Nonlinear stability of Kerr for small angular momentum

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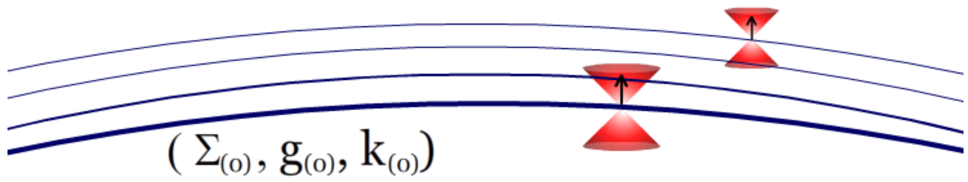
The Evolution Problem in General Relativity

Let $(\mathcal{M}^{1+3}, \mathbf{g})$ Lorentzian, and $\mathbf{Ric}(\mathbf{g})$ its Ricci tensor

Einstein vacuum equations: $\mathbf{Ric}(\mathbf{g}) = 0$

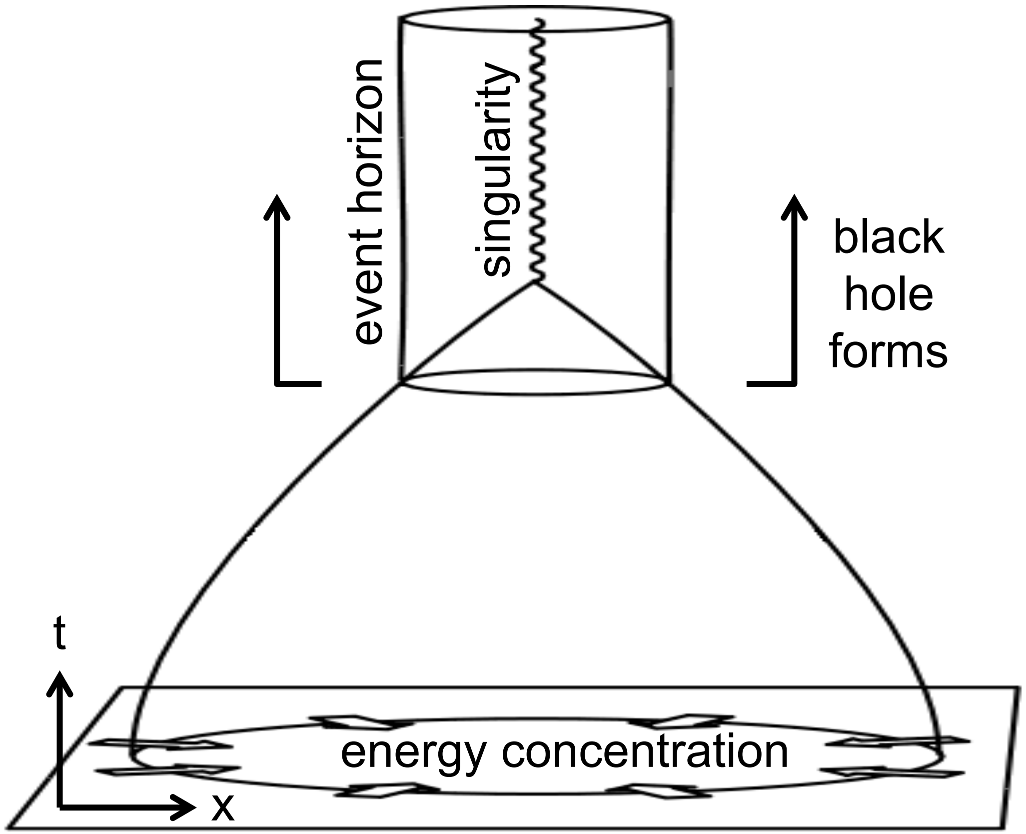
Evolution problem. Specify initial data $(\mathbf{g}_{(0)}, k_{(0)})$ on a given hypersurface $\Sigma_{(0)}$ and study its unique maximal globally hyperbolic development [Choquet-Bruhat 52'], [Choquet-Bruhat-Geroch 69']

$\mathbf{Ric}(\mathbf{g}) = 0$



The final state conjecture

What happens asymptotically to these maximal developments?



Final state conjecture. Large energy concentration may lead to the formation of a dynamical black hole settling down to a Kerr black hole

3 topics in the mathematical theory of black holes

In order to address the final state conjecture, one should first consider the following three problems:

1. **Collapse.** Can black holes form starting from reasonable initial data configurations? [Penrose 65'], [Christodoulou 08'], ...

2. **Rigidity.** Does the Kerr family of solutions exhaust all possible stationary vacuum black holes? [Carter 71'], [Robinson 75'], [Hawking 73'], [Alexakis-Ionescu-Klainerman 10'], ...

3. **Stability.** Is the Kerr family stable under arbitrary small perturbations?

The Kerr and the Schwarzschild solutions

Kerr metric given in Boyer-Lindquist (t, r, θ, φ) coordinates by

$$\mathbf{g}_{a,m} = -\frac{\rho^2 \Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2 (\sin\theta)^2}{\rho^2}\left(d\varphi - \frac{2amr}{\Sigma^2} dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2 (d\theta)^2$$

$$\Delta = r^2 + a^2 - 2mr,\ \rho^2 = r^2 + a^2 (\cos\theta)^2,\ \Sigma^2 = (r^2 + a^2)^2 - a^2 (\sin\theta)^2 \Delta$$

Kerr is stationary (∂_t is Killing) and axisymmetric (∂_φ is Killing)

For $|a| \leq m$, Kerr contains a black hole region $r \leq m + \sqrt{m^2 - a^2}$

The Schwarzschild metric is static and spherically symmetric and corresponds to the particular case $a = 0, m > 0$

$$\mathbf{g}_m = -\left(1 - \frac{2m}{r}\right)(dt)^2 + \left(1 - \frac{2m}{r}\right)^{-1}(dr)^2 + r^2\Big((d\theta)^2 + (\sin\theta)^2 (d\varphi)^2\Big)$$

Stability conjecture for the Kerr family

In the context of asymptotically flat solutions to the Einstein vacuum equation, we have the following conjecture:

Conjecture (Stability of the exterior region of Kerr). Small perturbations of given initial conditions of an exterior Kerr $\mathbf{g}_{a_0,m_0}$ with $|a_0| < m_0$ have maximal future developments converging to another exterior Kerr solution $\mathbf{g}_{a_f,m_f}$ with $|a_f| < m_f$

The particular case $m = a = 0$ corresponds to the stability of the Minkowski spacetime proved by [Christodoulou-Klainerman 93']

Stability of Minkowski

Minkowski: $(\mathbb{R}^{1+3}, \eta)$, $\quad \eta = -(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

The linear stability reduces to the understanding of $\Box\psi = 0$

Toy model problem for nonlinear stability: $\Box_{\mathbf{g}(\psi)}\psi = (\partial\psi)^2$

The toy model problem $\Box\psi = (\partial_t\psi)^2$ develops singularities in finite time $\Rightarrow$ specific structure of the quadratic nonlinear terms is essential!

Proof of the stability of Minkowski relies on:

1. A robust method to derive quantitative decay estimates for wave equations (vectorfield method, Klainerman Sobolev inequalities)

2. A procedure to exploit the null structure of the quadratic nonlinear terms

Basic quantities

Null frame. $e_3, e_4, (e_a)_{a=1,2}$

Connection coefficients. ξ, ω, χ, η, ζ, $\underline{\eta}$, $\underline{\chi}$, $\underline{\omega}$, $\underline{\xi}$ defined by:

$$\chi_{ab} = \mathbf{g}(\mathbf{D}_a e_4, e_b), \quad \underline{\chi}_{ab} = \mathbf{g}(\mathbf{D}_a e_3, e_b), \quad \xi_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_4, e_a), \cdots$$

Curvature components. α, β, $\rho, {}^\star\rho$, $\underline{\beta}$, $\underline{\alpha}$ defined by:

$$\alpha_{ab} = \mathbf{R}(e_a, e_4, e_b, e_4), \quad \beta_a = \frac{1}{2}\mathbf{R}(e_a, e_4, e_3, e_4), \quad \cdots$$

The following components vanish in Kerr in the principal null frame:

$$\widehat{\chi}_{Kerr} = \widehat{\underline{\chi}}_{Kerr} = \xi_{Kerr} = \underline{\xi}_{Kerr} = \beta_{Kerr} = \underline{\beta}_{Kerr} = \alpha_{Kerr} = \underline{\alpha}_{Kerr} = 0$$

Basic equations

Null structure eqts. (Transport)

$$\nabla_4 \Gamma = \mathbf{R} + \nabla \Gamma' + \Gamma \cdot \Gamma, \qquad \nabla_3 \Gamma = \mathbf{R} + \nabla \Gamma' + \Gamma \cdot \Gamma$$

Null structure eqts. (Codazzi)

$$\mathcal{D}\Gamma = \mathbf{R} + \nabla \Gamma' + \Gamma \cdot \Gamma$$

To recover the connection coefficients from the curvature requires additionally a choice of gauge

Null bianchi $\quad \nabla_4 \mathbf{R} = \nabla \mathbf{R}' + \Gamma \cdot \mathbf{R}, \quad \nabla_3 \mathbf{R} = \nabla \mathbf{R}' + \Gamma \cdot \mathbf{R}$

The linearized gravity system (LGS)

To get insight into the problem, look at the linearized Einstein equations around a Kerr solution which is the so called LGS, and prove decay for its solutions

We decompose Γ (connection) and R in the null frame $e_3, e_4, (e_a)_{a=1,2}$

$$\Gamma = \Gamma_{Kerr} + \check{\Gamma}, \quad \check{\Gamma} = O(\epsilon), \quad R = R_{Kerr} + \check{R}, \quad \check{R} = O(\epsilon)$$

The null structure and projected Bianchi equations take the form

$$\begin{aligned} \partial\check{\Gamma} + \Gamma_{Kerr} \cdot \check{\Gamma} &= \check{R} + O(\epsilon^2) \\ \partial\check{R} + \Gamma_{Kerr} \cdot \check{R} + R_{Kerr} \cdot \check{\Gamma} &= O(\epsilon^2) \end{aligned}$$

In the case of Kerr, LGS is a large and entangled system, whose analysis is non trivial

The Teukolsky scalars

In the 60's and 70's, a large physics literature devoted to the LGS [Regge-Wheeler], [Bardeen-Press], [Teukolsky], [Chandrasekhar],...

To analyze LGS, identify a quantity which decouples from the rest of the solution and from which one can in principle reconstruct the whole solution

For Kerr, Teukolsky (1973) showed that the curvature components α and $\underline{\alpha}$, in LGS, satisfy non conservative linear wave equations

$$\Big(\square_{\mathbf{g}_{a,m}} + A^\mu \partial_\mu + V\Big)\alpha = 0, \qquad \Big(\square_{\mathbf{g}_{a,m}} + \underline{A}^\mu \partial_\mu + \underline{V}\Big)\underline{\alpha} = 0$$

Goal: derive decay for α and $\underline{\alpha}$, and then control the rest of LGS by exhibiting a triangular structure

Obstructions to decay for LGS

We have $\mathbf{Ric}[\mathbf{g_{m,a}}] = \mathbf{0}$ for all $|a| < m$ and hence:

$$\delta\mathbf{Ric}\left[\frac{\partial\mathbf{g_{m,a}}}{\partial\mathbf{m}}\right] = \delta\mathbf{Ric}\left[\frac{\partial\mathbf{g_{m,a}}}{\partial\mathbf{a}}\right] = \mathbf{0}$$

This shows that $\partial_m \mathbf{g}_{m,a}$ and $\partial_a \mathbf{g}_{m,a}$ belong to the kernel of LGS

Due to general covariance of Einstein equations, we also have

$$\delta\mathbf{Ric}\left[\mathcal{L}_\mathbf{X}\mathbf{g_{m,a}}\right] = \mathbf{0}$$

for all vectorfields X so $\mathcal{L}_X \mathbf{g}_{m,a}$ also belongs to the kernel of LGS

General covariance of Einstein equations generates a kernel of LGS which has infinite dimension and is related at the nonlinear level to the infinitesimal motions of the center of mass of the Kerr solution

State of the art on the control of LGS

Simplest linear toy model $\square_{\mathbf{g}_{a,m}}\psi = 0$ is already highly non trivial

Teukolsky scalars satisfy a non conservative linear wave equation. LGS has no exponentially growing modes [Whiting 89'], but one cannot a priori establish, even formally, the boundedness of solutions

To obtain quantitative decay for Teukolsky scalars, one may rely on Chandrasekhar transform to transform Teukolsky in Regge Wheeler

Starting point of the control of LGS near Schwarzschild obtained by Dafermos-Holzegel-Rodnianski 16'

Control of LGS near Kerr with $|a| \ll m$ by Andersson, Bäckdahl, Blue and Ma 19' and by Häfner, Hintz and Vasy 19'

Nonlinear stability of Schwarzschild

To prove stability of Schwarzschild (particular case $a = 0$), one needs to restrict the class of perturbations. One possibility is to restrict to a specific class of symmetry excluding the case $a \neq 0$

Theorem [Klainerman-S, 18']. **The stability conjecture holds true for Schwarzschild in the axial polarized case**: small axial polarized perturbations of given initial conditions of an exterior Schwarzschild $\mathbf{g}_m$ with $m > 0$ have maximal future developments converging to another exterior Schwarzschild solution $\mathbf{g}_{m_f}$ with $m_f > 0$

Extended recently to a well-prepared codimension 3 set of initial data by Dafermos, Holzegel, Rodnianski and Taylor 21'

Nonlinear stability of Kerr for $|a| \ll 1$

Theorem [Klainerman-Sz, 21']. **The stability conjecture holds true for** $|a| \ll m$. Relies on:

- Modulation: Klainerman-Sz 19' (arXiv:1911.00697, arXiv:1912.12195), and Shen (preprint)
- Formalism for non integrable structures: Giorgi-Klainerman-Sz 20' (arXiv:2002.0274)
- Decay estimates, as well as statement of the result and strategy: Klainerman-Sz 21' (arXiv:2104.11857)
- Hyperbolic estimates: Giorgi-Klainerman-Sz (in preparation)

In the context of a positive cosmological constant, the stability of the Kerr de Sitter family for $|a| \ll m$ has been proved by Hintz-Vasy 16'

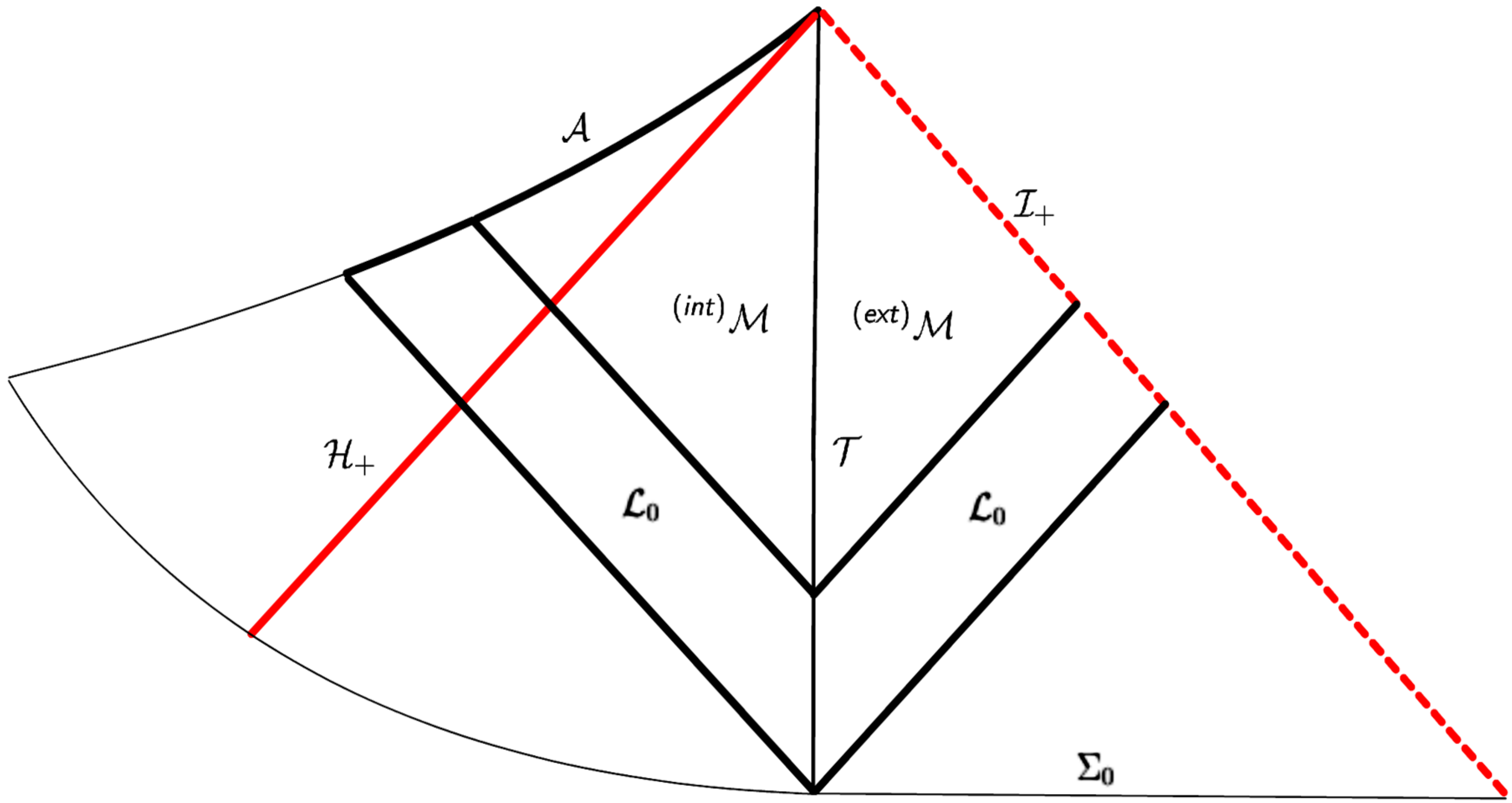
Properties of the constructed spacetimes

The perturbation of Kerr spacetimes constructed in the theorem satisfy the following properties:

- The future null infinity $\mathcal{I}_+$ is complete and satisfies a Bondy mass formula
- The spacetime possesses a black hole region with precise bounds on the location of its future event horizon $\mathcal{H}_+$
- The final mass m_f coincides with the final Bondi mass
- A suitable notion of quasilocal angular momentum on $\mathcal{I}_+$ converges to the final angular momentum a_f
- The gravitational wave recoil (or black hole kick) can be tracked

Penrose diagram of the final spacetime

${}^{(ext)}\mathcal{M}$ foliated by (u, r), ${}^{(int)}\mathcal{M}$ foliated by $(\underline{u}, r)$, $\mathcal{T} = \{r = r_0\}$ for some fixed $r_0 > r_+$, and $\mathcal{A} = \{r = r_+(1 - \delta)\}$



Estimates in ${}^{(ext)}\mathcal{M}$

On ${}^{(ext)}\mathcal{M}$, we have

$$
\begin{aligned}
|\alpha|, |\beta| &\lesssim \epsilon_0 \min\left\{\frac{1}{r^3(u+2r)^{\frac{1}{2}+\delta}}, \frac{1}{r^2(u+2r)^{1+\delta}}\right\}\\
|\check{\rho}|, |\widecheck{{}^\star\rho}| &\lesssim \epsilon_0 \min\left\{\frac{1}{r^3u^{\frac{1}{2}+\delta}}, \frac{1}{r^2u^{1+\delta}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\left|\widecheck{\mathrm{tr}\chi}\right| &\lesssim \frac{\epsilon_0}{r^2u^{1+\delta}}\\
|\widehat{\chi}|, |\check{\zeta}|, \left|\widecheck{\mathrm{tr}\underline{\chi}}\right| &\lesssim \epsilon_0 \min\left\{\frac{1}{r^2u^{\frac{1}{2}+\delta}}, \frac{1}{ru^{1+\delta}}\right\}
\end{aligned}
$$

$\check{\rho}$, $\widecheck{{}^\star\rho}$, $\widecheck{\mathrm{tr}\chi}$, $\check{\zeta}$, $\widecheck{\mathrm{tr}\underline{\chi}}$ denote quantities where the corresponding Kerr value has been subtracted

Further estimates in ${}^{(ext)}\mathcal{M}$ and ${}^{(int)}\mathcal{M}$

On ${}^{(ext)}\mathcal{M}$, we have also

$$|\underline{\beta}| \lesssim \frac{\epsilon_0}{r^2 u^{1+\delta}}, \qquad\qquad |\widehat{\underline{\chi}}|, |\underline{\alpha}| \lesssim \frac{\epsilon_0}{r u^{1+\delta}}$$

On ${}^{(int)}\mathcal{M}$, we have

$$\begin{aligned} |\alpha|, |\beta|, |\underline{\beta}|, |\underline{\alpha}|, |\widehat{\chi}|, |\widehat{\underline{\chi}}| &\lesssim \frac{\epsilon_0}{\underline{u}^{1+\delta}} \\ |\check{\rho}|, \left|\widecheck{{}^\star\rho}\right|, \left|\widecheck{\mathrm{tr}\chi}\right|, \left|\check{\zeta}\right|, \left|\widecheck{\mathrm{tr}\underline{\chi}}\right| &\lesssim \frac{\epsilon_0}{\underline{u}^{1+\delta}} \end{aligned}$$

The future event horizon $\mathcal{H}_+$ is in ${}^{(int)}\mathcal{M}$ and its location satisfies

$$r = m_f + \sqrt{m_f^2 - a_f^2} + O\left(\frac{\epsilon_0}{\underline{u}^{1+\delta}}\right) \text{ on } \mathcal{H}_+$$

Convergence of the metric

On ${}^{(ext)}\mathcal{M}$, in outgoing Eddington-Finkelstein coordinates (u, r, θ, φ)

$$\mathbf{g} = \mathbf{g}_{a_f, m_f, {}^{(ext)}\mathcal{M}} + O\left(\frac{\epsilon_0}{u^{1+\delta}}\right)$$

where $\mathbf{g}_{a_f, m_f, {}^{(ext)}\mathcal{M}}$ is the Kerr metric in (u, r, θ, φ) coordinates

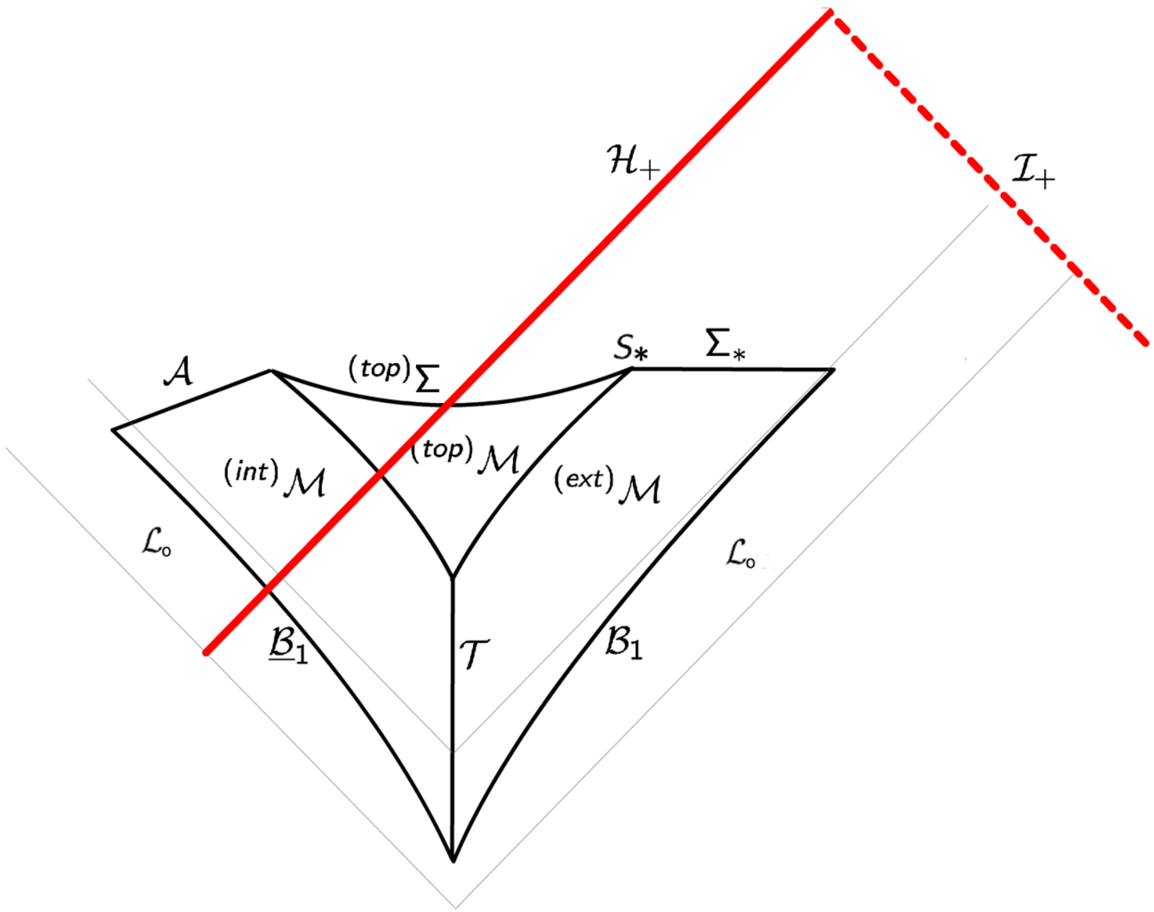
On ${}^{(int)}\mathcal{M}$, in ingoing Eddington-Finkelstein coordinates $(\underline{u}, r, \theta, \varphi)$

$$\mathbf{g} = \mathbf{g}_{a_f, m_f, {}^{(int)}\mathcal{M}} + O\left(\frac{\epsilon_0}{\underline{u}^{1+\delta}}\right)$$

where $\mathbf{g}_{a_f, m_f, {}^{(int)}\mathcal{M}}$ is the Kerr metric in $(\underline{u}, r, \theta, \varphi)$ coordinates

The continuity argument and the last slice

In the context of a continuity argument, we initialize the outgoing hypersurfaces $\{u = cst\}$ on a carefully chosen last slice Σ_*



Main ingredients of the proof

1. Control of α and $\underline{\alpha}$ using the Teukolsky equations and a Chandrasekhar type transform in case of the nonlinear problem

2. Reconstruct curvature, connection and metric components by exhibiting a triangular structure. Derive suitable decay solving
   (a) Transport equations along generator of $\{u = cst\}$, $\{\underline{u} = cst\}$
   (b) Elliptic systems of Hodge type on spheres $S(u, r)$, $S(\underline{u}, r)$

3. To deal with the presence of a kernel at the linearized level:
   (a) Define suitable constants (m, a) converging to (m_f, a_f)
   (b) Construct spheres tracking the gravitational wave recoil

4. Top derivatives: use induction on derivatives, energy-Morawetz estimates and Maxwell like character of the Bianchi identities

General Covariant Modulated (GCM) spheres

Spheres foliating outgoing hypersurfaces $\{u = cst\}$ should converge to the ones of Kerr in the limit thus tracking gravitational wave recoil

In [Klainerman-S 19'], we design a procedure to construct a unique 2-parameter family of preferred codimension 2 surfaces in general perturbations of Kerr

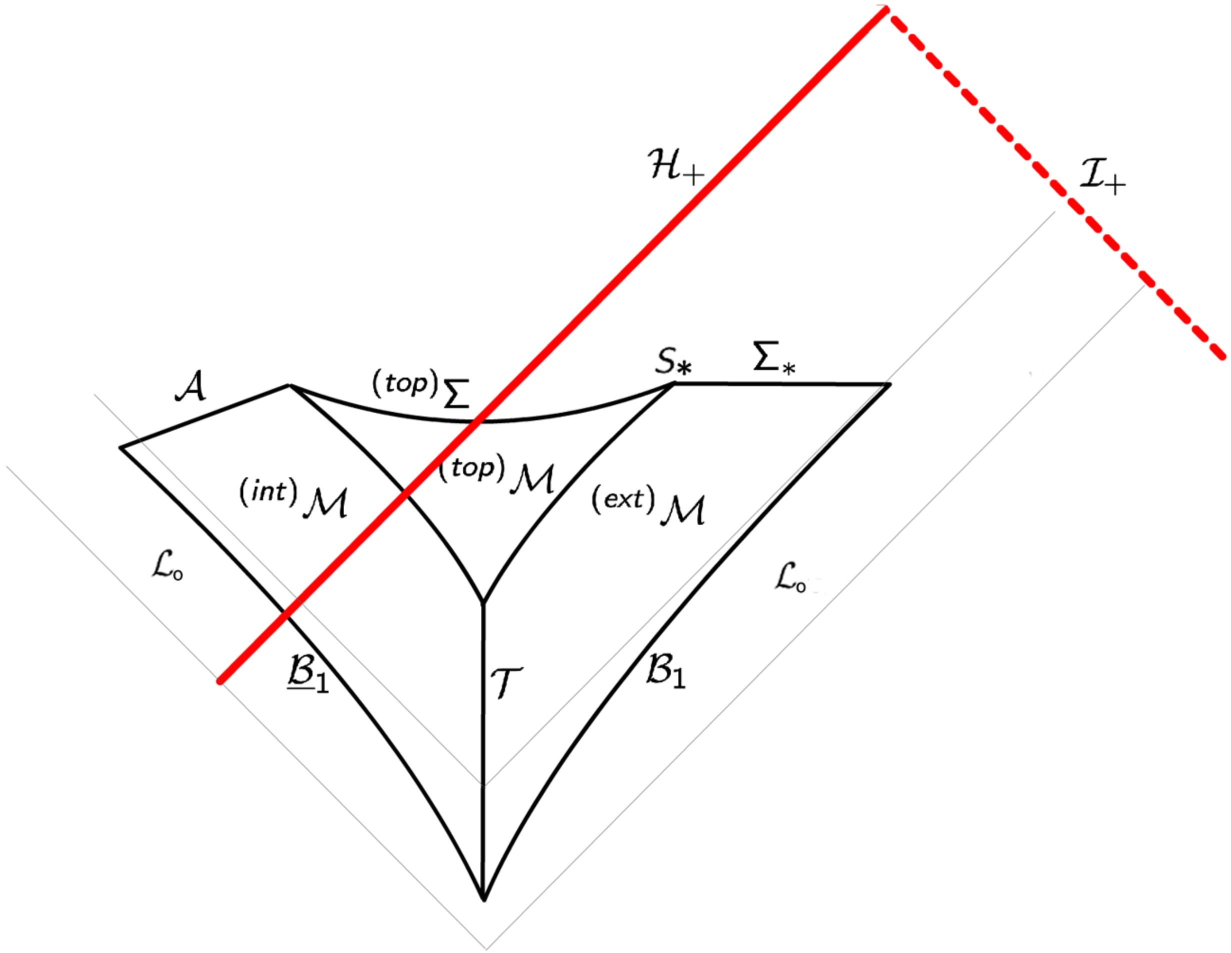
In the particular case of Kerr, this procedure recovers the 2-parameter family of Kerr spheres

Definition of GCM spheres relies on fixing well-chosen geometric quantities ($\mathrm{tr}\chi$, μ and $\mathrm{tr}\,\underline{\chi}$) to agree with their Kerr value

Intrinsic definition of mass (given by Hawking mass), of angular momentum, and tracking of the "axis" for such GCM spheres

The choice of the last slice

The last slice Σ_* is foliated by these GCM spheres



New challenges in the case $a \neq 0$

The principal null frame in Schwarzschild is integrable, but the principal null frame of Kerr is non integrable

- Lack of integrability of the principal null frames in Kerr: requires formalism for non integrable structures and new, non integrable gauges in perturbations of Kerr
- Lack of integrability of gauges problematic for control of Hodge type systems on spheres: construct an integrable structure and a mechanism to go back and forth between the two structures
- Decay and boundedness estimates are based on two different types of non-integrable gauges