	NL field of ϖ_μ	1	$\frac{d}{2}-1$	$\pi^+ = (\mathrm{n},\mathrm{n})\mathsf{s}_x^+ \frac{1}{2}\pi^+(x)$	2	$\frac{d}{2}+1$
$\delta = (\mathrm{g},\mathrm{a})\mathsf{s}_x \delta(x)$	anti-ghost of X_μ	1	$\frac{d}{2}-1$	$\delta^+ = (\mathrm{g},\mathrm{a})\mathsf{s}_x^+ \delta^+(x)$	2	$\frac{d}{2}+1$
$\bar X = (\mathrm{a},\mathrm{v}^\mu)\mathsf{s}_x \bar X_\mu(x)$	anti-ghost	2	$\frac{d}{2}$	$\bar X^+ = (\mathrm{g},\mathrm{v}^\mu)\mathsf{s}_x^+ \bar X_\mu(x)$	1	$\frac{d}{2}$
$\bar\beta = (\mathrm{a},\mathrm{n})\mathsf{s}_x \bar\beta(x)$	NL field of $\bar X_\mu$	2	$\frac{d}{2}$	$\bar\beta^+ = (\mathrm{g},\mathrm{n})\mathsf{s}_x^+ \bar\beta^+(x)$	1	$\frac{d}{2}$

Homotopy Double Copy: Fields and Action

- Given $\mathfrak{kinematics} \otimes_\tau \mathfrak{scalar}$, double-copy action completely determined:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{DC}} = &\tfrac{1}{2} h_{\mu\nu} \Box h^{\mu\nu} + \tfrac{1}{2}\varpi_\mu \Box \varpi^\mu + \xi^2(\partial^\mu \varpi_\mu)^2 + \tfrac{1}{2}\pi \Box \pi \\
&- 2\xi \varpi^\nu \Box^{\frac{1}{2}} \partial^\mu h_{\mu\nu} - 2\xi\pi \Box^{\frac{1}{2}} \partial_\mu \varpi^\mu + 2\xi^2 \pi \partial_\mu \partial_\nu h^{\mu\nu} \\
&- 2\bar{X}_\mu \Box X^\mu - \delta \Box \delta - 2\bar{\beta} \Box \beta \\
&+ \tfrac{1}{2} B_{\mu\nu} \Box B^{\mu\nu} - 2\bar{\Lambda}_\mu \Box \Lambda^\mu + \alpha_\mu \Box \alpha^\mu + \xi^2(\partial^\mu \alpha_\mu)^2 + \varepsilon \Box \varepsilon - \bar{\lambda} \Box \lambda - 2\bar{\gamma} \Box \gamma \\
&- 2\xi \alpha^\nu \Box^{\frac{1}{2}} \partial^\mu B_{\mu\nu} - 2\xi\gamma \Box^{\frac{1}{2}} \partial_\mu \bar{\Lambda}^\mu + 2\xi\bar{\gamma} \Box^{\frac{1}{2}} \partial_\mu \Lambda^\mu \\
&- 2\xi\beta \Box^{\frac{1}{2}} \partial_\mu \bar{X}^\mu + 2\xi\bar{\beta} \Box^{\frac{1}{2}} \partial_\mu X^\mu \; + \cdots
\end{aligned}
$$

- Canonical field redefinition to Fierz-Pauli + Kalb-Ramond + dilaton action

- Read off from action of double-copy BRST operator

Homotopy algebra of CK duality

- Michel Reiterer [1912.03110]: proof of on-shell tree-level CK duality for physical gluons via $BV_{\infty}^{\Box}$-algebra of Zeitlin-Costello complex!

Homotopy algebra of CK duality

- Michel Reiterer [1912.03110]: proof of on-shell tree-level CK duality for physical gluons via $BV_\infty^\square$-algebra of Zeitlin-Costello complex!
- Very special: only $D = 4$, no loop desiderata (gauge-fixing, ghosts etc)
- Work to appear [BJKMSW '21]:
  - Symmetric monoidal category of Hodge complexes (modules over twisted Hopf algebras with central element $\square$)

$$d^2 = h^2 = 0 \quad dh + hd = \square$$

  - $BV^\square$-operad: perfect BV CK duality (up to conterterms)
  - $BV_\infty^\square$-operad: auxiliaries integrated out

Future work

- AdS background [Zhou '21; Diwakar-Herderschee-Roiban-Teng '21 ...]$\to$ Hopf algebra of universal enveloping algebra of AdS isometries
- Bagger-Lambert-Gustavsson CK duality [Bargheer-He-McLoughlin '12; Huang-Johansson '12] $\to$ m-ary $BV^{\square}$ operads
- Matter coupling [Johansson-Ochirov '14] $\to$ many-sorted $BV^{\square}$ operads
- String theory (modular envelope over) $BV_{\infty}^{L_0}$

$$dh + hd = \square \quad \longrightarrow \quad \{Q, b\} = L_0$$

- Computational efficiency: purely tree-level calculations, one identity at any order (the rest follow axiomatically)... but Feynman diagrams
- Counterterms?

Thanks for listening