Topological recursion and quantisation of spectral curves

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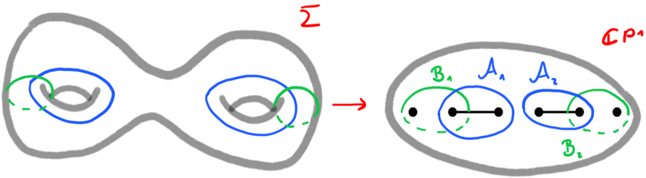
(based on joint work with B. Eynard, O. Marchal and N. Orantin)



Séminaire Darboux, LPTHE

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Outline

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1. Topological recursion and quantum curves
   - Topological recursion and its ramifications
   - Example: Witten's conjecture, Kontsevich's theorem and Airy
   - Quantum curves, history, context and examples
2. Spectral curves
3. Topological recursion and loop equations
4. Perturbative wave function and KZ equations
5. Non-perturbative wave functions KZ equations and Lax system
6. Link with isomonodromic systems
7. Questions and future work

Topological recursion (TR, Chekhov–Eynard–Orantin '04-'07)

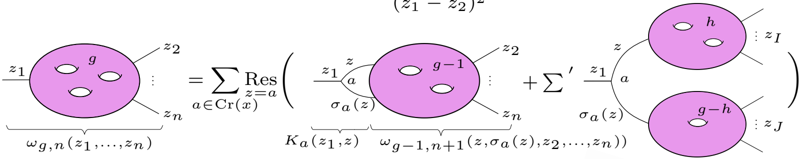
Goal: "Count surfaces $S_{g,n}$ of genus g with n boundaries (topology (g,n))."

Spectral curve

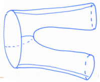
$$\mathbf{TR}: \begin{cases} \Sigma \text{ Riemann surface} \\ x\colon \Sigma \to \mathbb{CP}^1 \\ \omega_{0,1} = y\,dx \text{ 1-form} \quad (\text{discs}) \\ \omega_{0,2} \quad (1,1)\text{-form} \quad (\text{cylinders}) \end{cases} \xrightarrow[\substack{\textbf{recursion on} \\ |\chi(S_{g,n})| = 2g-2+n}]{} \begin{array}{l} \text{Differential forms} \\ \omega_{g,n}(z_1,\dots,z_n),\, z_i \in \Sigma, \\ \forall g,n \geq 0. \end{array}$$

- x finitely many simple ramification points $(\mathrm{Cr}(x))$ and y holomorphic around $a \in \mathrm{Cr}(x)$ and $dy(a) \neq 0 \Rightarrow$ Local involution σ around every ramification point: $x(z) = x(\sigma(z))$.
- $\omega_{0,2}$ symmetric bi-differential on $\Sigma \times \Sigma$ with only double poles along the diagonal and vanishing residues, that is when $z_1 \to z_2$

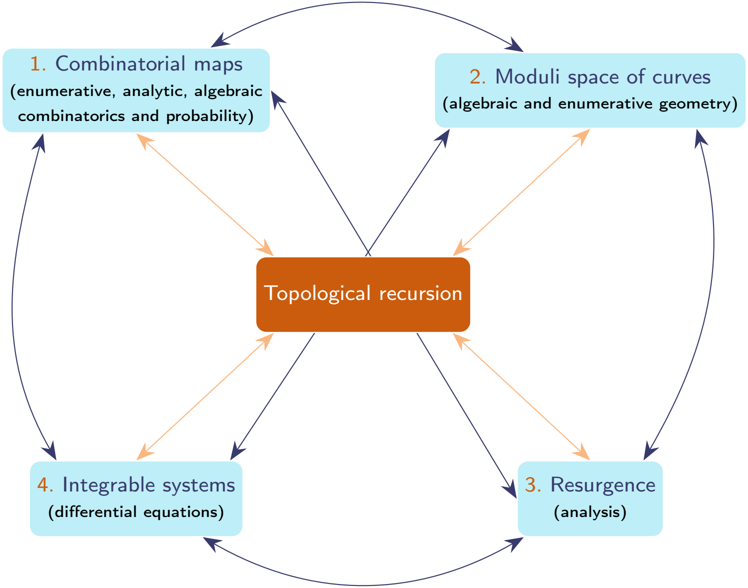
$$\omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2} + \overbrace{h(z_1,z_2)}^{\textbf{holomorphic}}.$$



- Terms in correspondence with the ways of cutting a **pair of pants** $(0,3)$ from $S_{g,n}$.



Connections



Properties and examples

- Interesting/powerful properties: $\omega_{g,n}$ are symmetric with poles at ramifications points, controlled deformations along families, dilaton equation, symplectic invariance, loop equations, modularity, integrability...

- For the Lambert curve $x = ye^{-y}$, TR provides simple Hurwitz numbers (Eynard–Mulase–Safnuk, '09, arXiv:0907.5224).
- For $y = \frac{-\sin(2\pi\sqrt{x})}{2\pi}$, TR gives Mirzakhani's recursion for Weil–Petersson volumes (of the moduli space of bordered hyperbolic surfaces), (Eynard–Orantin, '07, arXiv:0705.3600).
- TR on mirror curve of a toric CY3 computes its open Gromov–Witten theory (Bouchard–Klemm–Mariño–Pasquetti, '07, arXiv:0709.1453), (Fang–Liu–Zong, '16, arXiv:1604.07123).
- Chern–Simons theory on S^3 is governed by TR. Gopakumar–Ooguri–Vafa correspondence gives an A-model picture: GW of the resolved conifold, and B-model can be seen as TR on its Hori–Iqbal–Vafa mirror curve. (Brini, '17, hal-01474196).
- Statistical physics models on random maps: 1-hermitian matrix model, Ising model, Potts model, $O(\mathrm{n})$-loop model (Borot–Eynard, '09, arXiv:0910.5896), (Borot–Eynard–Orantin, '13, arXiv:1303.5808)...
- Semi-simple cohomological field theories and topological recursion (Dunin-Barkowski–Orantin–Shadrin–Spitz, '14, arXiv:1211.4021).
- Reconstruction of formal WKB expansions, integrability, isomonodromic systems (Borot–Eynard, '11, arXiv:1110.4936), (Eynard, '17, arXiv:1706.04938), (Eynard–G-F–Marchal–Orantin, '21, arXiv:2106.04339)...
- Conjecturally, for the A-polynomial of a knot as a spectral curve, TR computes the colored Jones polynomial of the knot (Borot–Eynard, '12, arXiv:1205.2261)).
- Extension to the non-perturbative world, resurgence theory: work in progress!

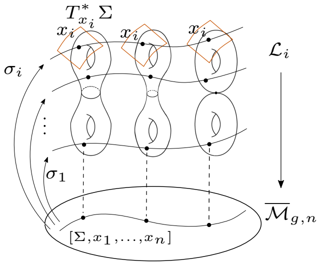
Outline

Moduli space of curves $\mathcal{M}_{g,n}$

For $g, n \geq 0$, with $2g - 2 + n > 0$, we define the moduli space:

$$\mathcal{M}_{g,n} := \left\{ \begin{array}{c} \text{curves of genus } g \text{ with } n \\ \text{marked points } x_1, \ldots, x_n \end{array} \right\} \Big/ \sim \ .$$

- $\overline{\mathcal{M}}_{g,n} \rightsquigarrow$ Deligne–Mumford compactification (including nodal curves).



$$\psi_i := c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}),$$
$$i = 1, \ldots, n.$$

Intersection numbers or correlators of psi classes:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \in \mathbb{Q},$$

which are zero unless $\sum_{i=1}^n d_i = \dim_{\mathbb{C}}(\overline{\mathcal{M}}_{g,n}) = 3g - 3 + n$.

Witten's conjecture

Generating series of intersection numbers of psi classes:

$$F(t_0, t_1, \ldots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(d_1,\ldots,d_n)} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle t_{d_1} \cdots t_{d_n}$$

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Conjecture: The series F satisfies the Korteweg–de Vries (KdV) hierarchy, the first equation of which is the classical KdV equation

$$\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3} \quad \left(U = \frac{\partial^2 F}{\partial t_0^2} \right),$$

and the string equation $\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i}$.

- Witten's motivation: Two different models of 2D quantum gravity should coincide.
- The conjecture uniquely determines F.

One explicit version of Witten's conjecture

Virasoro operators:

$$V_{-1} = -\frac{1}{2}\frac{\partial}{\partial t_0} + \frac{1}{2}\sum_{k=0}^{\infty} t_{k+1}\frac{\partial}{\partial t_k} + \frac{t_0^2}{4}, \quad V_0 = -\frac{3}{2}\frac{\partial}{\partial t_1} + \frac{1}{2}\sum_{k=0}^{\infty}(2k+1)t_k\frac{\partial}{\partial t_k} + \frac{1}{48},$$

and for $n > 0$,

$$V_n = -\frac{(2n+3)!!}{2}\frac{\partial}{\partial t_{n+1}} + \sum_{k=0}^{\infty}\frac{(2k+2n+1)!!}{2(2k-1)!!}t_k\frac{\partial}{\partial t_{k+n}} + \sum_{k_1+k_2=n-1}\frac{(2k_1+1)!!(2k_2+1)!!}{4}\frac{\partial^2}{\partial t_{k_1}\partial t_{k_2}}.$$

They satisfy the Virasoro relations:

$$[V_m, V_n] = (m-n)V_{m+n}.$$

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Theorem (equivalent to Witten's conjecture ('91))

For every integer $n \geq -1$, $V_n(\exp F) = 0$.

Witten's conjecture $\rightsquigarrow$ Kontsevich's theorem

1. Kontsevich maps and matrix model

TR ('07)

2. Intersection numbers
$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$$

3. Integrable hierarchy KdV

Witten's conjecture, '90

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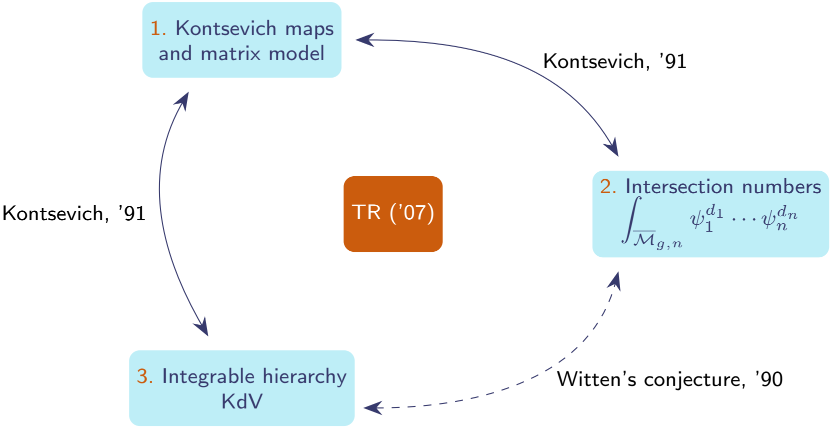

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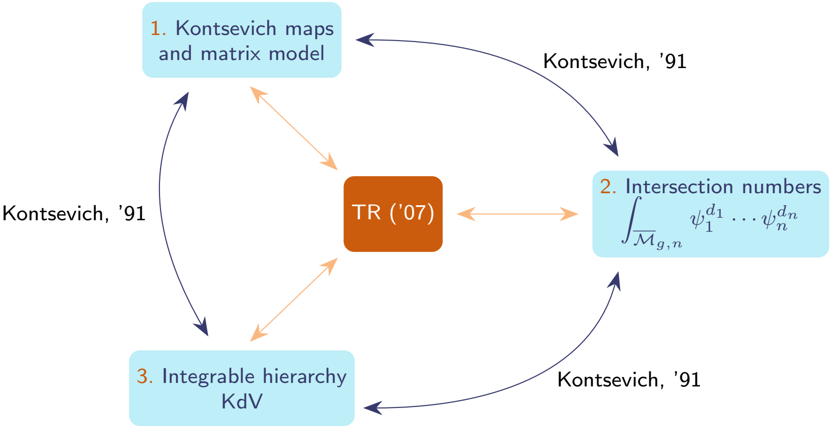

TR applied to the Airy curve $(x,y) = \left(\frac{z^2}{2}, z\right)$ produces

$$\omega_{g,n}(z_1,\ldots,z_n) = 2^{2-2g-n} \sum_{\sum_i d_i = 3g-3+n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) \prod_{i=1}^{n} \frac{(2d_i+1)!!\,dz_i}{z_i^{2d_i+2}}.$$

Airy differential equation

- Airy function $\mathsf{Ai}(\lambda) \rightsquigarrow \left(\frac{d^2}{d\lambda^2} - \lambda\right)\mathsf{Ai}(\lambda) = 0$. Asymptotic expansion (g.s. of intersection numbers), as $\lambda \to \infty$, of the form

$$\log \mathsf{Ai}(\lambda) - S_0(\lambda) - S_1(\lambda) = \sum_{m=2}^{\infty} S_m(\lambda),$$

where $S_0(\lambda) := -\frac{2}{3}\lambda^{\frac{3}{2}}$, $S_1(\lambda) := -\frac{1}{4}\log\lambda - \log(2\sqrt{\pi})$ and $\forall m \geq 2$

$$S_m(\lambda) := \frac{\lambda^{-\frac{3}{2}(m-1)}}{2^{m-1}} \sum_{\substack{h \geq 0\,,\, n>0 \\ 2h-2+n=m-1}} \frac{(-1)^n}{n!} \sum_{\mathbf{d}\in\mathbb{N}^n} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{h,n} \prod_{i=1}^{n} (2d_i - 1)!!\,.$$

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- Keep track of the Euler characteristics of the surfaces enumerated by introducing a formal parameter $\hbar$ through a rescaling of $\lambda \rightsquigarrow \psi^{\mathsf{Kont}}(\lambda, \hbar) := \mathsf{Ai}(\hbar^{-\frac{2}{3}}\lambda)$ satisfies

$$\left(\hbar^2 \frac{d^2}{d\lambda^2} - \lambda\right) \psi^{\mathsf{Kont}}(\lambda, \hbar) = 0$$

and admits an asymptotic expansion of the form

$$\log \psi^{\mathsf{Kont}}(\lambda, \hbar) - \hbar^{-1} S_0(\lambda) - S_1(\lambda) = \sum_{m=2}^{\infty} \hbar^{m-1} S_m(\lambda).$$

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- TR on the Airy spectral curve $y^2 - x = 0$ computes $Z^{\mathsf{Kont}}(\hbar,\mathbf{t})$ and $\psi^{\mathsf{Kont}}(\lambda,\hbar)$. The *quantum curve* $(\hbar^2 \frac{d^2}{d\lambda^2} - \lambda)\psi^{\mathsf{Kont}}(\lambda,\hbar) = 0$ can be constructed out of TR. Is this a general phenomenon?

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Presentation of the quantum curve conjecture

$P \in \mathbb{C}[x,y]$ and $\Sigma = \{(x,y) \in \mathbb{C}^2 \mid P(x,y) = 0\}$ plane curve of genus $\hat{g}$.

A quantization of Σ is a differential operator $\widehat{P}$ of the form

$$\widehat{P}(\widehat{x},\widehat{y};\hbar) = P_0(\widehat{x},\widehat{y}) + O(\hbar),$$

where $\widehat{x} = x\cdot$, $\widehat{y} = \hbar\frac{d}{dx}$, such that $P_0(x,y) = P(x,y)Q(x,y)$, for some $Q \in \mathbb{C}[x,y]$ (often 1).

- The operators $\widehat{x}$ and $\widehat{y}$ satisfy $[\widehat{y},\widehat{x}] = \hbar$.
- $\widehat{P}(\widehat{x},\widehat{y})\psi(x,\hbar) = 0$. Schrödinger equation: $\left(\hbar^2 \frac{d^2}{dx^2} - \widehat{R}(\widehat{x},\hbar)\right)\psi(z,\hbar) = 0$.

WKB asymptotic expansion $\rightsquigarrow \log\psi(x,\hbar) = \sum_{k\geq -1} \hbar^k S_k(x) \in \hbar^{-1}\mathbb{C}[[\hbar]]$.

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Question: Can we construct the operator $\widehat{P}$ and the solution ψ from P?

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Conjecture

Both $\widehat{P}$ and ψ can be constructed from Σ using topological recursion.

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Subtlety: We want $\widehat{P}$ to have a controlled pole structure, more precisely, to have the same pole structure as P.

First subtleties and comments

$$\widehat{P}(\widehat{x},\widehat{y})\psi(z,\hbar) = \left(\hbar^2 \frac{d^2}{dx^2} - \widehat{R}(\widehat{x},\hbar)\right)\psi(z,\hbar) = 0,$$

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- The differential equation is only satisfied on the plane curve $P(x,y)=0$.
- Higher order corrections in $\hbar$ are needed since $\left(\hbar\frac{d}{dx}\right)^2 \mapsto y^2 + O(\hbar)$ when acting on $\exp\left(\hbar^{-1}\int^z y dx\right)$.

History and literature

- Proved for many particular cases $\rightsquigarrow$ genus $\hat{g} = 0$ spectral curves.
- Bouchard–Eynard '17 $\rightsquigarrow$ spectral curves whose Newton polygon has $N_I := \#\{\text{interior points}\} = 0$ (Fact: $\hat{g} \leq N_I$).

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- Mariño–Eynard '08 $\rightsquigarrow$ Holomorphic, modular and background independent, non-perturbative partition functions.
- Borot–Eynard '12 $\rightsquigarrow$ Only non-perturbative wave functions can obey "good" quantum curves (for $\hat{g} > 0$).
- Eynard '17 $\rightsquigarrow$ General idea to construct integrable systems and their τ-functions from the geometry of the spectral curve.

History and literature

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- Eynard–GF–Marchal–Orantin '21 $\rightsquigarrow$ any algebraic curve with simple ramifications.

Beyond Airy: some meaningful generalisations

- $y^2 = x \rightsquigarrow$ Witten (conj) '90, Kontsevich '91, Airy, KW KdV tau function

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$$

$$\left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(z,\hbar) = 0$$

- $y^2 x = 1 \rightsquigarrow$ Norbury (conj) '17, [Chidambaram, Giacchetto, G-F, '22], Bessel, BGW KdV tau function

$$\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{d_1} \cdots \psi_n^{d_n}$$

$$\left(\hbar^2 \frac{d}{dx} x \frac{d}{dx} - 1\right)\psi(z,\hbar) = 0$$

- $y^r = x \rightsquigarrow$ Witten '93, Faber–Shadrin–Zvonkine, '10, rAiry, rKdV

$$\int_{\overline{\mathcal{M}}_{g,n}} W^r_{g,n}(a_1,\ldots,a_n) \psi_1^{d_1} \cdots \psi_n^{d_n}$$

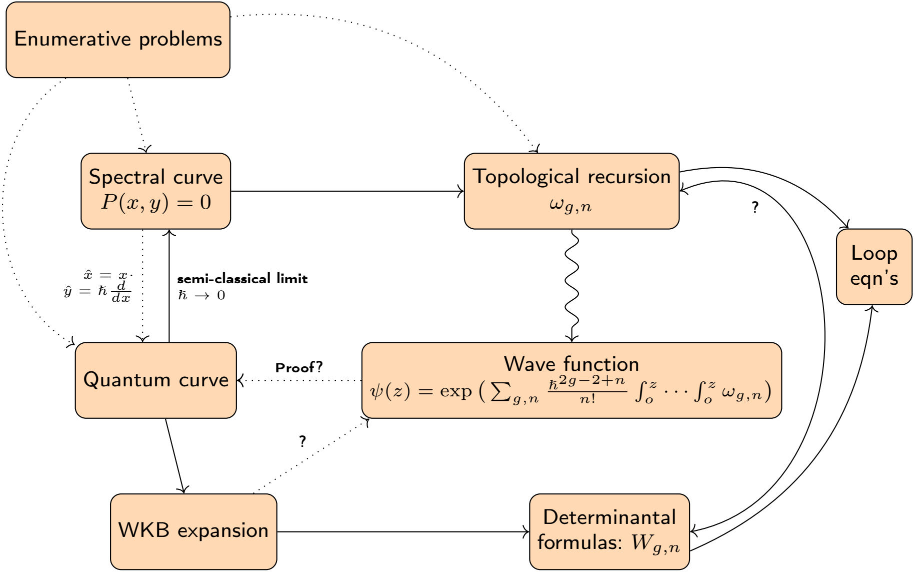
$$\left(\hbar^2 \frac{d^r}{dx^r} - x\right)\psi(z,\hbar) = 0$$

- $y^2 = x^3 + tx + V \rightsquigarrow$ **Painlevé I**, elliptic curve $(\hat{g} = 1)$

$$\int_{\overline{\mathcal{M}}_{g,n+m}} \psi_{n+1}^2 \cdots \psi_{n+m}^2 \psi_1^{d_1} \cdots \psi_n^{d_n}$$

$$\left(\hbar^2 \frac{d^2}{dx^2} - \left(x^3 + tx + V + \frac{\partial}{\partial t}\right)\right)\psi = 0$$

Context


Enumerative problems
Spectral curve
$P(x,y)=0$
Topological recursion
$\omega_{g,n}$
Loop eqn's
?
$\hat{x} = x \cdot$
$\hat{y} = \hbar \frac{d}{dx}$
semi-classical limit
$\hbar \to 0$
Quantum curve
Proof?
Wave function
$\psi(z) = \exp\left(\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int_o^z \cdots \int_o^z \omega_{g,n}\right)$
?
WKB expansion
Determinantal formulas: $W_{g,n}$

Outline

Input of topological recursion (TR) (Chekhov–Eynard–Orantin, '04-'07)

Input: *Spectral curve* $\mathcal{S} = (\Sigma, x, ydx, B)$:

- Σ Riemann surface of genus $\hat{g}$.
- Two meromorphic functions $x, y : \Sigma \to \mathbb{C} \Rightarrow P(x,y) = 0,\ P \in \mathbb{C}[x,y]$.
- Symplectic basis of non-contractible cycles $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{\hat{g}}$ on Σ.
- A symmetric bidifferential $B = \omega_{0,2}$ on $\Sigma \times \Sigma$ such that $\omega_{0,2}(z_1, z_2) \underset{z_2 \to z_1}{\sim} \frac{dz_1 dz_2}{(z_1 - z_2)^2}$ + holomorphic with vanishing $\mathcal{A}$-periods.

Output: $\omega_{\mathbf{g},\mathbf{n}}(\mathbf{z_1}, \ldots, \mathbf{z_n}) \in \mathrm{H}^0(\Sigma^n, (K_\Sigma(*\mathrm{Cr}(x)))^{\boxtimes n})^{\mathfrak{S}_n}$, for all $g, n \geq 0$.

Important properties: For $2g - 2 + n > 0$, the $\omega_{g,n}$ are symmetric meromorphic differentials with poles at ramifications points.

Regularity condition: $x : \Sigma \to \mathbb{C}$ meromorphic function with finitely many and simple ramification points (denoted $\mathcal{R}(x)$), and $y : \Sigma \to \mathbb{C}$ holomorphic on a neighborhood of every $a \in \mathcal{R}(x)$ and $dy(a) \neq 0 \Rightarrow$ Existence of a local involution σ around every ramification point: $x(z) = x(\sigma(z))$.

Spectral curves

N distinct points $\Lambda_1, \dots, \Lambda_N \in \mathbb{P}^1 \setminus \{\infty\}$. Let $\mathcal{H}_d(\Lambda_1, \dots, \Lambda_N, \infty)$ be the Hurwitz space of degree d ramified coverings $x\colon \Sigma \to \mathbb{P}^1$, where Σ is the Riemann surface:

$$\Sigma := \overline{\{(\lambda, y) \mid P(\lambda, y) = 0\}}$$

of genus $\hat{g}$, where $x(\lambda, y) := \lambda$ and

$$P(\lambda, y) = \sum_{l=0}^{d} (-1)^l y^{d-l} P_l(\lambda), \quad P_0(\lambda) = 1,$$

P_l being a rational function with possible poles at $\lambda \in \mathcal{P} := \{\Lambda_i\}_{i=1}^N \bigcup \{\infty\}$.

Classical spectral curve: $\rightsquigarrow (\Sigma, x)$.

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- Local coordinates (in the base): $\{\xi_q(\lambda)\}_{q \in \mathcal{P}}$ around $q \in \mathcal{P}$ are defined by

$$\forall\, i \in [\![1, N]\!] \,:\, \xi_{\Lambda_i}(\lambda) := (\lambda - \Lambda_i) \qquad \text{and} \qquad \xi_\infty(\lambda) := \lambda^{-1}.$$

- Local coordinates (in the cover): near any $p \in x^{-1}(q)$, let $d_p := \mathrm{ord}_p(\xi_q)$

$$\zeta_p(z) = \xi_q(x(z))^{\frac{1}{d_p}}.$$

$\{d_p\}_{p \in x^{-1}(q)}$ is called the ramification profile of q. We have $\sum_{p \in x^{-1}(P)} d_p = d$.

Admissible spectral curves

Expansion of the 1-form $\omega_{0,1} = ydx$ around any pole $p \in x^{-1}(\mathcal{P})$:

$$ydx = \sum_{k=0}^{r_p - 1} t_{p,k} \zeta_p^{-k-1} d\zeta_p + \text{analytic at } p.$$

The $t_{p,k}$'s are called the spectral times (or *KP times*).

Ramification points: $\mathcal{R}_0 := \left\{ p \in \Sigma \mid 1 + \mathrm{order}_p\, dx \neq \pm 1 \right\}$,

$$\mathcal{R} := \left\{ p \in \Sigma \mid dx(p) = 0 \, , \; x(p) \notin \mathcal{P} \right\} = \mathcal{R}_0 \setminus x^{-1}(\mathcal{P}).$$

Critical values: $x(\mathcal{R})$.

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Critical values: $x(\mathcal{R})$.

Definition (Admissible classical spectral curves)

A classical spectral curve (Σ, x) is *admissible* if:

- $P(\lambda, y) = 0$ is an irreducible algebraic curve;
- $a \in \mathcal{R}$ are simple, i.e. dx has only a simple zero at $a \in \mathcal{R}$;
- $\forall (a_i, a_j) \in \mathcal{R} \times \mathcal{R}$ with $a_i \neq a_j$, $x(a_i) \neq x(a_j)$;
- $\forall a \in \mathcal{R}$, $dy(a) \neq 0$;
- $\forall p \in x^{-1}(\mathcal{P})$ ramified, the 1-form ydx has a pole of degree $r_p \geq 3$ at p and $t_{p,r_p-2} \neq 0$.

Torelli marking and filling fractions

For any symplectic basis $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$ of $H_1(\Sigma, \mathbb{Z})$, let

$$B^{(\mathcal{A}_i,\mathcal{B}_i)_{i=1}^g} \in H^0(\Sigma^2, K_\Sigma^{\boxtimes 2}(2\Delta))^{\mathfrak{S}_2} \subset \mathcal{M}_2(\Sigma^2)$$

be the unique symmetric bidifferential on Σ^2 with a unique double pole on the diagonal Δ, without residue, bi-residue equal to 1 and normalized on the $\mathcal{A}$-cycles by

$$\forall\, i \in [\![1, g]\!]\,, \quad \oint_{z_1 \in \mathcal{A}_i} B^{(\mathcal{A}_i,\mathcal{B}_i)_{i=1}^g}(z_1, z_2) = 0.$$

Remark

Choice of Torelli marking can be thought of as a choice of polarisation from a geometric quantisation point of view.

Let $\left((\Sigma, x), (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g\right)$ be some admissible initial data. We define the tuple $(\epsilon_i)_{i=1}^g$ of *filling fractions* by

$$\forall\, i \in [\![1, g]\!]\,, \quad \epsilon_i := \frac{1}{2\pi i} \oint_{\mathcal{A}_i} y dx.$$

Outline

Properties of TR

- $\omega_{g,n}$ are invariant under permutations of their n arguments.
- $\omega_{0,1}(z_1)$ may only have poles at $x^{-1}(\mathcal{P})$. $\omega_{0,2}(z_1,z_2)$ may only have poles at $z_1 = z_2$. For $(h,n) \in \mathbb{N} \times \mathbb{N}^* \setminus \{(0,1),(0,2)\}$, $\omega_{h,n}(z_1,\dots,z_n)$ may only have poles at $z_i \in \mathcal{R}$, for $i \in [\![1,n]\!]$.
- For all $i \in [\![1,\hat{g}]\!]$,

$$\frac{\partial}{\partial \epsilon_i}\omega_{h,n}(z_1,\dots,z_n) = \oint_{z\in\mathcal{B}_i} \omega_{h,n+1}(z,z_1,\dots,z_n).$$

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Ramification points at poles:

- In the definition of TR, residues at $a \in \mathcal{R} = \mathcal{R}_0 \setminus x^{-1}(\mathcal{P})$.
- But the points of $\mathcal{P}$ could also be ramified (many interesting examples, like the Airy curve $y^2 = x$).
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Lemma (Ramified poles don't contribute for admissible curves)

Let $\omega'_{h,n}$ be the topological recursion differential forms defined by taking residues at all $a \in \mathcal{R}_0$ (including $a \in x^{-1}(\mathcal{P})$). If $\forall p \in x^{-1}(\mathcal{P})$, we have $r_p \geq 3$ and $t_{p,r_p-2} \neq 0$, then $\omega'_{h,n} = \omega_{h,n}$, and $\omega_{h,n}$ with $(h,n) \neq (0,1),(0,2)$ have poles only at $\mathcal{R} = \mathcal{R}_0 \setminus x^{-1}(\mathcal{P})$.

Loop equations

For $(h,n,l) \in \mathbb{N}^3$, $\lambda \in \mathbb{P}^1$ and $\mathbf{z} := (z_1,\dots,z_n) \in \Sigma^n$,

$$Q^{(l)}_{h,n+1}(\lambda;\mathbf{z}) := \sum_{\beta \underset{l}{\subseteq} x^{-1}(\lambda)} \sum_{\mu \in \mathcal{S}(\beta)} \sum_{\bigsqcup_{i=1}^{l(\mu)} J_i = \mathbf{z}} \sum_{\sum_{i=1}^{l(\mu)} g_i = h + l(\mu) - l} \left[\prod_{i=1}^{l(\mu)} \omega_{g_i,|\mu_i|+|J_i|}(\mu_i, J_i) \right],$$

differential with possible poles at $\lambda \in \mathcal{P} \cup x(\mathcal{R})$, $z_i \in \mathcal{R}$ and $z_i \in x^{-1}(\lambda)$.

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Particular cases:

- $Q^{(l)}_{0,1}(\lambda) = \sum_{\beta \underset{l}{\subseteq} x^{-1}(\lambda)} \prod_{z \in \beta} \omega_{0,1}(z) = P_l(\lambda)\,(d\lambda)^l$.
- $Q^{(l)}_{0,2}(\lambda; z_1) = \sum_{\beta \underset{l}{\subseteq} x^{-1}(\lambda)} \sum_{z \in \beta} \omega_{0,2}(z, z_1) \prod_{\substack{\tilde{z} \in \beta \\ \tilde{z} \neq z}} \omega_{0,1}(\tilde{z})$.

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Theorem (Loop equations)

The function $\lambda \mapsto \frac{Q^{(l)}_{h,n+1}(\lambda;\mathbf{z})}{(d\lambda)^l}$ *has no poles at* $\lambda \in x(\mathcal{R})$, $\forall \mathbf{z} \in (\Sigma \setminus \mathcal{R})^n$.

- $Q^{(1)}_{h,n+1}(\lambda;\mathbf{z}) = \sum_{z \in x^{-1}(\lambda)} \omega_{h,n+1}(z,\mathbf{z}) = \delta_{n,0}\delta_{h,0} P_1(\lambda) d\lambda + \delta_{n,1}\delta_{h,0} \frac{d\lambda\, dx(z_1)}{(\lambda - x(z_1))^2}$.

Loop equations

$$\hat{Q}^{(l)}_{h,n+1}(z;\mathbf{z}) := \sum_{\beta \underset{l}{\subseteq} \left(x^{-1}(x(z))\setminus\{z\}\right)} \sum_{\mu\in\mathcal{S}(\beta)} \sum_{\bigsqcup_{i=1}^{l(\mu)} J_i = \mathbf{z}} \sum_{\sum_{i=1}^{l(\mu)} g_i = h+l(\mu)-l} \prod_{i=1}^{l(\mu)} \omega_{g_i,|\mu_i|+|J_i|}(\mu_i, J_i)$$

Possible poles $\rightsquigarrow$ z with $x(z)\in x(\mathcal{R})$, $z\in x^{-1}(\mathcal{P})$, and $z_i \in \mathcal{R}\cup\left(x^{-1}\left(x(z)\right)\setminus\{z\}\right)$.

Lemma

For $\mathbf{z} := (z_1,\dots,z_n)\in\Sigma^n$ such that $x(z_i)\neq x(z_j)$ for any $i\neq j$, the functions

$$\widetilde{Q}^{(l)}_{h,n+1}(\lambda;\mathbf{z}) := \frac{Q^{(l)}_{h,n+1}(\lambda;\mathbf{z})}{(d\lambda)^l} - \sum_{j=1}^{n} d_{z_j}\left(\frac{1}{\lambda - x(z_j)}\frac{\hat{Q}^{(l-1)}_{h,n}(z_j;\mathbf{z}\setminus\{z_j\})}{(dx(z_j))^{l-1}}\right)$$

are rational functions of λ with no poles at $\lambda\in x(\mathcal{R})$ and at $\lambda\in\bigcup_{i=1}^{n}\{x(z_i)\}$.

For $z\in\Sigma\setminus\left(\mathcal{R}\bigcup x^{-1}(\mathcal{P})\right)$ and $\mathbf{z}\in\left[\Sigma\setminus\left(\mathcal{R}\bigcup x^{-1}(x(z))\right)\right]^n$, we have

$$Q^{(l)}_{h;n+1}(x(z);\mathbf{z}) = \hat{Q}^{(l)}_{h;n+1}(z;\mathbf{z}) + \hat{Q}^{(l-1)}_{h-1;n+2}(z;z,\mathbf{z}) + \sum_{A\sqcup B=\mathbf{z}} \sum_{h_1+h_2=h} \hat{Q}^{(l-1)}_{h_1,|A|+1}(z;A)\,\omega_{h_2,|B|+1}(z,B).$$

Outline

Perturbative wave function over a divisor

$D = \sum_{i=1}^{s} \alpha_i [p_i]$ a generic divisor (of degree$= \sum_i \alpha_i = 0$) on $\widetilde{\Sigma_{\mathcal{P}}}$, $\Sigma_{\mathcal{P}} := \Sigma \setminus x^{-1}(\mathcal{P})$.

Perturbative wave function $\psi(D,\hbar) = \psi_{0,i}(D,\hbar)$ associated to D:

$$\exp\left(\sum_{h\geq 0}\sum_{n\geq 0} \frac{\hbar^{2h-2+n}}{n!} \int_D \cdots \int_D \left(\omega_{h,n}(z_1,\dots,z_n) - \delta_{h,0}\delta_{n,2}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\right)\right).$$

$$e^{-\hbar^{-2}\omega_{0,0}} e^{-\hbar^{-1}\int_D \omega_{0,1}} \psi(D,\hbar) \in \mathbb{C}[[\hbar]].$$

Perturbative wave function over a divisor

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$$\exp\left(\sum_{h\geq 0}\sum_{n\geq 0}\frac{\hbar^{2h-2+n}}{n!}\int_D\cdots\int_D\left(\omega_{h,n}(z_1,\ldots,z_n)-\delta_{h,0}\delta_{n,2}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\right)\right).$$

$$e^{-\hbar^{-2}\omega_{0,0}}e^{-\hbar^{-1}\int_D \omega_{0,1}}\psi(D,\hbar) \in \mathbb{C}[[\hbar]].$$

$$\forall\, i \in [\![1, s]\!]\,,\ l \geq 1\ :\ \psi_{l,i}(D,\hbar) := \left[\sum_{h\geq 0}\sum_{n\geq 0}\frac{\hbar^{2h+n}}{n!}\overbrace{\int_D\cdots\int_D}^{n}\frac{\hat{Q}^{(l)}_{h,n+1}(p_i;\cdot)}{(dx(p_i))^l}\right]\psi(D,\hbar).$$

Perturbative partition function $Z(\hbar) = \psi(D = \emptyset, \hbar)$:

$$Z(\hbar) := \exp\left(\sum_{h\geq 0}\hbar^{2h-2}\omega_{h,0}\right), \text{ with } e^{-\hbar^{-2}\omega_{0,0}}Z(\hbar) \in \mathbb{C}[[\hbar]].$$

Remark

Wave functions are meant to be solutions to a differential equation; the partition function is expected to play the role of an associated tau function from the point of view of isomonodromic or integrable systems.

KZ equations

Loop equations $\Rightarrow$ Knizhnik–Zamolodchikov (KZ) equations:

Theorem (General KZ equations)

For $i \in [\![1,s]\!]$ *and* $l \in [\![0,d-1]\!]$,

$$\frac{\hbar}{\alpha_i}\frac{d\psi_{l,i}(D,\hbar)}{dx(p_i)} = -\psi_{l+1,i}(D,\hbar) - \hbar \sum_{j\in[\![1,s]\!]\setminus\{i\}} \alpha_j \frac{\psi_{l,i}(D,\hbar)-\psi_{l,j}(D,\hbar)}{x(p_i)-x(p_j)}$$

$$+\sum_{h\geq 0}\sum_{n\geq 0}\frac{\hbar^{2h+n}}{n!}\int_{z_1\in D}\cdots\int_{z_n\in D}\widetilde{Q}^{(l+1)}_{h,n+1}(x(p_i);\mathbf{z})\,\psi(D,\hbar)$$

$$+\left(\frac{1}{\alpha_i}-\alpha_i\right)\left[\sum_{(h,n)\in\mathbb{N}^2}\frac{\hbar^{2h+n+1}}{n!}\overbrace{\int_D\cdots\int_D}^{n}\frac{d}{dx(p_i)}\left(\frac{\hat{Q}^{(l)}_{h,n+1}(p_i;\cdot)}{(dx(p_i))^l}\right)\right]\psi(D,\hbar).$$

If $\alpha_i = \pm 1$,

$$\frac{\hbar}{\alpha_i}\frac{d\psi_{l,i}(D,\hbar)}{dx(p_i)} = -\psi_{l+1,i}(D,\hbar) - \hbar \sum_{j\in[\![1,s]\!]\setminus\{i\}} \alpha_j \frac{\psi_{l,i}(D,\hbar)-\psi_{l,j}(D,\hbar)}{x(p_i)-x(p_j)}$$

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Regularised KZ equations

Let $z \in \widetilde{\Sigma_{\mathcal{P}}}$ be a generic point and $x^{-1}(\infty) = \{\infty^{(\alpha)}\}_{\alpha \in [\![1,\ell_\infty]\!]}$.
When $D = [z] - [p_2]$, $\psi(D,\hbar)$ has an essential singularity as $p_2 \to \infty^{(\alpha)}$.
Need to regularise the wave functions: $\psi_l^{\text{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar)$.

Theorem (KZ equations for regularized wave functions)

For $\alpha \in [\![1,\ell_\infty]\!]$, $l \in [\![0, d-1]\!]$, the regularised wave functions satisfy

$$\hbar \frac{d}{dx(z)} \psi_l^{\text{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar) + \psi_{l+1}^{\text{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar)$$

$$= \Bigg[\sum_{h \geq 0} \sum_{n \geq 0} \frac{\hbar^{2h+n}}{n!} \sum_{P \in \mathcal{P}} \sum_{k \in S_P^{(l+1)}} \xi_P(x(z))^{-k} \operatorname*{Res}_{\lambda \to P} \xi_P(\lambda)^{k-1} \, d\xi_P(\lambda)$$

$$\int_{z_1 = \infty^{(\alpha)}}^{z_1 = z} \cdots \int_{z_n = \infty^{(\alpha)}}^{z_n = z} \frac{Q_{h,n+1}^{(l+1)}(\lambda; \mathbf{z})}{(d\lambda)^{l+1}} \Bigg] \psi^{\text{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar).$$

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- RHS of KZ equations uses residues, i.e. integrals.
- Can be re-written using generalised integrals, i.e. linear operators $\mathcal{I}_{\mathcal{C}_{p,k}}$.
- $\mathcal{I}_{\mathcal{C}_{p,k}}$ is expected to correspond to $\partial_{t_{p,k}}$. Valid for $d = 2$.
- Action of these operators defined only on a sub-algebra generated by $\int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_1} \omega_{h,n}$: **algebra of symbols**.
- Need to check that these operators never act on something else.
- Avoid the technicality of defining the action on all differentials on Σ.

Generalised cycles and algebra of symbols

Generalized cycles: $\mathcal{E} := \left\{\mathcal{C}_{p,k}\right\}_{p\in\Sigma, k\in\mathbb{Z}} \cup \left\{\mathcal{C}_o^p\right\}_{p\in\Sigma} \cup \{\mathcal{A}_i, \mathcal{B}_i\}_{i=1}^g$, where the integration of a meromorphic form ω along such cycles is defined as:

- $\forall\ p \in \Sigma$, and $\forall\ k \in \mathbb{Z}$,

$$\int_{\mathcal{C}_{p,k}} \quad : \quad \omega \mapsto \operatorname*{Res}_{p} \zeta_p^{-k}\, \omega\, .$$

- Let γ be a Jordan arc from a point $o \in \Sigma$ to a point $p \in \Sigma$.

$$\int_{\mathcal{C}_o^p} \quad : \quad \omega \mapsto \int_\gamma \omega$$

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Commutative algebra freely generated by a set of symbols consisting of a pair (h,n) and a symbol $\int_{C_1}\cdots\int_{C_n}$, labeled by generalised cycles $C_i \in \mathcal{E}$:

$$\check{\mathcal{W}} = \mathbb{C}\left[\left\{\int_{C_1}\cdots\int_{C_n}\omega_{h,n}\right\}_{h,n\geq 0}\right] \quad / \text{ (cycle linearity relations)}.$$

Evaluation map:

$$\begin{array}{rccc} \text{ev}: & \check{\mathcal{W}} & \to & \mathbb{C} \\ & \int_{C_1}\cdots\int_{C_n}\omega_{h,n} & \mapsto & \int_{z_1\in C_1}\cdots\int_{z_n\in C_n}\omega_{h,n}(z_1,\dots,z_n). \end{array}$$

$\mathcal{W} \rightsquigarrow$ extension to formal Laurent power series in $\hbar$, exponentials and inverses.

KZ equations with linear operators

Operators $(\mathcal{I}_C)_{C\in\mathcal{E}}$ acting on $\mathcal{W}$:

$$\forall\,(h,n)\in\mathbb{N}^2\;:\;\mathcal{I}_C\left[\int_{C_1}\cdots\int_{C_n}\omega_{h,n}\right]:=\int_{C_1}\cdots\int_{C_n}\int_C\omega_{h,n+1}.$$

Re-writing the RHS of the KZ equations with a multi-linear operator $\widetilde{\mathcal{L}}_l(x(z))$ that uses $\mathcal{I}_{\mathcal{C}_{p,k}}$ $\rightsquigarrow$ new system of KZ equations, for $\alpha\in[\![1,\ell_\infty]\!]$, $l\in[\![0,d-1]\!]$:

$$\hbar\frac{d}{dx(z)}\psi_l^{\mathrm{reg}}([z]-[\infty^{(\alpha)}])+\psi_{l+1}^{\mathrm{reg}}([z]-[\infty^{(\alpha)}])$$
$$=\mathrm{ev.}\;\widetilde{\mathcal{L}}_l(x(z))\left[\psi^{\mathrm{reg\ symbol}}([z]-[\infty^{(\alpha)}])\right].$$

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Degree 2 case (hyperelliptic):

$$\boxed{P(x,y)=R(x)-y^2=0,\ \text{with}\ R(x)\in\mathbb{C}(x)}$$

$x:\Sigma\to\mathbb{CP}^1$ is a double cover and we have a global involution

$$(x,y)\mapsto(x,-y).$$

Remark

In degree 2, the operators $\mathcal{I}_{\mathcal{C}_{p,k}}$ can be interpreted as derivatives with respect to the moduli of the classical spectral curve $\partial_{t_{p,k}}$.

KZ equations for $d = 2 \rightsquigarrow$ system of PDEs

Theorem (Eynard–GF,'19)

For $k = 1, 2$,

$$\hbar^2 \left(\frac{d^2}{dx_k^2} + \sum_{i \neq k} \frac{\frac{d}{dx_k} - \frac{d}{dx_i}}{x_k - x_i} \right) \psi = \left(R(x_k) + \mathcal{L}(x_k) \right) \psi.$$

$\zeta_\infty \in x^{-1}(\infty)$ and $\zeta_l \in x^{-1}(\Lambda_l)$ poles of $\omega_{0,1}$ of orders m_∞ and m_l, $l = 1, \ldots, N$, respectively. Let $d_\infty := \mathrm{ord}_{\zeta_\infty}(x)$. Operator $\mathcal{L}(x) = \mathcal{L}_\infty(x) + \mathcal{L}_\Lambda(x)$:

$$\mathcal{L}_\infty(x) = \sum_{j=1-2d_\infty}^{m_\infty} t_{\zeta_\infty, j} \sum_{k=0}^{\frac{1-j}{d_\infty}-2} x^k \left(-\frac{j}{d_\infty} - k - 2 \right) \frac{\partial}{\partial t_{\zeta_\infty, j + d_\infty(k+2)}},$$

$$\mathcal{L}_\Lambda(x) = \sum_{l=1}^{N} \left(\frac{1}{x - \lambda_l} \frac{\partial}{\partial \lambda_l} + \sum_{j=1}^{m_l - 1} t_{\zeta_l, j} \sum_{k=1}^{j} (x - \lambda_l)^{-(k+1)} (j + 1 - k) \frac{\partial}{\partial t_{\zeta_l, j+1-k}} \right).$$

Example

In the Airy case, $y^2 = x$, we have only one pole, at $\zeta_i = \infty$, of degree $m_i = 3$, with $d_i = -2$. The sum is empty and $\mathcal{L}(x) = 0$.

Airy and elliptic cases for two-point divisors

Divisor $D = [z_1] - [z_2]$:

- **PDEs for Airy curve**: $y^2 = x$. We had $\mathcal{L}(x) = 0$.

$$\begin{cases} \hbar^2 \left(\frac{d^2}{dx_1^2} + \frac{\frac{d}{dx_1} - \frac{d}{dx_2}}{x_1 - x_2} \right) \psi & = x_1 \psi, \\ \hbar^2 \left(\frac{d^2}{dx_2^2} + \frac{\frac{d}{dx_1} - \frac{d}{dx_2}}{x_1 - x_2} \right) \psi & = x_2 \psi. \end{cases}$$

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More generally, admissible curves considered in Bouchard–Eynard, '17 (empty Newton polygon) are those for which $\mathcal{L}(x) = 0$.

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More generally, admissible curves considered in Bouchard–Eynard, '17 (empty Newton polygon) are those for which $\mathcal{L}(x) = 0$.

- **PDEs for elliptic curve**: $R(x(z)) = y(z)^2 = x^3 + tx + V$, with

$$-V = \int_{\mathcal{B}_{\infty,1}} \omega_{0,1} = \frac{\partial}{\partial t_{\infty,1}} \omega_{0,0} = -\frac{\partial}{\partial t} \omega_{0,0}$$

$\Rightarrow R(x(z)) = x^3 + tx + \frac{\partial}{\partial t} \omega_{0,0}$.
We have $\mathcal{L}(x) = \frac{\partial}{\partial t}$.

$$\left(\hbar^2 \frac{d^2}{dx_k^2} + \hbar^2 \frac{\frac{d}{dx_1} - \frac{d}{dx_2}}{x_1 - x_2} \right) \psi = \left(x_k^3 + t x_k + V + \frac{\partial}{\partial t} \right) \psi,$$

for $k = 1, 2$.

Monodromies of the perturbative wave function $\rightsquigarrow$ bad monodromies

Problem for genus $\hat{g} > 0$: $\int_o^z \cdots \int_o^z \omega_{g,n}$ are not invariant after z goes around a cycle. Very bad monodromies when z goes around a $\mathcal{B}_i$ (first type cycle).

Lemma

$$\forall\, p \in x^{-1}(\mathcal{P}) \,:\, \psi_l([z+\mathcal{C}_p]-[\infty^{(\alpha)}],\hbar) = (-1)^{\delta_{p,\infty^{(\alpha)}}}\, e^{\frac{2\pi i t_{p,0}}{\hbar}}\, \psi_l([z]-[\infty^{(\alpha)}],\hbar),$$

$$\forall\, j \in [\![1,\hat{g}]\!] \,:\, \psi_l([z+\mathcal{A}_j]-[\infty^{(\alpha)}],\hbar) = e^{\frac{2\pi i \epsilon_j}{\hbar}}\, \psi_l([z]-[\infty^{(\alpha)}],\hbar),$$

where $\mathcal{C}_p$ $(=\mathcal{C}_{p,0})$ *is a small circle around* p*, and*

$$\psi(D+\mathcal{B}_j,\hbar) = \exp\left(\sum_{(h,n,m)\in\mathbb{N}^3} \frac{\hbar^{2h-2+n+m}}{n!m!} \overbrace{\int_D \cdots \int_D}^{n} \overbrace{\int_{\mathcal{B}_j} \cdots \int_{\mathcal{B}_j}}^{m} \omega_{h,n+m}\right).$$

Since the $\mathcal{B}_j$ period of $\omega_{h,n+1}$ is equal to the variation of $\omega_{h,n}$ wrt $\epsilon_j := \oint_{\mathcal{A}_j} \omega_{0,1}$,

$$\psi(D+\mathcal{B}_j,\hbar) = \exp\left(\sum_{(h,n)\in\mathbb{N}^2} \frac{\hbar^{2h-2+n}}{n!} \overbrace{\int_D \cdots \int_D}^{n} \sum_{m\geq 0} \frac{1}{m!}\left(\hbar\frac{\partial}{\partial \epsilon_j}\right)^m \omega_{h,n}\right) \Rightarrow$$

$$\psi_l([z+\mathcal{B}_j]-[\infty^{(\alpha)}],\hbar) = e^{\hbar\frac{\partial}{\partial\epsilon_j}}\, \psi_l([z]-[\infty^{(\alpha)}],\hbar) = \psi_l([z]-[\infty^{(\alpha)}],\hbar,\epsilon_j \to \epsilon_j+\hbar).$$

Outline

Summing over the lattice

Remark

Our KZ equations do not depend on $z \in \widetilde{\Sigma}$ but only on its image $x(z) \Rightarrow$
For any finite family of c_γ, the following sum satisfies the same KZ equations

$$\psi_l([z] - [\infty^{(\alpha)}], \hbar, \{c_\gamma\}) := \sum_{\gamma \in \pi_1(\Sigma \setminus x^{-1}(\mathcal{P}))} c_\gamma \; \psi_l([z] + \gamma - [\infty^{(\alpha)}], \hbar).$$

Goal: Build solutions to the same KZ equations but with better monodromies along the $\mathcal{B}_i$-cycles.

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Goal: Build solutions to the same KZ equations but with better monodromies along the $\mathcal{B}_i$-cycles.

Strategy: Sum over $\gamma = \sum_{i=1}^{g} n_i \mathcal{B}_i$, i.e. $\epsilon_i \to \epsilon_i + \hbar$. Formally $\rightsquigarrow$ discrete Fourier transform of the perturbative wave function:

$$\psi_l^{\infty^{(\alpha)}}(z, \hbar; \boldsymbol{\epsilon}, \boldsymbol{\rho}) := \sum_{\mathbf{n} \in \mathbb{Z}^g} e^{\frac{2\pi i}{\hbar} \sum_{j=1}^{g} \rho_j n_j} \ \psi_l([z] - [\infty^{(\alpha)}], \hbar, \epsilon + \hbar \mathbf{n}).$$

Trans-series with special ordering

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Remark (Limitations)

- *Filling fraction* $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_g) \rightsquigarrow$ *not a global coordinate on the space of classical spectral curves with fixed spectral times (only a local coordinate).*
- *Not a finite sum* $\rightsquigarrow$ *not necessarily defined in* $\mathcal{W}$.

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We need a special ordering of the trans-monomials:

$$\sum_{r\geq 0}\sum_{\mathbf{n}\in\mathbb{Z}^g} F_{\mathbf{n},r}\hbar^r e^{\frac{1}{\hbar}\sum_{j=1}^{g} n_j v_j}.$$

The partial sums $\sum_{\mathbf{n}\in\mathbb{Z}^g} F_{\mathbf{n},r} e^{\frac{1}{\hbar}\sum_{j=1}^{g} n_j v_j}$ will give rise to theta functions (through convergent series in the spirit of the trans-asymptotics of Costin–Costin, '10).
Equalities: coefficient by coefficient in the trans-monomials.

Non-perturbative wave functions

Riemann matrix of periods of Σ: $\tau_{i,j} = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{B}_i} \int_{\mathcal{B}_j} \omega_{0,2}$, $\forall (i,j) \in [\![1,\hat{g}]\!]^2$.

Riemann theta function (analytic function of $\mathbf{v} \in \mathbb{C}^{\hat{g}}$) and its derivatives:

$$\Theta^{(i_1,\dots,i_k)}(\mathbf{v},\tau) = \sum_{(n_1,\dots,n_g)\in\mathbb{Z}^{\hat{g}}} e^{2\pi \mathrm{i} \sum_{i=1}^{\hat{g}} n_i v_i} e^{\pi \mathrm{i} \sum_{(i,j)\in[\![1,\hat{g}]\!]^2} n_i \tau_{i,j} n_j} \prod_{j=1}^{k} n_{i_j}.$$

For $D = [z] - [\infty^{(\alpha)}]$, we define the non-perturbative wave function

$$\psi_{\mathrm{NP}}(D;\hbar,\boldsymbol{\rho}) := e^{\hbar^{-2}\omega_{0,0}+\omega_{1,0}} e^{\hbar^{-1}\int_D \omega_{0,1}} \frac{1}{E(D)} \sum_{r=0}^{\infty} \hbar^r G^{(r)}(D;\boldsymbol{\rho}),$$

where E is the prime form on Σ,

$$G^{(r)}(D;\boldsymbol{\rho}) := \sum_{k=0}^{3r} \sum_{i_1,\dots,i_k\in[\![1,\hat{g}]\!]^k} \Theta^{(i_1,\dots,i_k)}(\mathbf{v},\tau) G^{(r)}_{(i_1,\dots,i_k)}(D)$$

and where $v_j := \frac{\rho_j+\varphi_j}{\hbar} + \mu_j^{(\alpha)}(z)$, $\mathbf{v} = (v_1,\dots,v_{\hat{g}})$, with

$$\varphi_j := \frac{1}{2\pi i} \oint_{\mathcal{B}_j} \omega_{0,1} \quad \text{and} \quad \mu_j^{(\alpha)}(z) := \frac{1}{2\pi i} \int_D \oint_{\mathcal{B}_j} \omega_{0,2}.$$

Same KZ equations and good monodromies

- Non-perturbative wave functions satisfy the same KZ equations as their perturbative partners.

$$\hbar \frac{d\psi^{\infty^{(\alpha)}}_{l,\mathrm{NP}}(z,\hbar,\boldsymbol{\rho})}{dx(z)} + \psi^{\infty^{(\alpha)}}_{l+1,\mathrm{NP}}(z,\hbar,\boldsymbol{\rho}) =$$

$$\sum_{P\in\mathcal{P}} \sum_{k\in S_P^{(l+1)}} \xi_P^{-k}(x(z))\mathrm{ev.}\left[\widetilde{\mathcal{L}}_{P,k,l}\,\psi^{\infty^{(\alpha)},\,\mathrm{symbol}}_{0,\mathrm{NP}}(z,\hbar,\boldsymbol{\rho})\right].$$

- Non-perturbative wave functions $\rightsquigarrow$ simple monodromy properties.

For $j \in [\![1,\hat{g}]\!]$, we have

$$\psi^{\infty^{(\alpha)}}_{l,\mathrm{NP}}(z+\mathcal{A}_j,\hbar,\boldsymbol{\rho}) = e^{\frac{2\pi i\epsilon_j}{\hbar}}\,\psi^{\infty^{(\alpha)}}_{l,\mathrm{NP}}(z,\hbar,\boldsymbol{\rho}),$$

$$\psi^{\infty^{(\alpha)}}_{l,\mathrm{NP}}(z+\mathcal{B}_j,\hbar,\boldsymbol{\rho}) = e^{-\frac{2\pi i\rho_j}{\hbar}}\,\psi^{\infty^{(\alpha)}}_{l,\mathrm{NP}}(z,\hbar,\boldsymbol{\rho})$$

and $\forall\ p \in x^{-1}(\mathcal{P})$

$$\psi^{\infty^{(\alpha)}}_{l,\mathrm{NP}}(z+\mathcal{C}_p,\hbar,\boldsymbol{\rho}) = (-1)^{\delta_{p,\infty^{(\alpha)}}}\ e^{\frac{2\pi i t_{p,0}}{\hbar}}\,\psi^{\infty^{(\alpha)}}_{l,\mathrm{NP}}(z,\hbar,\boldsymbol{\rho}).$$

Lax systems

For $l \geq 0$, we define

$$\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho}) := \mathrm{ev.} \sum_{\beta \underset{l}{\subseteq} \left(x^{-1}(x(z)) \setminus \{z\}\right)} \frac{1}{l!} \left(\prod_{j=1}^{l} \mathcal{I}_{\mathcal{C}_{\beta_j,1}} \right) \psi_{\mathrm{NP}}^{\mathrm{symbol}}(D;\hbar,\boldsymbol{\rho}).$$

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For $l \geq 0$, we define

$$\psi_{l,\mathrm{NP}}^{\infty(\alpha)}(z,\hbar,\boldsymbol{\rho}) \coloneqq \mathrm{ev.} \sum_{\beta \subseteq_l (x^{-1}(x(z))\setminus\{z\})} \frac{1}{l!} \left(\prod_{j=1}^{l} \mathcal{I}_{\mathcal{C}_{\beta_j,1}} \right) \psi_{\mathrm{NP}}^{\mathrm{symbol}}(D;\hbar,\boldsymbol{\rho}).$$

We use them to define a $d \times d$ matrix

$$\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar,\boldsymbol{\rho}) \coloneqq \left[\psi_{l-1,\mathrm{NP}}^{\infty(\alpha)}(z^{(\beta)}(\lambda),\hbar,\boldsymbol{\rho}) \right]_{1 \leq l,\beta \leq d},$$

where $z^{(\beta)}(\lambda)$ denotes the β^{th} preimage by x of λ.

Lax systems

$$\widetilde{\mathcal{L}}_l(x(z)) = \sum_{P\in\mathcal{P}} \sum_{k\in S_P^{(l+1)}} \xi_P(x(z))^{-k} \widetilde{\mathcal{L}}_{P,k,l}, \;\; \mathcal{L}_{P,k,l} := \widetilde{\mathcal{L}}_{P,k,l} - P_{P,k}^{(l+1)}.$$

Theorem (ODE and Lax system)

Let $\hat{L}(\lambda,\hbar) := -\widehat{P}(\lambda) + \hbar \sum_{P\in\mathcal{P}} \sum_{k\in\mathbb{N}} \xi_P^{-k}(\lambda)\widehat{\Delta}_{P,k}(\lambda,\hbar)$. *Then,*

$$\hbar \frac{d\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)}{d\lambda} = \hat{L}(\lambda,\hbar)\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar),$$

where

$$\widehat{P}(\lambda) := \begin{bmatrix} -P_1(\lambda) & 1 & 0 & \dots & 0 \\ -P_2(\lambda) & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & dots & \vdots \\ -P_{d-1}(\lambda) & 0 & 0 & \dots & 1 \\ -P_d(\lambda) & 0 & 0 & \dots & 0 \end{bmatrix}$$

For any $P \in \mathcal{P}$, $k \in \mathbb{N}$, $l \in [\![0, d-1]\!]$, *one has the auxiliary systems*

$$\hbar^{-1}\mathrm{ev}.\mathcal{L}_{P,k,l}\widehat{\Psi}_{\mathrm{NP}}^{\mathrm{symbol}}(\lambda,\hbar) = \widehat{A}_{P,k,l}(\lambda,\hbar)\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar),$$

where $\hat{L}(\lambda,\hbar)$ *and* $\widehat{A}_{P,k,l}(\lambda,\hbar)$ *are* $\hbar$*-trans-series functions that are rational functions of* λ*, with no poles at critical values* $\lambda \in x(\mathcal{R})$*.*

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- (1) $\rightsquigarrow$ linear differential system of size $d \times d$ whose formal fundamental solution can be computed by TR, with poles at the poles of the leading WKB term...
- $\hat{L}(\lambda,\hbar)$ has poles only at $\lambda \in \mathcal{P}$ and at zeros of the Wronskian $\det \widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)$, *apparent singularities* of the system (can be computed thanks to the KZ eqns).

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- Most technical proof $\rightsquigarrow$ by induction on the order of the transseries.
- Proof uses admissibility conditions (distinct critical values, smooth simple ramification points) $\rightsquigarrow$ should adapt without them but involving more technical computations.

4 different interesting gauges

None of the gauge transformations modify the first line of the wave functions matrix (used to define the quantum curve).

- **Gauge $\widehat{\Psi}$**: Natural gauge coming from KZ equations and provides compatible auxiliary systems $\left(\mathcal{L}_{P,k,l}\right)_{P\in\mathcal{P},l\in[\![0,d-1]\!],k\in S_P^{(l+1)}}$.
- **Gauge $\widetilde{\Psi}$** ($\hbar^0$ gauge transformation from $\widehat{\Psi}$): Leading order in $\hbar$ of $\widetilde{L}$ is companion-like $\rightsquigarrow$ the classical spectral curve is directly recovered from its last line.
- **Gauge Ψ**: Corresponding Lax matrix L is companion-like at all orders in $\hbar$ $\rightsquigarrow$ both the quantum and classical curves are directly read from the last line of L and its $\hbar \to 0$ limit. Natural framework for Darboux coordinates and isomonodromic deformations.
- **Gauge $\check{\Psi}$**: Lax matrix $\check{L}$ has no apparent singularities. This allows to interpret $\check{L}(\lambda,\hbar)d\lambda$ as an $\hbar$-familly of Higgs fields giving rise to a flow in the corresponding Hitchin system.

Outline

Spectral curves from integrable systems

Definition

Let $\hbar \frac{\partial}{\partial x}\Psi(x,\hbar) = \mathcal{L}(x,\hbar)\Psi(x,\hbar)$ be a (2×2) differential system (with $\det \Psi = 1$). We define the classical spectral curve associated to it by

$$P(x,y) \coloneqq \lim_{\hbar\to 0} \det(y\mathrm{Id} - \mathcal{L}(x,\hbar)) = 0,$$

which gives a polynomial equation. For a non-zero genus curve, this must be completed with a choice of symplectic basis of cycles and a bidifferential B.

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Different approach:

- $\hbar$-differential system.
- Define the classical spectral curve associated to it.
- Show that interesting quantities from the point of view of the differential system may be reconstructed from topological recursion applied to this classical spectral curve.
- Proof by showing that the differential system satisfies the topological type property (Bergère–Borot–Eynard '15).

Isomonodromic deformations

We consider isomonodromic deformations of the linear differential equation $\partial_x - \mathcal{L}(x)$, which depend on a number of continuous parameters t_k (times):

$$\begin{cases} \hbar \dfrac{\partial}{\partial x} \Psi(x, t_k; \hbar) = \mathcal{L}(x, t_k; \hbar) \Psi(x, t_k; \hbar), \\ \hbar \dfrac{\partial}{\partial t_k} \Psi(x, t_k; \hbar) = \mathcal{R}_k(x, t_k; \hbar) \Psi(x, t_k; \hbar) \end{cases}$$

We call such a (compatible integrable) system an isomonodromic system.

$$\frac{\partial^2}{\partial t_k \partial x} \Psi = \frac{\partial^2}{\partial x \partial t_k} \Psi \Leftrightarrow \hbar \frac{\partial \mathcal{L}}{\partial t_k} - \hbar \frac{\partial \mathcal{R}_k}{\partial x} + [\mathcal{L}, \mathcal{R}_k] = 0 \text{ (zero-curvature equation).}$$

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Consider the deformed spectral curve

$$P(x, y; \hbar) = \det(y \, \mathrm{Id} - \mathcal{L}(x, t_k; \hbar)) = P_0(x, y) + \sum_{m \geq 1} \hbar^m P_m(x, y).$$

Classical spectral curve $\rightsquigarrow P_0(x, y)$ (family of curves parametrized by t_k's).

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Remark

Painlevé equations $\rightsquigarrow$ *Isomonodromic deformations. Painlevé property* $\rightsquigarrow$ *Solutions have no movable singularities other than poles. Classification of all second order differential equations with the Painlevé property* $\rightsquigarrow$ 50 *solutions and only* 6 *which could not be integrated from already known functions.*

Painlevé I

In the family of elliptic curves $y^2 = x^3 + tx + V$, taking $t = -3u_0^2$ and $V = 2u_0^3$, amounts to pinching the $\mathcal{B}$-cycle (first kind). So in this case, we have genus $\hat{g} = 0$ and the **spectral curve** admits a rational parametrization:

$$\begin{cases} \Sigma = \mathbb{C}P^1, & x(z) = z^2 - 2u_0,\ y(z) = z^3 - 3u_0 z, \\ ydx = (z^3 - 3u_0 z)2zdz, & B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z^2)^2}. \end{cases}$$

TR: Witten–Kontsevich intersection numbers $\rightsquigarrow \omega_{g,n}(z_1, \dots, z_n) =$

$$\sum_{d_1,\dots,d_n} \frac{6^{2-2g-n} u_0^{5-5g-2n}}{(3g-3+n-\sum_i d_i)!} \left\langle \tau_2^{3g-3+n-\sum_i d_i} \tau_{d_1} \cdots \tau_{d_n} \right\rangle_g \prod_{i=1}^n \frac{u_0^{d_i} (2d_i+1)!! dz_i}{z_i^{2d_i+1}}.$$

$$n = 0 \rightsquigarrow \mathcal{F}_g = \omega_{g,0} = u_0^{5-5g} \frac{6^{2-2g}}{(3g-3)!} \left\langle \tau_2^{3g-3} \right\rangle_g = (-t/3)^{\frac{5-5g}{2}} \frac{6^{2-2g}}{(3g-3)!} \left\langle \tau_2^{3g-3} \right\rangle_g.$$

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Then $U(t) = u_0 + \frac{\hbar^2}{48t^2} + \sum_{g\geq 2} \hbar^{2g} \frac{\partial^2 \mathcal{F}_g}{\partial t^2}$ satisfies the Painlevé I equation $\frac{\hbar^2}{2} \frac{\partial^2}{\partial t^2} U + 3U^2 = -t$, which is the compatibility equation of the Lax pair

$$\mathcal{L}(x, t; \hbar) := \begin{pmatrix} \frac{\hbar}{2}\dot{U} & x - U \\ (x-U)(x+2U) + \frac{\hbar^2}{2} dotU & -\frac{\hbar}{2}\dot{U} \end{pmatrix} \text{ and } \mathcal{R}(x, t; \hbar) := \begin{pmatrix} 0 & 1 \\ x + 2U & 0 \end{pmatrix}.$$

From the PDE found we can get that $\psi_\pm(x) = e^{\sum_{g,n} \frac{(\pm 1)^n \hbar^{2g-2+n}}{n!} \int \cdots \int \omega_{g,n}}$:

$$\left(\hbar \frac{\partial}{\partial x} - \mathcal{L}(x, t; \hbar)\right) \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0\,, \quad \left(\hbar \frac{\partial}{\partial t} - \mathcal{R}(x, t; \hbar)\right) \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0.$$

Practical computations to quantise a classical spectral curve

1. Write down the KZ equations satisfied by the non-perturbative wave function.
2. Expand these KZ equations around each pole $\lambda \to P \in \mathcal{P} \rightsquigarrow$ expression of the coefficients of the asymptotic expansion of $\psi_{0,\mathrm{NP}}^{(\infty^{(\alpha)})}$ in terms of the action of the operators $\mathcal{I}_C$.
3. Use the latter expressions to compute the Wronskian of the system thanks to its expansion around its poles. This allows to compute the position of the apparent singularities $(q_i(\hbar))_{i=1}^{d}$.
4. Write down the linear system and the associated quantum curve, and use the compatibility of the system to recover its properties.

Example

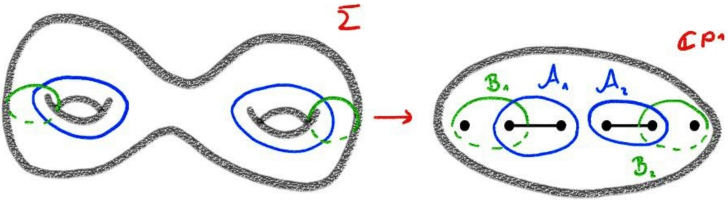
- Reconstruction via TR of a 2-parameter family of formal transseries solutions to Painlevé 2 and quantisation. Classical spectral curve: $y^2 - P_1(\lambda)y + P_2(\lambda) = 0$, where $P_1(\lambda) = P^{(1)}_{\infty,2}\lambda^2 + P^{(1)}_{\infty,1}\lambda + P^{(1)}_{\infty,0}$ and $P_2(\lambda) = P^{(2)}_{\infty,4}\lambda^4 + P^{(2)}_{\infty,3}\lambda^3 + P^{(2)}_{\infty,2}\lambda^2 + P^{(2)}_{\infty,1}\lambda + P^{(2)}_{\infty,0}$.
- Quantisation of a degree 3, genus 1 classical spectral curve with a single singularity at infinity: $y^3 - (P^{(1)}_{\infty,1}\lambda + P^{(1)}_{\infty,0})y^2 + (P^{(2)}_{\infty,2}\lambda^2 + P^{(2)}_{\infty,1}\lambda + P^{(2)}_{\infty,0})y - P^{(3)}_{\infty,3}\lambda^3 - P^{(3)}_{\infty,2}\lambda^2 - P^{(3)}_{\infty,1}\lambda - P^{(3)}_{\infty,0} = 0$.

Outline

Future work

- Ongoing: More conceptual proof of the QC conjecture?
- Explore the connection with summability, exact WKB, Stokes phenomenon and resurgence. **Conjecture:** There exist values of ε and $\hbar$ making the transseries involved summable.
- **Conjecture:** The non-perturbative partition function is a tau function.
- How does the connection built as $d - \mathcal{L}(x,\hbar)dx/\hbar$ depend on the choice of cycles $(\mathcal{A}_i, \mathcal{B}_i)$?
- Upgrade to trans-algebraic spectral curves (essential singularities) with the work of Bouchard–Kramer–Weller?
- Interesting enumerative geometry in higher genus TR problems?
- Get rid of admissibility conditions?
- Relation to the topological type property approach (can that be proved for higher genus spectral curves?).
- Extend the result to a ramified covering of surfaces other than $\mathbb{C}P^1$.
- Generalization to difference equations? (Subtleties including K_2 condition of Gukov–Sułkowski '12?). Non-algebraic curves, such as $P(e^x, e^y)$ (important for volume conjecture).
- General relation between Virasoro constraints (or even Kontsevich–Soibelman '17, ABCD of Andersen–Borot–Chekhov–Orantin '17) and quantum curves.

Merci beaucoup pour votre attention!



Articles:

- *From topological recursion to wave functions and PDEs quantizing hyperelliptic curves*, with B. Eynard, arXiv:1911.07795 (2019)
- *Quantizing generic algebraic spectral curves via topological recursion*, with B. Eynard, O. Marchal, N. Orantin, arXiv:2106.04339 (2021)