Rencontres

Théoriciennes

Modular graph forms and iterated integrals in string amplitudes

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Oliver Schlotterer (Uppsala University)

based on work with E. D'Hoker, D. Dorigoni, J. Gerken,

M. Hidding, A. Kleinschmidt, C. Mafra, B. Pioline, B. Verbeek

06.01.2022

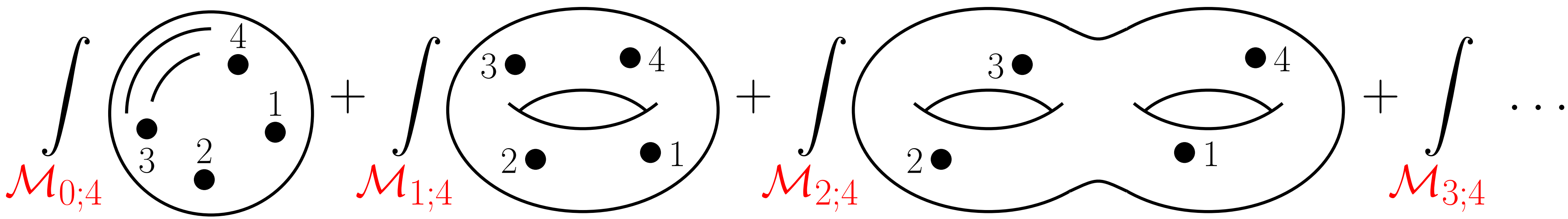
Key idea in this talk

String perturbation theory $\Longrightarrow$ generating series of ...

... iterated integrals (open strings) and

... single-valued periods / functions & modular forms (closed strings)

upon low-energy expansion w.r.t. $s_{ij} = \alpha' k_i \cdot k_j$ (with ext. momenta k_i)



$\mathcal{M}_{g;n}$ = moduli space of n-punctured compact genus-g Riemann surfaces

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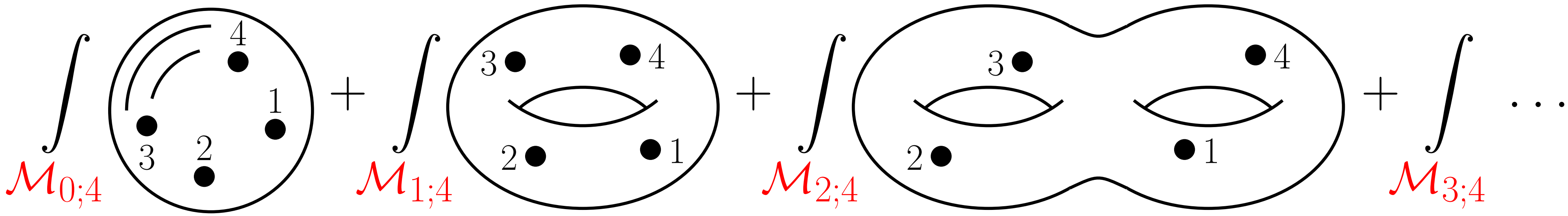

Amplitude of genus-g surface Σ_g with moduli Ω & Green fct. $\mathcal{G}_{\Sigma_g}$

$$\mathcal{A}_{\Sigma_g}(\{1,2,\ldots,n\}) \sim \int_{\mathcal{M}_{g;n}} \exp\Bigg(\sum_{1\leq i<j}^{n} s_{ij}\, \mathcal{G}_{\Sigma_g}(z_i, z_j|\Omega) \Bigg) \times \Bigg(\begin{matrix} \text{theory-} \\ \text{dependent} \end{matrix} \Bigg)$$

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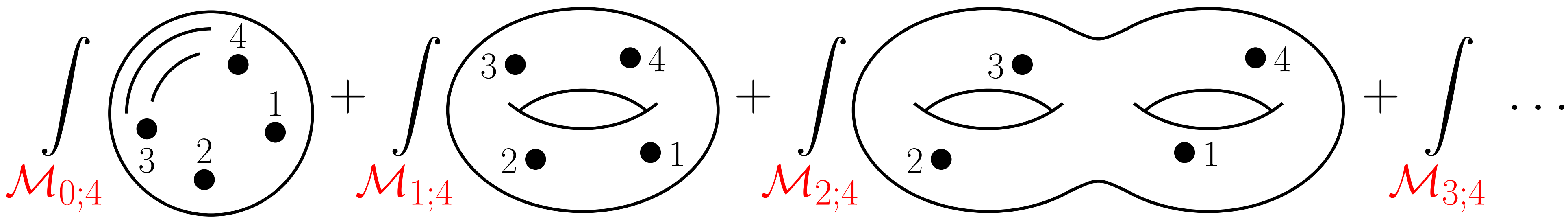
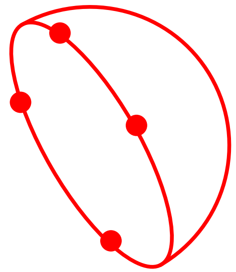


Today: String amplitudes as mathematical laboratory

$\longrightarrow$ apologies for not elaborating on the more physics-oriented motivations

for string amplitudes (string dualities, gravity as (gauge theory)$^{\otimes 2}$, ...)

Periods of configuration spaces in α'-expansion

	open strings	closed strings
tree level	$\int_{\text{all } z_j}$  multiple zeta values (MZVs)	
one loop		

Definition of multiple zeta values (MZVs) with $n_j \in \mathbb{N}$ and $n_r \geq 2$

$$\zeta_{n_1,n_2,\ldots,n_r} = \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r}$$

Periods of configuration spaces in α'-expansion

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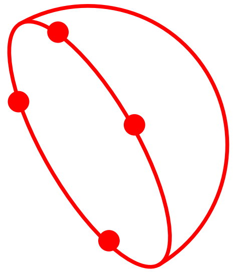
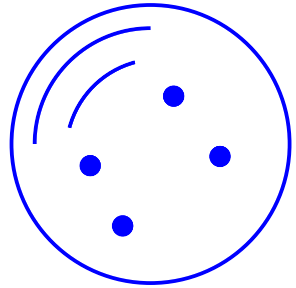
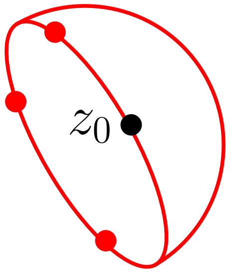
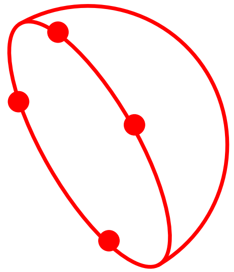
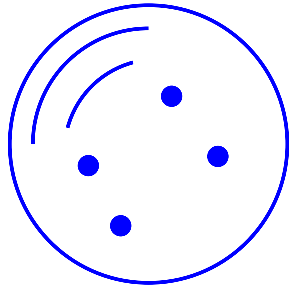
tree level	$\int_{\text{all } z_j}$  multiple zeta values (MZVs)	$\int_{\text{all } z_j}$  single-valued MZVs **[Brown, Schnetz '13]**
one loop		

Examples of single-valued MZVs

$$\text{sv}\,\zeta_{2k} = 0\,, \quad \text{sv}\,\zeta_{2k+1} = 2\zeta_{2k+1}\,, \quad \text{sv}\,\zeta_{3,5} = -10\zeta_3\zeta_5\,, \quad \text{etc.}$$

Periods of configuration spaces in α'-expansion

	open strings		closed strings	
tree level	$\int$ not z_0  multiple poly-logarithms(z_0)	$\int$ all z_j  multiple zeta values (MZVs)		$\int$ all z_j  single-valued MZVs **[Brown, Schnetz '13]**
one loop				

Not yet integrating z_0: multiple polylogarithms (yield MZVs as $z_0 \to 1$)

$$\int_{0<z_1<z_2<\ldots<z_r<z_0} \mathrm{d}\log(z_1 - a_1)\,\mathrm{d}\log(z_2 - a_2)\ \ldots\ \mathrm{d}\log(z_r - a_r)$$

Periods of configuration spaces in α'-expansion

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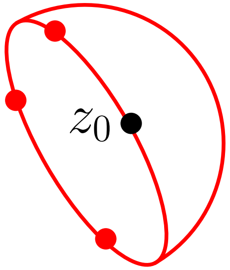
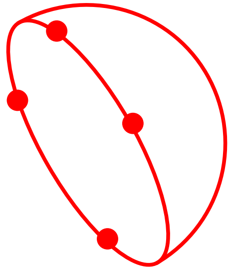
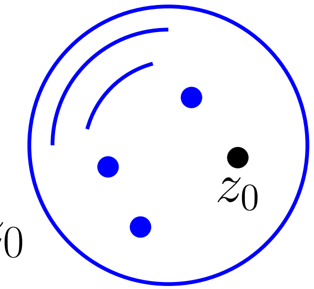
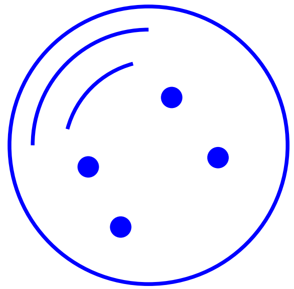
tree level	$\int$ not z_0  multiple polylogarithms(z_0)	$\int$ all z_j  multiple zeta values (MZVs)	$\int$ not z_0  single-valued polylogs(z_0) **[Brown '04]**	$\int$ all z_j  single-valued MZVs **[Brown, Schnetz '13]**
one loop				

At tree level: closed strings from single-valued map of open-string data

[OS, Stieberger '12; Stieberger '13; Stieberger, Taylor '14; OS, Schnetz '18; Brown, Dupont '18 & '19; Vanhove, Zerbini '18]

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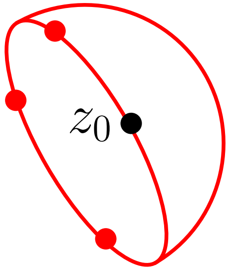
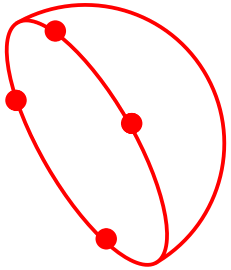
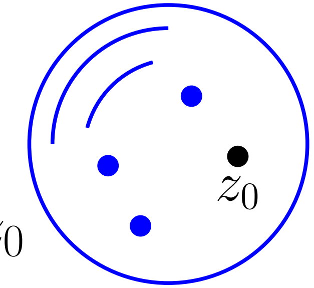
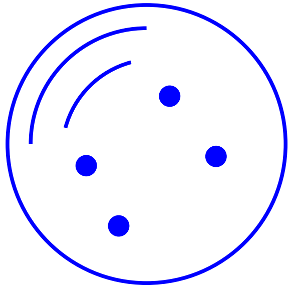
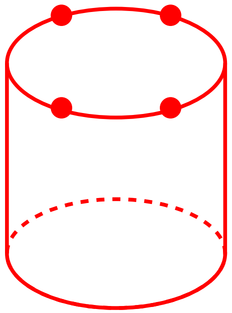
tree level	$\int_{\text{not } z_0}$  multiple polylogarithms(z_0)	$\int_{\text{all } z_j}$  multiple zeta values (MZVs)	$\int_{\text{not } z_0}$  single-valued polylogs(z_0) **[Brown '04]**	$\int_{\text{all } z_j}$  single-valued MZVs **[Brown, Schnetz '13]**
one loop		$\int_{\text{all } z_j}$  elliptic MZVs **[Enriquez '13]**		

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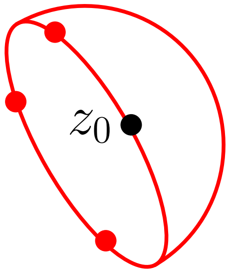
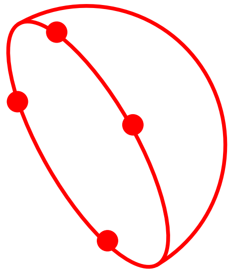
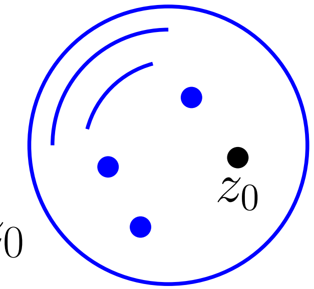
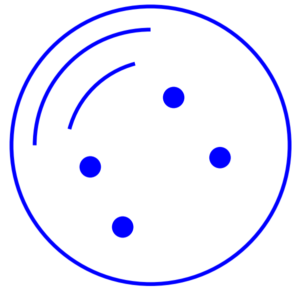
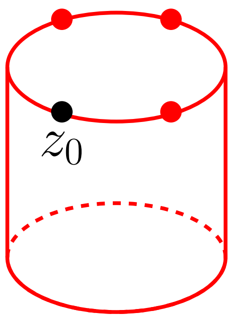
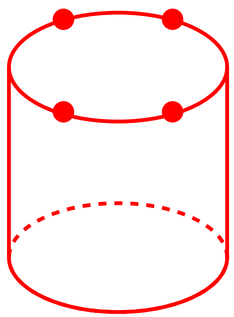
tree level	$\int_{\text{not } z_0}$  multiple poly-logarithms(z_0)	$\int_{\text{all } z_j}$  multiple zeta values (MZVs)	$\int_{\text{not } z_0}$  single-valued poly-logs(z_0) **[Brown '04]**	$\int_{\text{all } z_j}$  single-valued MZVs **[Brown, Schnetz '13]**
one loop	$\int_{\text{not } z_0}$  elliptic polylogs(z_0) **[Brown, Levin '11]**	$\int_{\text{all } z_j}$  elliptic MZVs **[Enriquez '13]**		

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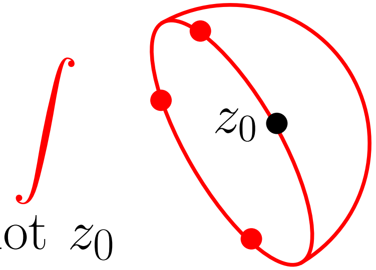
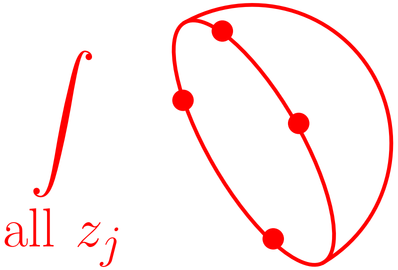
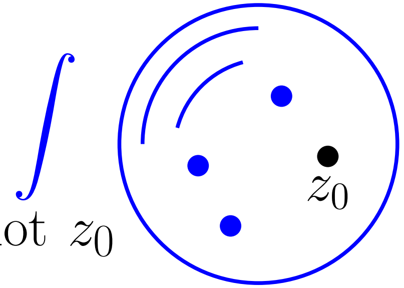
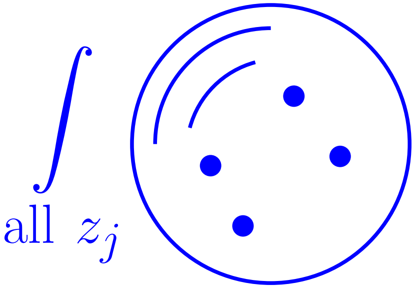
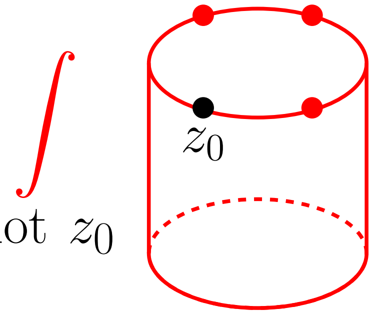
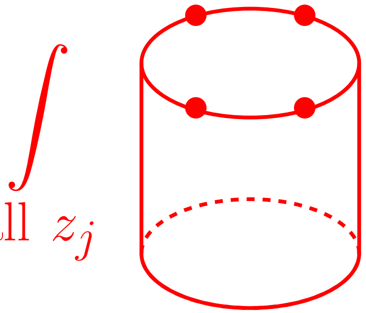
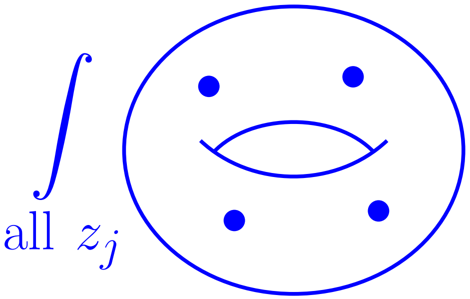
tree level	 not z_0 — multiple poly-logarithms(z_0)	 all z_j — multiple zeta values (MZVs)	 not z_0 — single-valued poly-logs(z_0) **[Brown '04]**	 all z_j — single-valued MZVs **[Brown, Schnetz '13]**
one loop	 not z_0 — elliptic polylogs(z_0) **[Brown, Levin '11]**	 all z_j — elliptic MZVs **[Enriquez '13]**		 all z_j — modular graph forms **[DGGV '15, DG '16]**

Also at genus one, $\exists$ evidence / proposal for sv map: open $\rightarrow$ closed strings

[Brown '14/'17; Zerbini '15; Brödel, OS, Zerbini '18; Gerken, Kleinschmidt, OS '18/'20; Panzer '18; Zagier, Zerbini '19; Gerken, Kleinschmidt, Mafra, OS, Verbeek '20]

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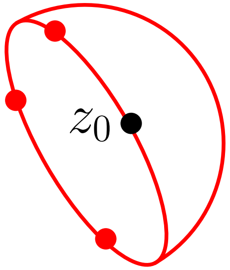
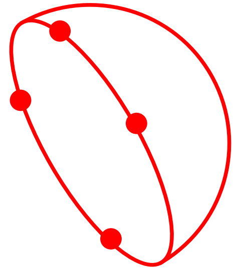
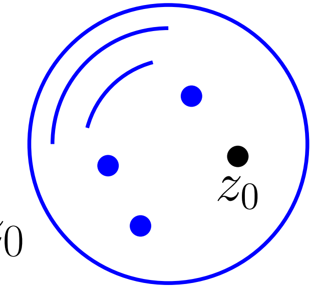
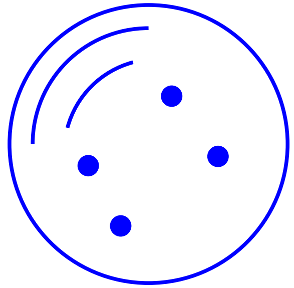
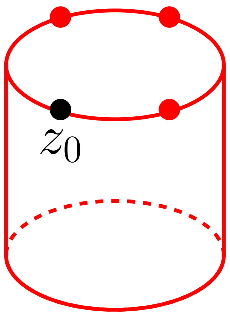
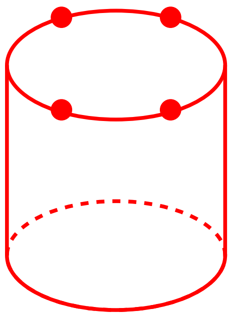
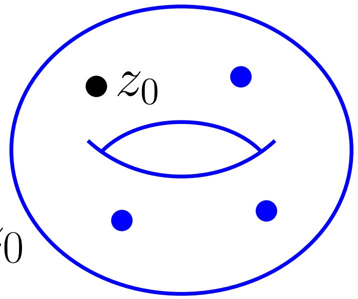
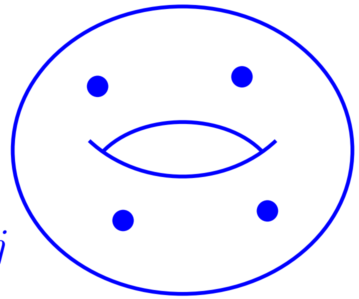
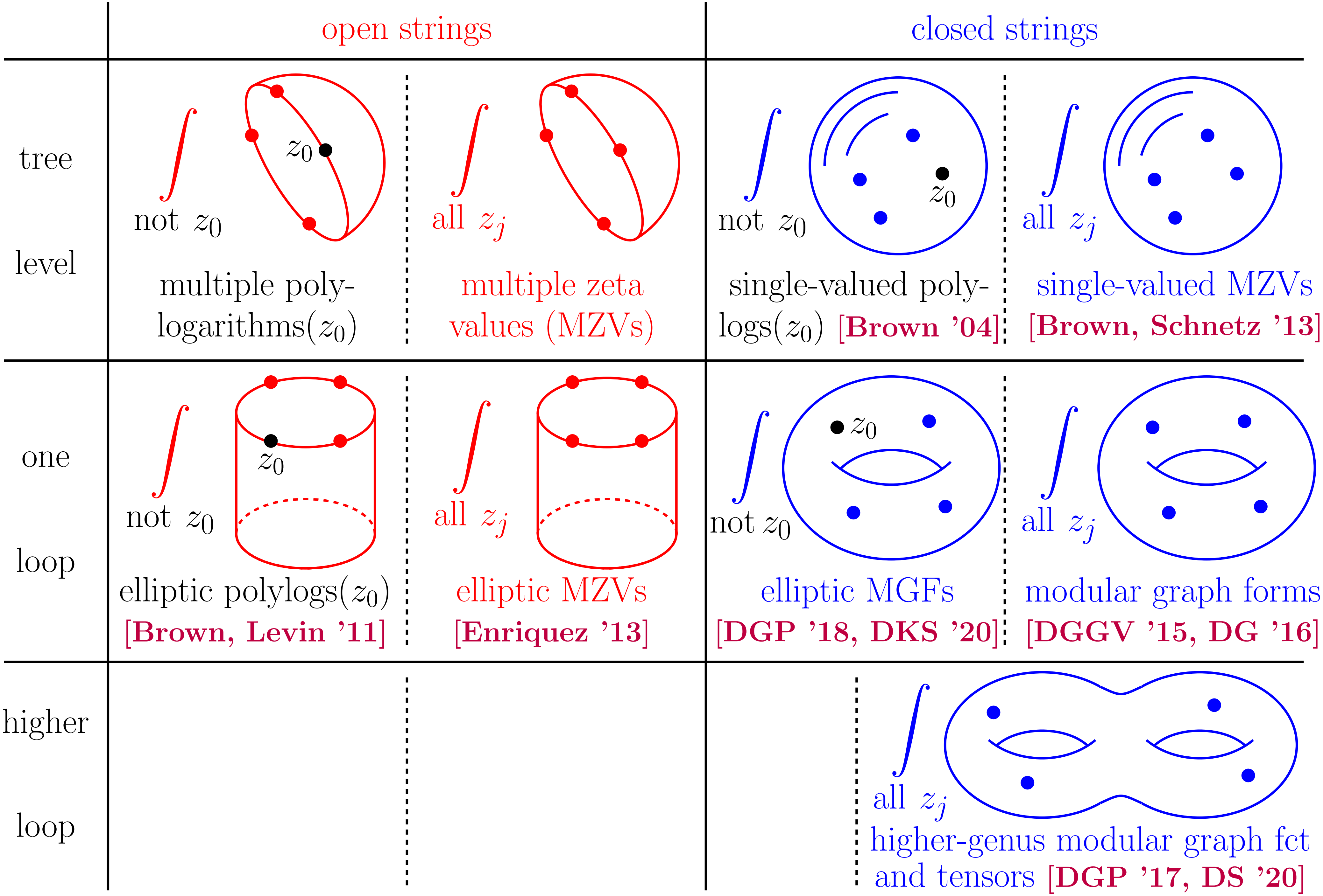
tree level	$\int$ not z_0  multiple poly-logarithms(z_0)	$\int$ all z_j  multiple zeta values (MZVs)	$\int$ not z_0  single-valued poly-logs(z_0) **[Brown '04]**	$\int$ all z_j  single-valued MZVs **[Brown, Schnetz '13]**
one loop	$\int$ not z_0  elliptic polylogs(z_0) **[Brown, Levin '11]**	$\int$ all z_j  elliptic MZVs **[Enriquez '13]**	$\int$ not z_0  elliptic MGFs **[DGP '18, DKS '20]**	$\int$ all z_j  modular graph forms **[DGGV '15, DG '16]**


open strings
closed strings
tree
level
not z_0
z_0
multiple poly-
logarithms(z_0)
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single-valued poly-
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[Brown, Schnetz '13]
one
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z_0
elliptic MGFs
[DGP '18, DKS '20]
all z_j
modular graph forms
[DGGV '15, DG '16]
higher
loop
all z_j
higher-genus modular graph fct
and tensors [DGP '17, DS '20]

Outline

I. Genus-zero recap

II. Elliptic polylogarithms and MZVs

[Brown, Levin 1110.6917; Brödel, Mafra, Matthes, OS 1412.5535]

III. Modular graph forms (MGFs)

[D'Hoker, Gürdogan, Green, Vanhove 1512.06779; D'Hoker, Green 1603.00839]

[Gerken, Kleinschmidt, OS 2004.05156]

IV. Elliptic modular graph forms

[D'Hoker, Green, Pioline 1806.02691; D'Hoker, Kleinschmidt, OS 2012.09198]

[D'Hoker, Hidding, Kleinschmidt, OS, Verbeek: to appear]

V. Conclusions & Outlook

I. Genus-zero recap

I. 1 Multiple polylogarithms and multiple zeta values

From the Green function on the sphere $\mathcal{G}_{S^2}(z_i, z_j) \sim \log|z_i - z_j|$,

obtain multiple polylogarithms and MZVs upon iterated integration

$$G(\underbrace{a_1, a_2, \ldots, a_w}_{\text{say } a_i \in \{0,1\}}; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_w; t)$$

with (transcendental) weight w and $G(\emptyset; z) = 1$ and

$$\zeta_{n_1,n_2,\ldots,n_r} = \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r} \qquad (\text{with } n_r \geq 2)$$

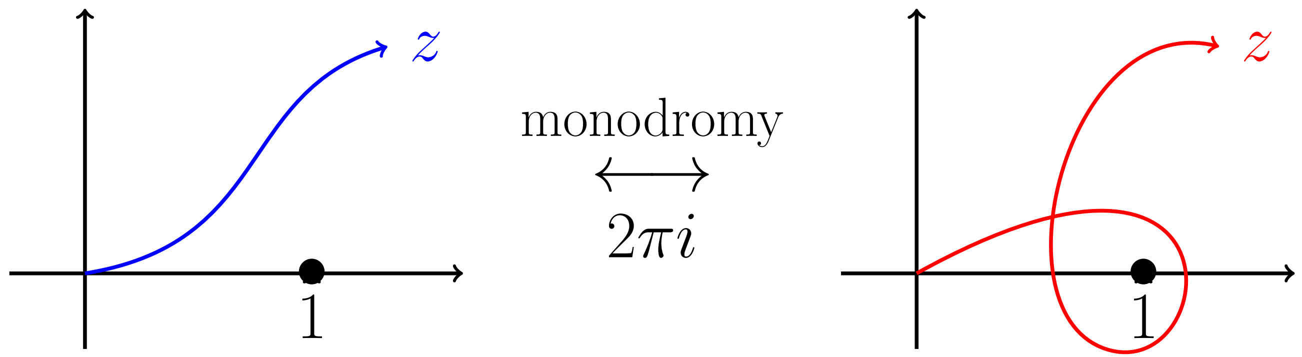
$$= (-1)^r G(\underbrace{0,0\ldots0,1}_{n_r}, \ldots, \underbrace{0,0\ldots0,1}_{n_2}, \underbrace{0,0\ldots0,1}_{n_1}; z{=}1)$$

Assign regularized values such as $G(0; z) = \log(z)$ compatible with

"shuffle-multiplication", e.g. $G(a; z)G(b; z) = G(a, b; z) + G(b, a; z)$.

I. 2 Single-valued polylogarithms: construction

Polylogarithms are notoriously multivalued under monodromies:



$G(1;z) = \log(1-z)$ $\qquad$ $G(1;z) + 2\pi i$

Can cancel monodromies by adding complex conjugates

$$\text{e.g.} \quad G^{\mathrm{SV}}(1;z) = \log(1-z) + \log(1-\bar{z}) = \log|1-z|^2$$

while preserving (only) the *holomorphic* differential equations

$$\partial_z G(a_1, a_2, \ldots, a_w; z) = \frac{1}{z-a_1} G(a_2, \ldots, a_w; z) \qquad \text{“meromorphic”}$$

$$\partial_z G^{\mathrm{SV}}(a_1, a_2, \ldots, a_w; z) = \frac{1}{z-a_1} G^{\mathrm{SV}}(a_2, \ldots, a_w; z) \qquad \text{“single-valued”}$$

I. 2 Single-valued polylogarithms: construction

Explicit construction of single-valued polylogarithms via generating series

$$G^{\mathrm{SV}}(a_1,a_2,\ldots,a_w;z)=\sum_{j=0}^{w}G(a_1,a_2,\ldots,a_j;z)\overline{G(a_w,a_{w-1},\ldots,a_{j+1};z)}$$

$$+\ \text{corrections}\ \zeta_{\ldots}G(\ldots;z)\overline{G(\ldots;z)}\ @\ w\geq 4$$

[Brown 2004]

e.g. $G^{\mathrm{SV}}(0,0,1,1;z)=G(0,0,1,1;z)+G(0,0,1;z)\overline{G(1;z)}$

$$+\,G(0,0;z)\overline{G(1,1;z)}+G(0;z)\overline{G(1,1,0;z)}+\overline{G(1,1,0,0;z)}+2\zeta_3\,\overline{G(1;z)}$$

I. 2 Single-valued polylogarithms: construction

Explicit construction of single-valued polylogarithms via generating series

$$G^{\rm SV}(a_1,a_2,\ldots,a_w;z) = \sum_{j=0}^{w} G(a_1,a_2,\ldots,a_j;z)\overline{G(a_w,a_{w-1},\ldots,a_{j+1};z)}$$

$$+ \text{corrections } \zeta_{\ldots} G(\ldots;z)\overline{G(\ldots;z)} \text{ @ } w \geq 4$$

[Brown 2004]

e.g. $G^{\rm SV}(0,0,1,1;z) = G(0,0,1,1;z) + G(0,0,1;z)\overline{G(1;z)}$

$$+ G(0,0;z)\overline{G(1,1;z)} + G(0;z)\overline{G(1,1,0;z)} + \overline{G(1,1,0,0;z)} + 2\zeta_3\,\overline{G(1;z)}$$

Define single-valued MZVs (svMZVs) via evaluation $G^{\rm SV}(\ldots;z{=}1)$

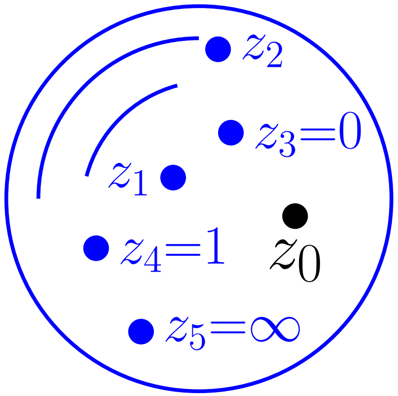
$$\zeta^{\rm SV}_{n_1,n_2,\ldots,n_r} = (-1)^r G^{\rm SV}(\underbrace{0,0\ldots0,1}_{n_r},\ldots,\underbrace{0,0\ldots0,1}_{n_2},\underbrace{0,0\ldots0,1}_{n_1};z{=}1)$$

e.g. $\zeta^{\rm SV}_{2k} = 0\,, \quad \zeta^{\rm SV}_{2k+1} = 2\zeta_{2k+1}\,, \quad \zeta^{\rm SV}_{3,5} = -10\zeta_3\zeta_5$ **[Brown, Schnetz '13]**

I. 3 Single-valued polylogarithms: application

Single-valued polylogarithms in multiple variables @ closed-string tree level

$$\int_{\substack{z_1,z_2 \\ \text{not } z_0}}$$



$$\prod_{0\leq i<j}^{4} |z_i - z_j|^{s_{ij}} \Longrightarrow \begin{cases} \mathbb{Q}[\text{svMZV}]\times G^{\text{sv}}(\ldots;z_0) \\ \text{@ each order in } \alpha' \sim s_{ij} \end{cases}$$

Final integration over $z_0 \Longrightarrow$ svMZV in α'-expansion of closed-string trees!

In fact, closed strings from single-valued map $\zeta \rightarrow \zeta^{\text{sv}}$ of open-string data

[OS, Stieberger '12; Stieberger '13; Stieberger, Taylor '14; OS, Schnetz '18; Brown, Dupont '18 & '19; Vanhove, Zerbini '18]

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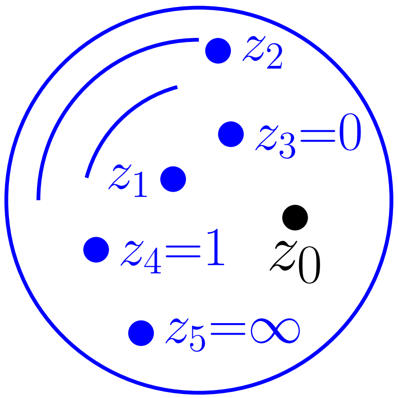

$$\prod_{0\leq i<j}^{4} |z_i - z_j|^{s_{ij}} \Longrightarrow \begin{cases} \mathbb{Q}[\text{svMZV}] \times G^{\text{sv}}(\ldots;z_0) \\ @ \text{ each order in } \alpha' \sim s_{ij} \end{cases}$$

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[OS, Stieberger '12; Stieberger '13; Stieberger, Taylor '14; OS, Schnetz '18; Brown, Dupont '18 & '19; Vanhove, Zerbini '18]

Also in multi-Regge kinematics of $\mathcal{N}=4$ SYM amplitudes, prominent appearance of $G^{\text{sv}}(a_1,\ldots,a_w;z)$, also in more variables $a_j \neq 0,1$!

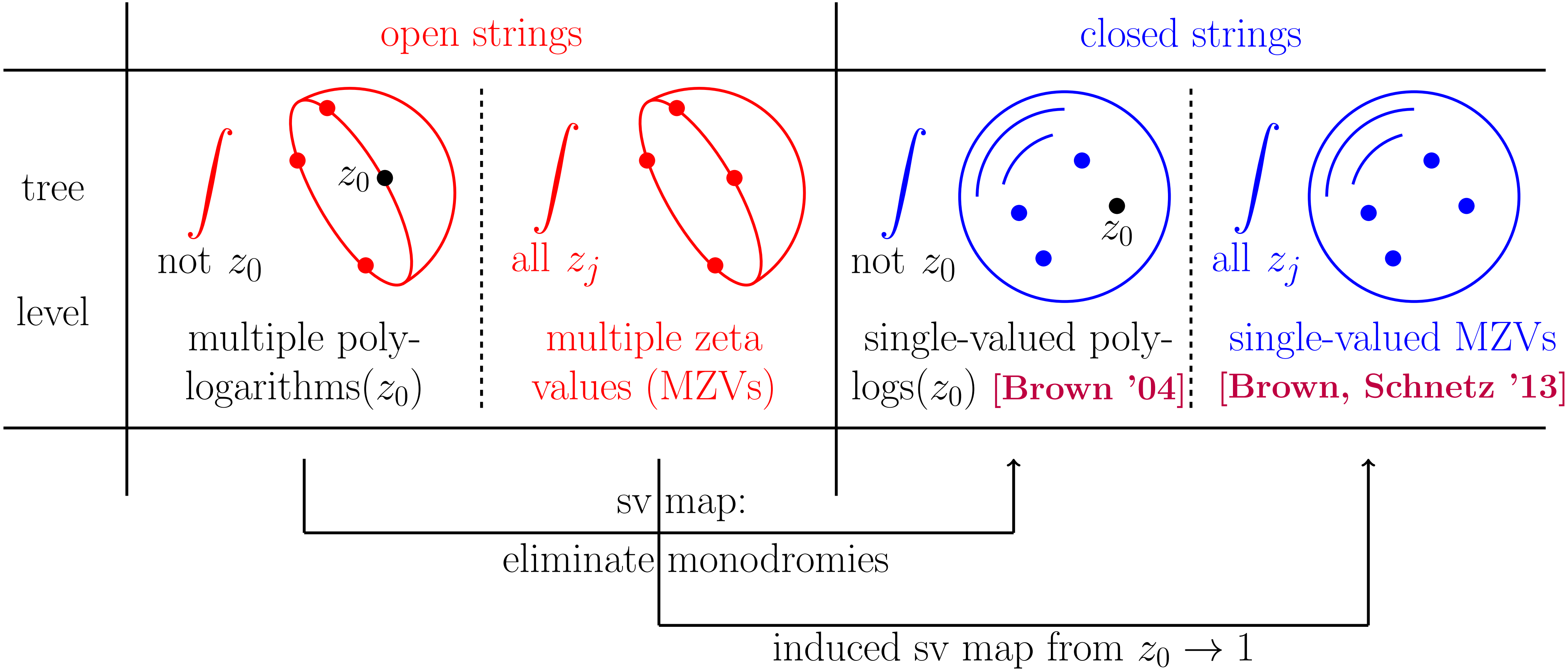
[Dixon, Duhr, Pennington '12, 13; Broedel, Sprenger '15, 16]

[DelDuca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek '16-19]

I. 4 Genus-zero summary

Moral: In order to integrate *all* punctures z_j in string tree-level amplitudes,

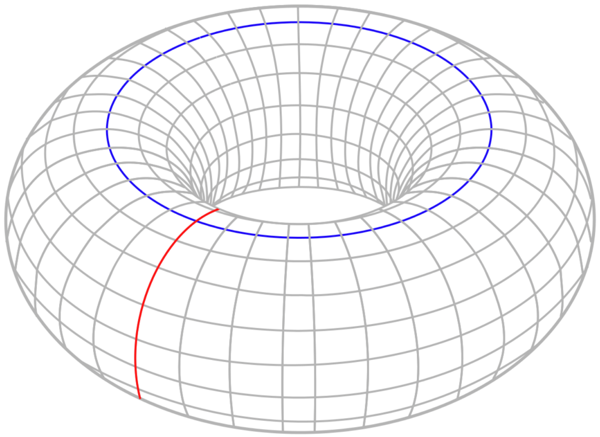
it is essential to know about intermediate results with z_0 unintegrated

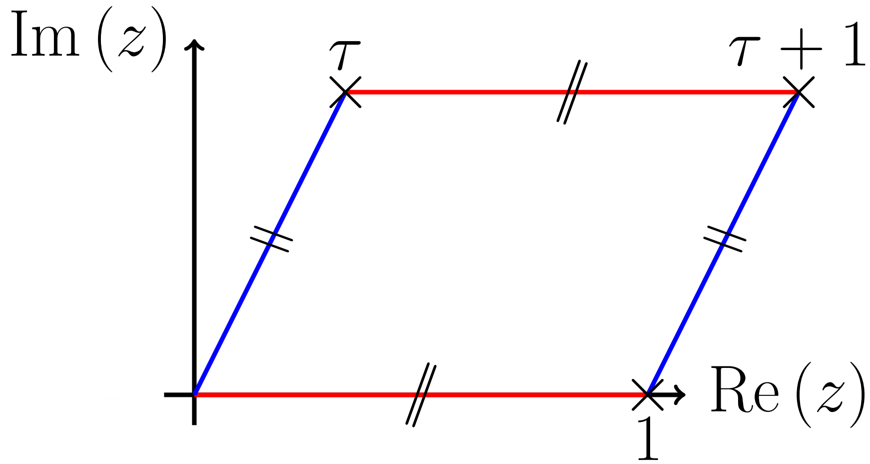


II. Elliptic polylogarithms and MZVs

II. 1 Basics of functions on a torus

Now study (meromorphic) interated integrals on a torus $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ @ $\mathrm{Im}\,\tau > 0$





Kernels $\frac{1}{z-a}$ of genus-zero polylogs generalize to ∞ many $f^{(k=0,1,2,\ldots)}(z-a,\tau)$

subject to periodicities $f^{(k)}(z+1,\tau) = f^{(k)}(z,\tau) = f^{(k)}(z+\tau,\tau)$

and modularity $f^{(k)}(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f^{(k)}(z,\tau)$,

i.e. weight $(k,0)$ under modular group $\mathrm{SL}_2(\mathbb{Z}) \ni \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$

II. 2 Kronecker-Eisenstein series

Generating function of kernels $f^{(k)}(z-a,\tau)$ that generalize $\frac{1}{z-a}$:

non-holomorphic Kronecker-Eisenstein series

$$\exp\left(2\pi i\eta\,\frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}\right)\frac{\theta_1'(0,\tau)\theta_1(z+\eta,\tau)}{\theta_1(z,\tau)\theta_1(\eta,\tau)}=\sum_{k=0}^{\infty}\eta^{k-1}f^{(k)}(z,\tau)$$

with the odd Jacobi theta function ($q=e^{2\pi i\tau}$)

$$\theta_1(z,\tau)=2q^{1/8}\sin(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi iz}q^n)(1-e^{-2\pi iz}q^n)$$

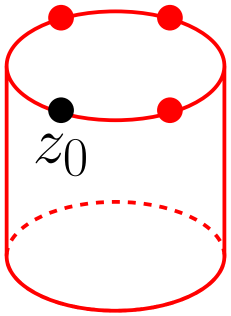
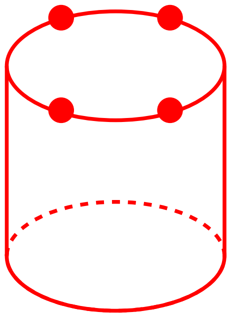
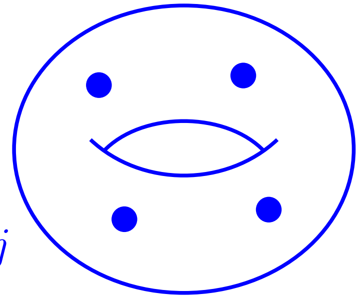
[Brown, Levin 1110.6917]

Simplest examples $f^{(0)}(z,\tau)=1$ and $f^{(1)}(z,\tau)=\partial_z\log\theta_1(z,\tau)+2\pi i\frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}$

such that $f^{(1)}(z,\tau)=\frac{1}{z}+\mathcal{O}(z,\bar{z})$ and all $f^{(k\neq 1)}$ are non-singular on $\mathbb{C}$.

Note: non-holomorphicity as a price for double-periodicity.

II. 3 Elliptic polylogarithms and MZV: construction

	open strings		closed strings	
one loop	$\int_{\text{not } z_0}$  elliptic polylogs(z_0) **[Brown, Levin '11]**	$\int_{\text{all } z_j}$  elliptic MZVs **[Enriquez '13]**		$\int_{\text{all } z_j}$ 

Open strings at one loop $\Longrightarrow$ elliptic polylogarithms $\Gamma(\ldots; z, \tau)$

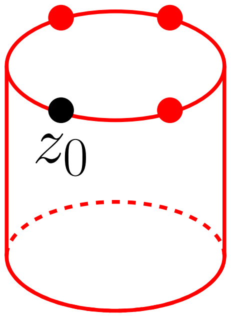
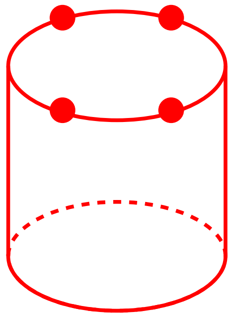
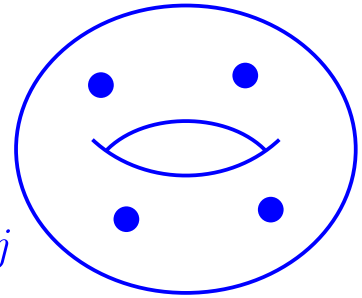
restricted to A-cycle $z \in (0,1) \equiv$ cylinder boundary

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ a_1 & a_2 & \ldots & a_r \end{smallmatrix} ; z, \tau \right) = \int_0^z \mathrm{d}t \, f^{(n_1)}(t-a_1, \tau) \, \Gamma\left(\begin{smallmatrix} n_2 & \ldots & n_r \\ a_2 & \ldots & a_r \end{smallmatrix} ; t, \tau \right)$$

ubiquitous in state-of-the-art evaluations of Feynman integrals

[e.g. Bloch, Kerr, Vanhove; Brödel, Duhr, Dulat, Penante, Tancredi; Abreu, Adams, Bogner, Chaubey, Marzucca, Müller-Stach, Walden, Weinzierl etc.]

II. 3 Elliptic polylogarithms and MZV: construction

	open strings		closed strings	
one loop	$\int$ not z_0  elliptic polylogs(z_0) **[Brown, Levin '11]**	$\int$ all z_j  elliptic MZVs **[Enriquez '13]**		$\int$ all z_j 

Open strings at one loop $\Longrightarrow$ elliptic polylogarithms $\Gamma(\ldots;z,\tau)$

restricted to A-cycle $z \in (0,1) \equiv$ cylinder boundary

$$\Gamma\big({}^{n_1\ n_2\ \ldots\ n_r}_{a_1\ a_2\ \ldots\ a_r} ;z,\tau\big) = \int_0^z \mathrm{d}t\, f^{(n_1)}(t-a_1,\tau)\, \Gamma\big({}^{n_2\ \ldots\ n_r}_{a_2\ \ldots\ a_r} ;t,\tau\big)$$

Evaluation at $z=1$ $\Longrightarrow$ (A-cycle) elliptic multiple zeta values (eMZVs)

$$\omega(n_1,n_2,\ldots,n_r|\tau) = \Gamma\big({}^{n_r\ \ldots\ n_2\ n_1}_{0\ \ \ldots\ \ 0\ \ \ 0} ;z{=}1,\tau\big)$$

II. 4 Elliptic polylogarithms and string amplitudes

Appearance of $\Gamma(\ldots;z,\tau)$ in one-loop open-string amplitude

$$\mathcal{A}_{\rm cyl}(\{1,2,\ldots,n\}) \sim \int_{\mathcal{M}_{1;n}} \exp\Bigg(\sum_{1\leq i<j}^{n} s_{ij}\, \mathcal{G}_{\rm cyl}(z_i,z_j|\tau)\Bigg) \times \left(\begin{matrix} \text{theory-} \\ \text{dependent} \end{matrix} \right)$$

- cylinder Green function → elliptic polylogs & eMZV

$$\mathcal{G}_{\rm cyl}(z_i,z_j|\tau) = \omega(1,0|\tau) - \Gamma\Big({}^{1}_{z_j} \,;z_i,\tau\Big) - \Gamma\Big({}^{1}_{0} \,;z_j,\tau\Big)$$

- theory-dependent factor is a polynomial in Kronecker-Eisenstein kernels

$f^{(n)}(z_i{-}z_j,\tau)$ and holomorphic Eisenstein series $\mathrm{G}_k(\tau)$

$$\mathrm{G}_k(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^k} = -f^{(k)}(z{=}0,\tau)$$

(true for bosonic and supersymmetric strings & het. strings on torus)

[**Brödel, Mafra, Matthes, OS 1412.5535; Gerken, Kleinschmidt, OS 1811.02548**]

II. 5 Elliptic MZVs as iterated Eisenstein integrals

Instead of iterated z_j-integral representation inherited from $\Gamma(\ldots; z, \tau)$

$$\omega(n_1, n_2, \ldots, n_r|\tau) = \int\limits_{0<z_1<\ldots<z_r<1} \mathrm{d}z_1\, f^{(n_1)}(z_1,\tau)\, \mathrm{d}z_2\, f^{(n_2)}(z_2,\tau)\, \ldots\, \mathrm{d}z_r\, f^{(n_r)}(z_r,\tau)$$

can write eMZVs as iterated τ_j-integrals over holo. Eisenstein series

$$\text{eMZVs} \quad \leftrightarrow \quad \mathbb{Q}[\text{MZV}] \int_{\tau}^{i\infty} \mathrm{d}\tau_1\, \mathrm{G}_{k_1}(\tau_1)\tau_1^{j_1} \int_{\tau_1}^{i\infty} \mathrm{d}\tau_2\, \mathrm{G}_{k_2}(\tau_2)\tau_2^{j_2} \int_{\tau_2}^{i\infty} \ldots$$

[Enriquez 1301.3042; Brödel, Matthes, OS 1507.02254]

- expose all relations among eMZVs (with coefficients in $\mathbb{Q}[\text{MZV}]$)
- powers of τ_i bounded by $0 \leq j_\ell \leq k_\ell - 2 \Longrightarrow$ good modular properties
- tangential base point regularization $\int_{\tau}^{i\infty} \tau_\ell^j \mathrm{d}\tau_\ell = \frac{-\tau^{j+1}}{j+1}$

  for zero-mode contribution $\mathrm{G}_k(\tau) = 2\zeta_k + \mathcal{O}(q)$ **[Brown 1407.5167]**

III. Modular graph forms (MGFs)

III. 1 MGFs as discretized Feynman integrals on torus

Expose double-periodicity $z \cong z+1 \cong z+\tau$ of functions on a torus via

double Fourier expansion in comoving coord's $u, v \in (0,1)$ of $z = u\tau+v$

- Kronecker-Eisenstein coefficients

$$f^{(k)}(z,\tau) = -\sum_{(m,n)\neq(0,0)} \frac{e^{2\pi i(nu-mv)}}{(m\tau+n)^k}$$

- torus Green function $\mathcal{G}(z,\tau) = \mathcal{G}_{T^2}(z,0|\tau)$

$$\mathcal{G}(z,\tau) = \frac{\operatorname{Im}\tau}{\pi} \sum_{(m,n)\neq(0,0)} \frac{e^{2\pi i(nu-mv)}}{|m\tau+n|^2}$$

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$$f^{(k)}(z,\tau) = -\sum_{p\in\Lambda'} \frac{e^{2\pi i \langle p,z\rangle}}{p^k}$$

- torus Green function $\mathcal{G}(z,\tau) = \mathcal{G}_{T^2}(z,0|\tau)$

$$\mathcal{G}(z,\tau) = \frac{\operatorname{Im}\tau}{\pi} \sum_{p\in\Lambda'} \frac{e^{2\pi i \langle p,z\rangle}}{|p|^2}$$

Gather $m, n \in \mathbb{Z}$ in discrete torus momentum $p = m\tau+n$ on lattice

$\Lambda' = (\mathbb{Z} + \tau\mathbb{Z}) \setminus \{0\}$ and abbreviate exponents via $\langle p, z\rangle = nu - mv$

III. 2 Modular graph forms from closed strings @ 1 loop

Four-point closed-string amplitude at one loop (gravitons in type IIA/B)

$$\mathcal{A}_{T^2}(\{1,2,3,4\}) = \left|s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\right|^2 \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^2} \, J(s_{ij},\tau)$$

$$J(s_{ij},\tau) = \left(\prod_{j=1}^{4} \int_{T^2} \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau} \right) \exp\left(\sum_{i<j}^{4} s_{ij} \, \mathcal{G}(z_i - z_j, \tau) \right)$$

modular invariant

[Brink, Green, Schwarz 1982]

- fund. domain $\mathcal{F}$ of modular group $\mathrm{SL}_2(\mathbb{Z})$ and torus $T^2 = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$
- as before: Fourier expansion of the Green function

$$\mathcal{G}(z,\tau) = \frac{\operatorname{Im}\tau}{\pi} \sum_{p\in\Lambda'} \frac{e^{2\pi i \langle p,z\rangle}}{|p|^2}$$

- coeff's in α'-expansion of $J(s_{ij},\tau)$ are dubbed modular graph forms

[Green, Vanhove 9910056; Green, Russo, Vanhove 0801.0322]

[D'Hoker, Gürdogan, Green, Vanhove 1512.06779]

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modular invariant

[Brink, Green, Schwarz 1982]

- fund. domain $\mathcal{F}$ of modular group $\mathrm{SL}_2(\mathbb{Z})$ and torus $T^2 = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$
- disclaimer: α'-expanding $J(s_{ij},\tau)$ at fixed τ does not capture

  discontinuities $\sim \log(\alpha' s_{ij})$ of $\mathcal{A}_{T^2}$ from $\tau \to i\infty$,

  need separate expansion method for "non-analytic part"

[Green, Russo, Vanhove 0801.0322; D'Hoker, Green, Vanhove 1502.06698]

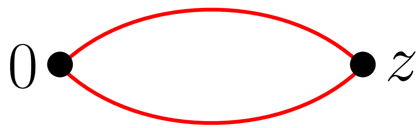
[D'Hoker, Green 1906.01652; Edison, Guillen, Johansson, OS, Teng 2107.08009]

III. 3 Simplest examples and relations of MGFs

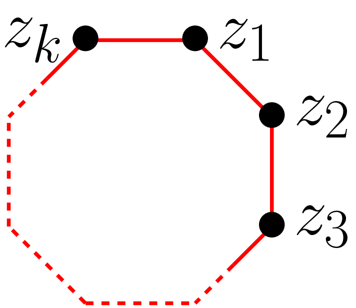
MGFs $\ni$ integrate polynomials in $\underbrace{\mathcal{G}(z_{ij}{=}z_i{-}z_j,\tau)}_{\text{edge } z_i\to z_j}$ over $\underbrace{z_1, z_2, \ldots \in T^2}_{\text{vertices}}$

- by the absence of zero-modes in $p \in \Lambda' = (\mathbb{Z}{+}\tau\mathbb{Z}) \setminus \{0\}$, 1-particle reducible graphs vanish $\int \mathrm{d}^2 z\, \mathcal{G}(z,\tau) = 0$, so simplest nonzero MGF is

$$\int \frac{\mathrm{d}^2 z}{\operatorname{Im}\tau} \mathcal{G}(z,\tau)^2 = \left(\frac{\operatorname{Im}\tau}{\pi}\right)^2 \sum_{p\in\Lambda'} \frac{1}{|p|^4}$$



- more generally, 1-loop graphs on $T^2 \Rightarrow$ non-holo. Eisenstein series E_k

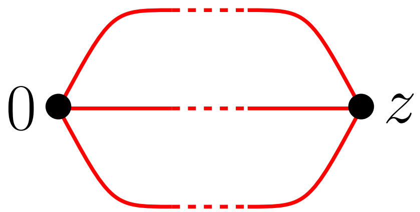


$$\int \left(\prod_{j=1}^{k} \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau}\right) \mathcal{G}(z_{12},\tau)\mathcal{G}(z_{23},\tau)\ldots\mathcal{G}(z_{k1},\tau)$$

$$= \left(\frac{\operatorname{Im}\tau}{\pi}\right)^k \sum_{p\in\Lambda'} \frac{1}{|p|^{2k}} = \mathrm{E}_k(\tau)$$

III. 3 Simplest examples and relations of MGFs

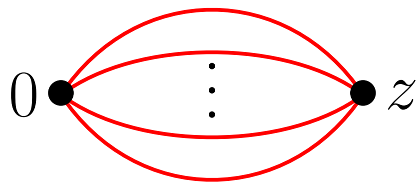
Beyond 1-loop graphs on the torus, get nested lattice sums, e.g.



$$\leftrightarrow \quad C_{a,b,c}(\tau) = \left(\frac{\operatorname{Im}\tau}{\pi}\right)^{a+b+c} \sum_{p_1,p_2,p_3\in\Lambda'} \frac{\delta(p_1+p_2+p_3)}{|p_1|^{2a}|p_2|^{2b}|p_3|^{2c}}$$

Higher-loop graphs often simplify to lower loop and MZVs, e.g.

$$\int \frac{\mathrm{d}^2 z}{\operatorname{Im}\tau}\mathcal{G}(z,\tau)^3 = C_{1,1,1}(\tau) = \mathrm{E}_3(\tau) + \zeta_3$$



$$\int \frac{\mathrm{d}^2 z}{\operatorname{Im}\tau}\mathcal{G}(z,\tau)^4 = 24C_{2,1,1}(\tau) - 18\mathrm{E}_4(\tau) + 3\mathrm{E}_2(\tau)^2$$

$$\int \frac{\mathrm{d}^2 z}{\operatorname{Im}\tau}\mathcal{G}(z,\tau)^5 = 60C_{3,1,1}(\tau) + 10\mathrm{E}_2(\tau)C_{1,1,1}(\tau) - 48\mathrm{E}_5(\tau) + 16\zeta_5$$

[Zagier '08; D'Hoker, Green, Vanhove '15; D'Hoker, Green '16; D'Hoker, Kaidi '16]

Problem: How to anticipate such relations?

What is the set of independent MGFs (over $\mathbb{Q}[\mathrm{MZV}]$)?

III. 4 MGFs from iterated Eisenstein integrals

Recall: eMZV relations are exposed by holo. iterated Eisenstein integrals

$$\text{eMZVs} \quad \leftrightarrow \quad \mathbb{Q}[\text{MZV}] \int_\tau^{i\infty} d\tau_1 \, \mathrm{G}_{k_1}(\tau_1)\tau_1^{j_1} \int_{\tau_1}^{i\infty} d\tau_2 \, \mathrm{G}_{k_2}(\tau_2)\tau_2^{j_2} \int_{\tau_2}^{i\infty} \ldots$$

Repeat strategy for MGFs: $\mathcal{G}(z,\tau)$ is real analytic $\Rightarrow$ also need cplx. conj.

depth one: for $j = 0, 1, 2, \ldots, k-2$ and $k \geq 4$, define

$$\beta^{\text{sv}}\Big[\begin{smallmatrix} j \\ k \end{smallmatrix}; \tau\Big] = \frac{(2\pi i)^{-1}}{(4\pi \text{Im}\,\tau)^{k-2-j}} \Big\{ \int_\tau^{i\infty} d\tau_1 \, (\tau - \tau_1)^{k-2-j} (\bar\tau - \tau_1)^j \mathrm{G}_k(\tau_1) - \int_{\bar\tau}^{-i\infty} d\bar\tau_1 \, (\tau - \bar\tau_1)^{k-2-j} (\bar\tau - \bar\tau_1)^j \overline{\mathrm{G}_k(\tau_1)} \Big\}$$

$\longrightarrow$ recover non-holomorphic Eisenstein series via

$$\mathrm{E}_k(\tau) = \frac{(2k-1)!}{(k-1)!^2} \Big\{ -\beta^{\text{sv}}\Big[\begin{smallmatrix} k-1 \\ 2k \end{smallmatrix}; \tau\Big] + \underbrace{\frac{2\zeta_{2k-1}}{(2k-1)(4\pi \text{Im}\,\tau)^{k-1}}}_{\text{mod. invariant completion of } \beta^{\text{sv}}} \Big\}$$

III. 4 MGFs from iterated Eisenstein integrals

depth two: mimic genus-zero formula for sv polylogs at weight two

$$G^{\rm SV}(a_2,a_1;z) = G(a_2,a_1;z) + G(a_2;z)\overline{G(a_1;z)} + \overline{G(a_1,a_2;z)}$$

$$\beta^{\rm sv}\Big[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix} ;\tau \Big] = \text{“}\zeta - \text{corrections”} + \frac{(2\pi i)^{-2}}{(4\pi {\rm Im}\,\tau)^{k_1+k_2-4-j_1-j_2}}$$

$$\times \bigg\{ \int^{i\infty}_{\tau} {\rm d}\tau_2 \, (\tau{-}\tau_2)^{k_2-2-j_2} (\bar\tau{-}\tau_2)^{j_2} {\rm G}_{k_2}(\tau_2) \int^{i\infty}_{\tau_2} {\rm d}\tau_1 \, (\tau{-}\tau_1)^{k_1-2-j_1} (\bar\tau{-}\tau_1)^{j_1} {\rm G}_{k_1}(\tau_1)$$

$$- \int^{i\infty}_{\tau} {\rm d}\tau_2 \, (\tau{-}\tau_2)^{k_2-2-j_2} (\bar\tau{-}\tau_2)^{j_2} {\rm G}_{k_2}(\tau_2) \int^{-i\infty}_{\bar\tau} {\rm d}\bar\tau_1 \, (\tau{-}\bar\tau_1)^{k_1-2-j_1} (\bar\tau{-}\bar\tau_1)^{j_1} \overline{{\rm G}_{k_1}(\tau_1)}$$

$$+ \int^{-i\infty}_{\bar\tau} {\rm d}\bar\tau_1 \, (\tau{-}\bar\tau_1)^{k_1-2-j_1} (\bar\tau{-}\bar\tau_1)^{j_1} \overline{{\rm G}_{k_1}(\tau_1)} \int^{-i\infty}_{\bar\tau_1} {\rm d}\bar\tau_2 \, (\tau{-}\bar\tau_2)^{k_2-2-j_2} (\bar\tau{-}\bar\tau_2)^{j_2} \overline{{\rm G}_{k_2}(\tau_2)} \bigg\}$$

$$\text{e.g.} \quad C_{2,1,1}(\tau) = \left(\frac{{\rm Im}\,\tau}{\pi} \right)^4 \sum_{p_1,p_2,p_3 \in \Lambda'} \frac{\delta(p_1+p_2+p_3)}{|p_1|^4 \, |p_2|^2 \, |p_3|^2}$$

$$= -18 \beta^{\rm sv}\big[\begin{smallmatrix} 2 & 0 \\ 4 & 4 \end{smallmatrix} ;\tau \big] + 12 \zeta_3 \beta^{\rm sv}\big[\begin{smallmatrix} 0 \\ 4 \end{smallmatrix} ;\tau \big] + \frac{5\zeta_5}{12\pi {\rm Im}\,\tau} - \frac{\zeta_3^2}{4\pi^2 {\rm Im}\,\tau^2} + \frac{9}{10} {\rm E}_4(\tau)$$

III. 4 MGFs from iterated Eisenstein integrals

Concerning the "ζ − corrections" in

$$\beta^{\rm sv}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix};\tau\right] = \text{“}\zeta - \text{corrections”} + \frac{(2\pi i)^{-2}}{(4\pi {\rm Im}\,\tau)^{k_1+k_2-4-j_1-j_2}}$$

$$\times \Big\{ \int_\tau^{i\infty} {\rm d}\tau_2\, (\tau-\tau_2)^{k_2-2-j_2} (\bar\tau-\tau_2)^{j_2} {\rm G}_{k_2}(\tau_2) \int_{\tau_2}^{i\infty} {\rm d}\tau_1\, (\tau-\tau_1)^{k_1-2-j_1} (\bar\tau-\tau_1)^{j_1} {\rm G}_{k_1}(\tau_1)$$

$$- \int_\tau^{i\infty} {\rm d}\tau_2\, (\tau-\tau_2)^{k_2-2-j_2} (\bar\tau-\tau_2)^{j_2} {\rm G}_{k_2}(\tau_2) \int_{\bar\tau}^{-i\infty} {\rm d}\bar\tau_1\, (\tau-\bar\tau_1)^{k_1-2-j_1} (\bar\tau-\bar\tau_1)^{j_1} \overline{{\rm G}_{k_1}(\tau_1)}$$

$$+ \int_{\bar\tau}^{-i\infty} {\rm d}\bar\tau_1\, (\tau-\bar\tau_1)^{k_1-2-j_1} (\bar\tau-\bar\tau_1)^{j_1} \overline{{\rm G}_{k_1}(\tau_1)} \int_{\bar\tau_1}^{-i\infty} {\rm d}\bar\tau_2\, (\tau-\bar\tau_2)^{k_2-2-j_2} (\bar\tau-\bar\tau_2)^{j_2} \overline{{\rm G}_{k_2}(\tau_2)} \Big\}$$

$\longrightarrow$ genus-1 analogue of the MZV corrections in $G^{\rm SV}$, e.g.

$$\beta^{\rm sv}\left[\begin{smallmatrix} 2 & 0 \\ 4 & 4 \end{smallmatrix};\tau\right] \Big|_{\zeta-{\rm corrections}} = \frac{-\zeta_3}{24\pi^3 {\rm Im}\,\tau} \int_{\bar\tau}^{-i\infty} {\rm d}\tau_1\, (\tau+\bar\tau-2\bar\tau_1)\big(\overline{{\rm G}_4(\tau_1)} - 2\zeta_4\big)$$

III. 4 MGFs from iterated Eisenstein integrals

Concerning the "ζ − corrections" in

$$\beta^{\rm sv}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix};\tau\right] = \text{“}\zeta - \text{corrections”} + \frac{(2\pi i)^{-2}}{(4\pi {\rm Im}\,\tau)^{k_1+k_2-4-j_1-j_2}}$$

$$\times \bigg\{ \int_\tau^{i\infty} {\rm d}\tau_2\, (\tau-\tau_2)^{k_2-2-j_2}(\bar\tau-\tau_2)^{j_2}{\rm G}_{k_2}(\tau_2) \int_{\tau_2}^{i\infty} {\rm d}\tau_1\, (\tau-\tau_1)^{k_1-2-j_1}(\bar\tau-\tau_1)^{j_1}{\rm G}_{k_1}(\tau_1)$$

$$- \int_\tau^{i\infty} {\rm d}\tau_2\, (\tau-\tau_2)^{k_2-2-j_2}(\bar\tau-\tau_2)^{j_2}{\rm G}_{k_2}(\tau_2) \int_{\bar\tau}^{-i\infty} {\rm d}\bar\tau_1\, (\tau-\bar\tau_1)^{k_1-2-j_1}(\bar\tau-\bar\tau_1)^{j_1}\overline{{\rm G}_{k_1}(\tau_1)}$$

$$+ \int_{\bar\tau}^{-i\infty} {\rm d}\bar\tau_1\, (\tau-\bar\tau_1)^{k_1-2-j_1}(\bar\tau-\bar\tau_1)^{j_1}\overline{{\rm G}_{k_1}(\tau_1)} \int_{\bar\tau_1}^{-i\infty} {\rm d}\bar\tau_2\, (\tau-\bar\tau_2)^{k_2-2-j_2}(\bar\tau-\bar\tau_2)^{j_2}\overline{{\rm G}_{k_2}(\tau_2)} \bigg\}$$

$\longrightarrow$ genus-1 analogue of the MZV corrections in $G^{\rm sv}$, e.g.

$$\beta^{\rm sv}\left[\begin{smallmatrix} 2 & 0 \\ 4 & 4 \end{smallmatrix};\tau\right] \Big|_{\zeta-\text{corrections}} = \frac{-\zeta_3}{24\pi^3 {\rm Im}\,\tau} \int_{\bar\tau}^{-i\infty} {\rm d}\bar\tau_1\, (\tau+\bar\tau-2\bar\tau_1)\Big(\overline{{\rm G}_4(\tau_1)} - 2\zeta_4\Big)$$

Key result: all MGFs are expressible via $\mathbb{Q}[{\rm MZV}, (\pi {\rm Im}\,\tau)^{-1}]$ combinations

of $\beta^{\rm sv}\left[\begin{smallmatrix} j_1 & j_2 & \dots & j_r \\ k_1 & k_2 & \dots & k_r \end{smallmatrix};\tau\right]$, exposing all algebraic relations and $q,\bar q$-expansions!

[Gerken, Kleinschmidt, OS 2004.05156]

III. 5 MGFs as single-valued eMZVs

Above real-analytic $\beta^{\rm SV}$ related to holo. iterated Eisenstein integrals

$$\beta\Big[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau\Big] = \frac{(2\pi i)^{-1}}{(-2\pi i\tau)^{k-j-2}} \int_\tau^{i\infty} \mathrm{d}\tau_1\, (\tau-\tau_1)^{k-j-2}(-\tau_1)^j \mathrm{G}_k(\tau_1)$$

obtained from formally setting $\bar\tau \to 0$ and $\overline{\mathrm{G}_k} \to 0$ in

$$\beta^{\rm SV}\Big[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau\Big] = \frac{(2\pi i)^{-1}}{(4\pi \mathrm{Im}\,\tau)^{k-2-j}} \bigg\{ \int_\tau^{i\infty} \mathrm{d}\tau_1\, (\tau-\tau_1)^{k-2-j}(\bar\tau-\tau_1)^j \mathrm{G}_k(\tau_1) \\ - \int_{\bar\tau}^{-i\infty} \mathrm{d}\bar\tau_1\, (\tau-\bar\tau_1)^{k-2-j}(\bar\tau-\bar\tau_1)^j \overline{\mathrm{G}_k(\tau_1)} \bigg\}$$

Same relation $\beta^{\rm SV}[\ldots] \to \beta[\ldots]$ at higher depth (without ζ-corrections)

Conversely, $\exists$ explicit proposal for single-valued map at genus 1 such that

$$\mathrm{SV}: \quad \tau \to \tau-\bar\tau\,, \qquad \beta[\ldots] \to \beta^{\rm SV}[\ldots]$$

[Gerken, Kleinschmidt, Mafra, OS, Verbeek 2010.10558]

III. 6 Modular graph forms with modular weight

Lattice sums with different holomorphic / antiholomorphic exponents

$$\frac{(\mathrm{Im}\,\tau)^{k+m}}{\pi^k} \sum_{p\in\Lambda'} \frac{1}{p^{k+m}\bar{p}^{k-m}} = \frac{(k-1)!}{(k+m-1)!} \nabla_\tau^m \mathrm{E}_k(\tau)$$

$\longrightarrow$ modular form of weight $(0,-2m)$, hence MGF = modular graph *form*

- from Maaß operators $\nabla_\tau = 2i(\mathrm{Im}\,\tau)^2\partial_\tau$ @ modular invariant MGFs

[D'Hoker, Green 1603.00839]

- from torus integrals over $f^{(n)}(z_{ij},\tau)$ and $\overline{f^{(n)}(z_{ij},\tau)}$ besides $\mathcal{G}(z_{ij},\tau)$

[Gerken, Kleinschmidt, OS 1811.02548]

III. 6 Modular graph forms with modular weight

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- from Maaß operators $\nabla_\tau = 2i(\mathrm{Im}\,\tau)^2\partial_\tau$ @ modular invariant MGFs

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- from torus integrals over $f^{(n)}(z_{ij},\tau)$ and $\overline{f^{(n)}(z_{ij},\tau)}$ besides $\mathcal{G}(z_{ij},\tau)$

[Gerken, Kleinschmidt, OS 1811.02548]

$\longrightarrow$ β^{sv} with more general $j=0,1,\ldots,k-2$, e.g.

$$(-4\pi\nabla_\tau)^m \mathrm{E}_k = \frac{(2k-1)!}{(k-1)!(k-1-m)!}\left\{-\beta^{\mathrm{sv}}\left[\begin{matrix} k-1+m \\ 2k \end{matrix}\right] + \frac{2\zeta_{2k-1}}{(2k-1)(4\pi\mathrm{Im}\,\tau)^{k-1-m}}\right\}$$

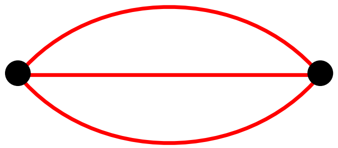
[Gerken, Kleinschmidt, OS 2004.05156]

III. 7 Counting modular graph forms

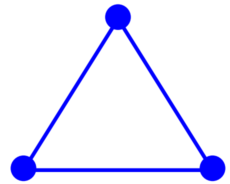
How many $\mathbb{Q}$[MZV] independent MGFs $\exists$ @ given $\underbrace{\text{transcendental weight } w}_{\text{roughly, \# of integrated } \mathcal{G}(z,\tau)}$?

- consider all partitions $w = k_1 + k_2 + \ldots$ with integers $k_j \geq 2$
- write down all $\beta^{\rm sv}\big[\begin{smallmatrix} j_1 & j_2 & \cdots & j_r \\ 2k_1 & 2k_2 & \ldots & 2k_r \end{smallmatrix}\big]$ subject to $\sum_{\ell=1}^r j_\ell = \sum_{\ell=1}^r (k_\ell - 1)$

  for modular invariance (other choices of $0 \leq j_\ell \leq 2k_\ell - 2$ cover their ∇_τ's)
- at $w=3$, for instance, only one admissible choice $\beta^{\rm sv}\big[\begin{smallmatrix} 2 \\ 6 \end{smallmatrix}\big]$, anticipating



$$\int \frac{d^2 z}{\operatorname{Im} \tau} \mathcal{G}(z,\tau)^3 = {\rm E}_3(\tau) + \zeta_3$$



- also products $\sim (\nabla_\tau {\rm E}_m)(\overline{\nabla}_\tau {\rm E}_n)$ of total mod. weight $(0,0)$ are counted
- starting from $w=7$, $\exists$ dropouts of certain $\beta^{\rm sv}$ from MGFs governed by

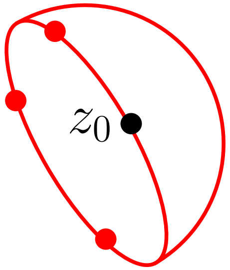
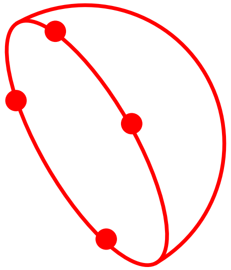
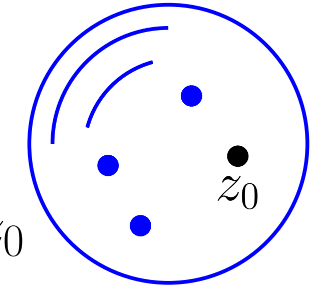
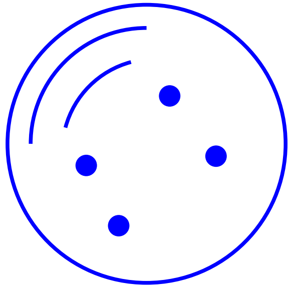
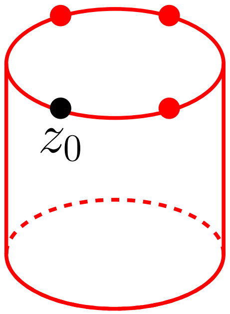
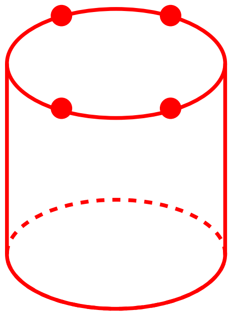
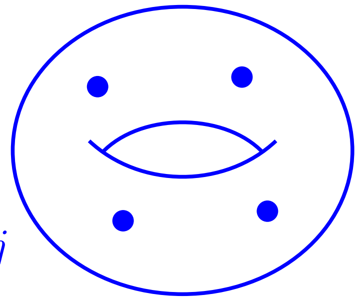
  rel's $[\epsilon_4, \epsilon_{10}] = 3[\epsilon_6, \epsilon_8]$ among generators $\{\epsilon_{k \in 2\mathbb{N}}\}$ **[Tsunogai, Pollack, . . .]**

[Gerken, Kleinschmidt, OS 2004.05156; Dorigoni, Kleinschmidt, OS 2109.05017/18]

IV. Elliptic modular graph forms

IV. 1 Elliptic MGFs and the bigger picture

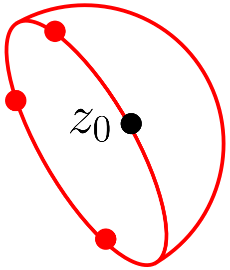
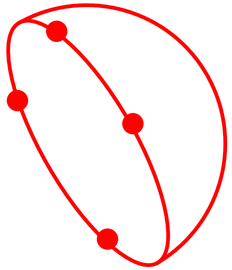
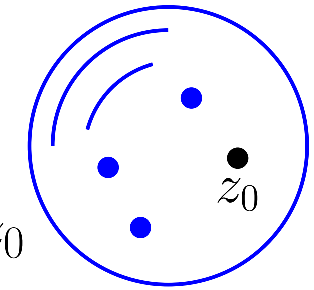
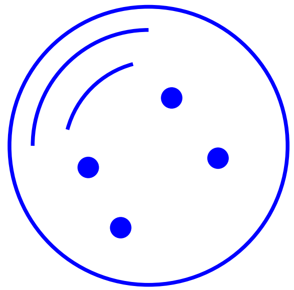
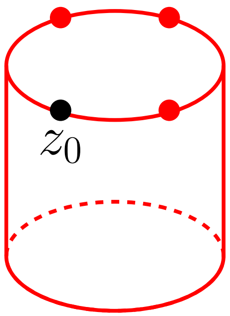
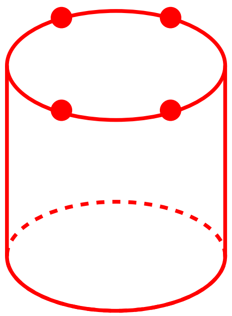
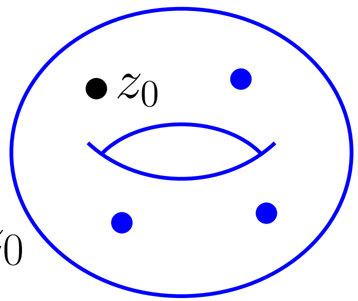
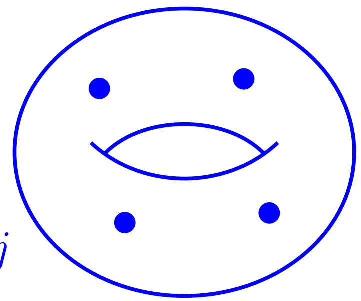
Back to the cast of characters

	open strings		closed strings	
tree level	$\int_{\text{not } z_0}$  multiple poly-logarithms(z_0)	$\int_{\text{all } z_j}$  multiple zeta values (MZVs)	$\int_{\text{not } z_0}$  single-valued poly-logs(z_0) **[Brown '04]**	$\int_{\text{all } z_j}$  single-valued MZVs **[Brown, Schnetz '13]**
one loop	$\int_{\text{not } z_0}$  elliptic polylogs(z_0) **[Brown, Levin '11]**	$\int_{\text{all } z_j}$  elliptic MZVs **[Enriquez '13]**		$\int_{\text{all } z_j}$  modular graph forms **[DGGV '15, DG '16]**

to be thought of as
single-valued eMZVs

IV. 1 Elliptic MGFs and the bigger picture

Back to the cast of characters

	open strings		closed strings	
tree level	$\int$ not z_0  multiple poly-logarithms(z_0)	$\int$ all z_j  multiple zeta values (MZVs)	$\int$ not z_0  single-valued poly-logs(z_0) **[Brown '04]**	$\int$ all z_j  single-valued MZVs **[Brown, Schnetz '13]**
one loop	$\int$ not z_0  elliptic polylogs(z_0) **[Brown, Levin '11]**	$\int$ all z_j  elliptic MZVs **[Enriquez '13]**	$\int$ not z_0  elliptic MGFs **[DGP '18, DKS '20]**	$\int$ all z_j  modular graph forms **[DGGV '15, DG '16]**

Elliptic MGFs: by analogy: single-valued elliptic polylogarithms

Modular graph forms: to be thought of as single-valued eMZVs

IV. 2 Zagier's single-valued elliptic polylogarithms

Simplest elliptic MGF is the torus Green function

$$\mathcal{G}(z,\tau) = \frac{\mathrm{Im}\,\tau}{\pi} \sum_{p\in\Lambda'} \frac{e^{2\pi i \langle p,z\rangle}}{|p|^2}\,, \qquad \langle p,z\rangle = nu - mv\,, \qquad \begin{cases} z = u\tau + v \\ p = m\tau + n \end{cases}$$

More general exponents of $p, \bar{p} \Rightarrow$ Zagier's single-valued elliptic polylogs

$$\mathcal{D}^+\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \frac{(\mathrm{Im}\,\tau)^a}{\pi^b} \sum_{p\in\Lambda'} \frac{e^{2\pi i\langle p,z\rangle}}{p^a\,\bar{p}^b}$$

[Zagier 1990]

- expressible via (meromorphic versions of) $\Gamma\left(\begin{smallmatrix} n_1 & \ldots & n_r \\ a_1 & \ldots & a_r \end{smallmatrix} ; z,\tau \right)$ & cplx. conj.

[Broedel, Kaderli 1906.11857]

- special choices of a, b yield

$$\mathcal{D}^+\begin{bmatrix} 1 \\ 1 \end{bmatrix}(z,\tau) = \mathcal{G}(z,\tau)\,, \qquad \mathcal{D}^+\begin{bmatrix} a \\ 0 \end{bmatrix}(z,\tau) = -(\mathrm{Im}\,\tau)^a f^{(a)}(z,\tau)$$

- evaluation at $z=0$ yields $\nabla_\tau^m \mathrm{E}_k \longrightarrow$ MGFs as single-valued eMZVs

[D'Hoker, Green, Gürdogan, Vanhove 1512.06779]

IV. 3 General definition and properties of elliptic MGFs

General elliptic MGFs $\rightarrow$ torus integrals of $f^{(k)}(z_{ij},\tau), \overline{f^{(k)}(z_{ij},\tau)}, \mathcal{G}(z_{ij},\tau)$

over *subsets* of the torus punctures $z_i, z_j, \ldots \Rightarrow$ fcts. of τ & unintegrated z's

- mod. forms transforming with $(c\tau{+}d)^r(c\bar\tau{+}d)^s$ under $(z,\tau) \rightarrow (\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d})$
- in general obtain nested lattice sums with insertions of characters $e^{2\pi i\langle p_j,z_k\rangle}$,

  e.g. simplest generalization of Zagier's $\mathcal{D}^+\begin{bmatrix} a \\ b \end{bmatrix}$ are

$$\mathcal{C}^+\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \frac{(\mathrm{Im}\,\tau)^{a_1+a_2+a_3}}{\pi^{b_1+b_2+b_3}} \sum_{p_1,p_2,p_3\in\Lambda'} \frac{\delta(p_1{+}p_2{+}p_3)\prod_{k=1}^3 e^{2\pi i\langle p_k,z_k\rangle}}{p_1^{a_1}\,\bar p_1^{b_1}\,p_2^{a_2}\,\bar p_2^{b_2}\,p_3^{a_3}\,\bar p_3^{b_3}}$$

- (repeated) z-derivatives yield $f^{(k)}(z_{ij},\tau)$ times simpler elliptic MGFs

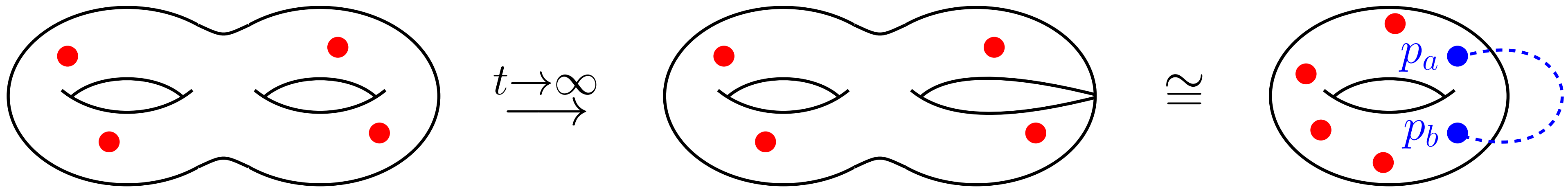
  $\longrightarrow$ view them as single-valued elliptic polylogs at arbitrary depth

[D'Hoker, Kleinschmidt, OS 2012.09198]

IV. 3 General definition and properties of elliptic MGFs

- elliptic MGFs firstly discussed in non-separating degeneration of genus-two MGFs ($z = p_b - p_a \leftrightarrow$ off-diagonal entry Ω_{12} of period matrix)

[D'Hoker, Green, Pioline 1806.02691]



- like MGFs, $\exists$ rich network of algebraic & differential rel's over $\mathbb{Q}$[MZV]
  - follow from degenerations of genus-two relations

[D'Hoker, Mafra, Pioline, OS 2008.08687]

  - growing body of genus-one techniques to derive relations

[Basu 2009.02221, 2010.08331; D'Hoker, Kleinschmidt, OS 2012.09198]

[D'Hoker, Hidding, Kleinschmidt, OS, Verbeek: to appear]

IV. 4 Iterated-integral representations of elliptic MGFs

τ-derivatives of elliptic MGFs at fixed u, v (rather than fixed $z = u\tau+v$)

$\longrightarrow$ $\mathrm{G}_k(\tau)$ or $f^{(k)}(u\tau+v,\tau)$ times simpler elliptic MGFs

$\Longrightarrow$ can find iterated-integral representations similar to $\beta^{\rm sv}$ for MGFs

Need additional kernels $f^{(k)}(u\tau+v,\tau)$ besides $\mathrm{G}_k(\tau)$, for instance

$$\beta^{\rm sv}\begin{bmatrix} j \\ k \\ z \end{bmatrix} = \frac{-(2\pi i)^{-1}}{(4\pi \mathrm{Im}\,\tau)^{k-2-j}} \bigg\{ \int_\tau^{i\infty} \mathrm{d}\tau_1\, (\tau-\tau_1)^{k-2-j} (\bar\tau-\tau_1)^j f^{(k)}(u\tau_1+v,\tau_1) - \int_{\bar\tau}^{-i\infty} \mathrm{d}\bar\tau_1\, (\tau-\bar\tau_1)^{k-2-j} (\bar\tau-\bar\tau_1)^j \overline{f^{(k)}(u\tau_1+v,\tau_1)} \bigg\}$$

reproduces Zagier's single-valued elliptic polylogs (integrating at fixed u, v)

$$\mathcal{D}^+\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = -\frac{(2i)^{b-a}(a+b-1)!}{(a-1)!(b-1)!}\,\beta^{\rm sv}\begin{bmatrix} a-1 \\ a+b \\ z \end{bmatrix}$$

[D'Hoker, Hidding, Kleinschmidt, OS, Verbeek: to appear]

IV. 4 Iterated-integral representations of elliptic MGFs

With similar definition for depth-two integrals $\beta^{\rm sv}\begin{bmatrix} j_1 & j_2 \\ k_1 & k_2 \\ z & z \end{bmatrix}$ or $\beta^{\rm sv}\begin{bmatrix} j_1 & j_2 \\ k_1 & k_2 \\ z & \end{bmatrix}$ & $\beta^{\rm sv}\begin{bmatrix} j_1 & j_2 \\ k_1 & k_2 \\ & z \end{bmatrix}$ in case of mixed kernels G_k and $f^{(k)}$ find canonical representations at higher depth such as

$$\left(\frac{\mathrm{Im}\,\tau}{\pi}\right)^4 \sum_{p_1,p_2,p_3\in\Lambda'} \frac{\delta(p_1{+}p_2{+}p_3)e^{2\pi i\langle p_1,z\rangle}}{|p_1|^4\,|p_2|^2\,|p_3|^2} = 14\beta^{\rm sv}\begin{bmatrix} 3 \\ 8 \end{bmatrix} - 140\beta^{\rm sv}\begin{bmatrix} 3 \\ 8 \\ z \end{bmatrix}$$
$$-18\beta^{\rm sv}\begin{bmatrix} 2 & 0 \\ 4 & 4 \\ & z \end{bmatrix} - 18\beta^{\rm sv}\begin{bmatrix} 2 & 0 \\ 4 & 4 \\ z & \end{bmatrix} + 18\beta^{\rm sv}\begin{bmatrix} 2 & 0 \\ 4 & 4 \\ z & z \end{bmatrix}$$
$$+ 12\zeta_3\beta^{\rm sv}\begin{bmatrix} 0 \\ 4 \\ z \end{bmatrix} - \frac{(u^2{-}u{+}\frac{1}{6})\zeta_5}{2\pi\mathrm{Im}\,\tau} - \frac{\zeta_7}{16(\pi\mathrm{Im}\,\tau)^3}$$

Again exposes counting of independent elliptic MGFs, all algebraic rel's and $q,\bar{q}$-expansions including powers of $q^u,\bar{q}^u$ with $z = u\tau{+}v$.

Conclusion & Outlook

- string amplitudes as natural habitat of polylogs on Riemann surfaces
- modular graph forms (MGFs) = rich family of non-holo. modular forms arising from integrating closed-string Green functions over the torus
- rewriting MGFs via iterated Eisenstein integrals & their cplx. conjugates $\Longrightarrow$ expose systematics of their relations / counting / q-expansions
- for computer algebra and introductory reading on MGFs, see

  [Mathematica package: Gerken 2007.05476, PhD thesis Gerken 2011.08647]
- work in progress: relate MGFs to Browns non-holo. modular forms

  [Brown 1407.5167, 1707.01230, 1708.03354]

and special thanks to BioNTtech!