Cosmological Scattering Equations

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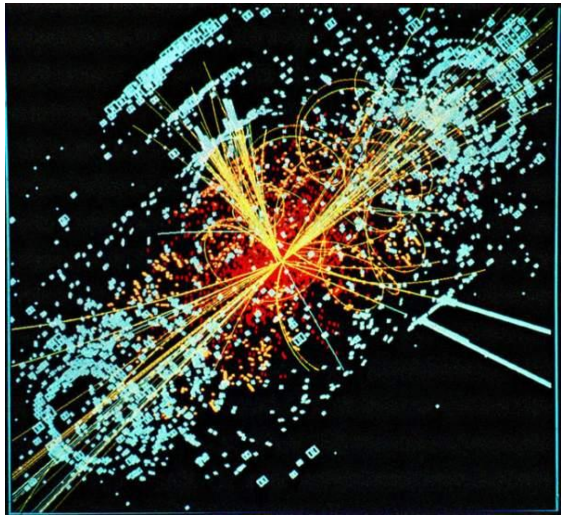
(with C. Armstrong, H. Gomez, R. Lipinski Jusinskas, J. Mei)

Introduction

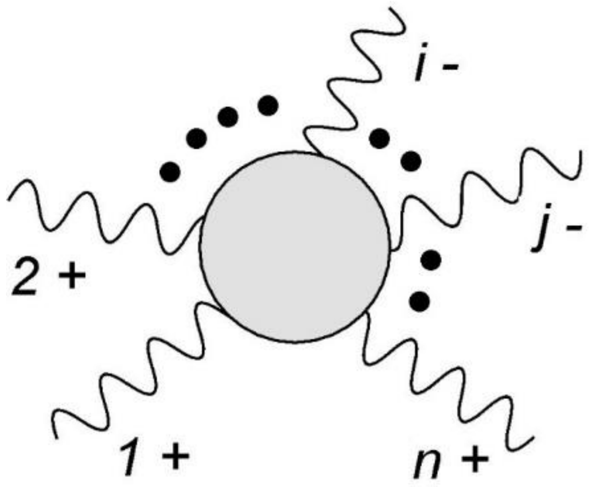
- Scattering amplitudes are the basic observables measured at colliders like the LHC.



- They also have a rich mathematical structure which is interesting in its own right.

Twistor String Theory

- Parke-Taylor:



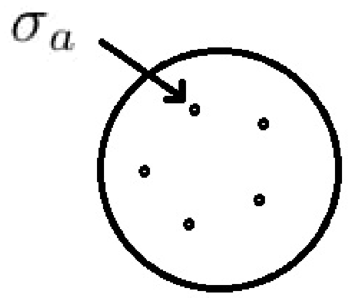
$$\mathcal{A}_n = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

- Nair,Berkovitz,Witten: derived tree-level N=4 SYM amplitudes from twistor string theory

Scattering Equations

- Cachazo,He,Yuan: Extended to general theory of massless particles

$$S_a = \sum_{a\neq b} \frac{2\,k_a \cdot k_b}{\sigma_{ab}} = 0$$

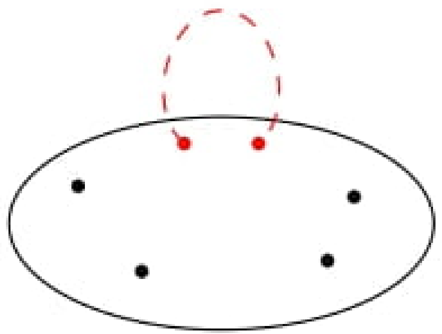


- First discovered in tensionless limit of string theory (Fairlie,Roberts,Gross,Mende)
- Universal structure:

$$\mathcal{A}_n = \int_\gamma \prod_{\substack{a=1\\ a\neq b,c,d}}^{n} \mathrm{d}\sigma_a \,(S_a)^{-1}\,(\sigma_{bc}\sigma_{cd}\sigma_{db})^2\,\mathcal{I}_n$$

Applications

- New representations of loops amplitudes (Geyer,Mason,Monteiro,Tourkine,Baadsgaard,Bjerrum-Bohr,Bourjaily,Caron-Huot,Daamgaard,Feng)



- Makes double copy manifest: GR = YM^2, special Galileon = NLSM^2, and many other relations (Cachazo,He,Yuan)

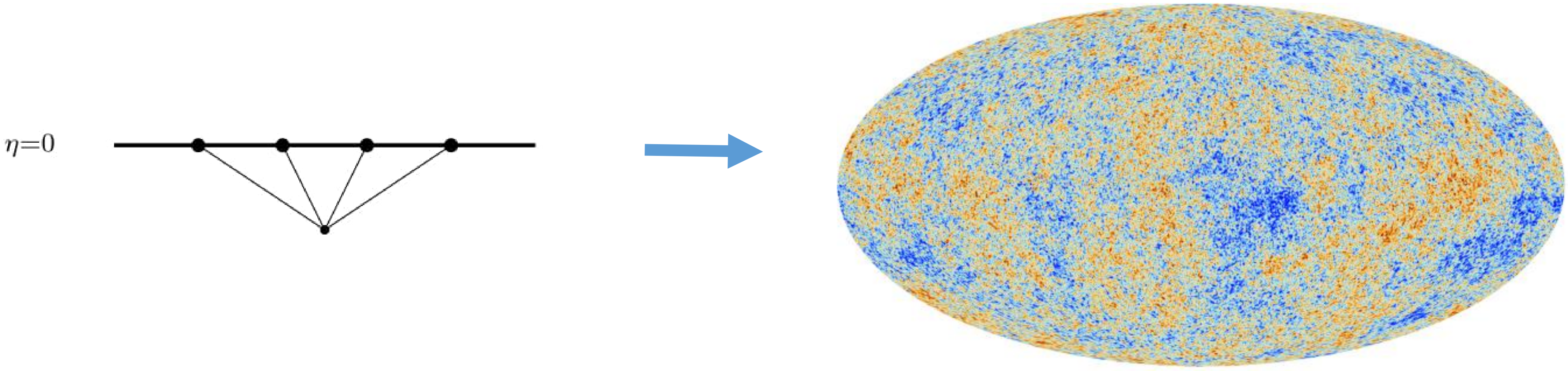
Questions:

- Can this progress be extended to cosmological observables?
- What does this teach us about amplitudes? Are SE and double copy fundamental or just an artefact of flat space?
- Other tools have recently developed for cosmological observables inspired by amplitudes: Mellin-Barnes (Sleight,Torrona), factorisation (Arkani-Hamed,Baumann,Chen,Pueyo,Joyce,Lee,Pimentel), unitarity (Hillman,Pajer,Goodhew,Jazayeri,Meltzer)

Cosmological Observables

- Inflation: early Universe approximately described by dS4. CMB comes from correlations on future boundary

$$ds^2 = \frac{-d\eta^2 + (dx^i)^2}{\eta^2}, \quad -\infty < \eta < 0$$



Wavefunction of the Universe

- In-in correlators (Maldacena):

$$\left\langle \phi(\vec{k}_1)...\phi(\vec{k}_n) \right\rangle = \frac{\int \mathcal{D}\phi\, \phi(\vec{k}_1)...\phi(\vec{k}_n) \left|\Psi\left[\phi\right]\right|^2}{\int \mathcal{D}\phi \left|\Psi\left[\phi\right]\right|^2}$$

- Wavefunction:

$$\ln \Psi\left[\phi\right] = -\sum_{n=2}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \frac{\mathrm{d}^d k_i}{(2\pi)^d} \Psi_n\left(\vec{k}_1, ...\vec{k}_n\right) \phi(\vec{k}_1)...\phi(\vec{k}_n)$$

- Ψ_n can be treated like CFT correlator in the future boundary and computed from Witten diagrams (Maldacena,Pimentel,McFadden,Skenderis)

dS momentum space

- We Fourier transform boundary correlators to momentum space, which is standard for cosmology and useful for studying soft limits and factorisation. Momentum conserved along boundary, but energy not conserved in bulk.
- boundary conformal generators (annihilate Ψ_n):

$$P^i = k^i, \qquad K_i = k_i \partial^j \partial_j - 2k^j \partial_j \partial_i - 2(d-\Delta)\partial_i,$$
$$D = k^i \partial_i + (d-\Delta), \qquad M_{ij} = (k_i \partial_j - k_j \partial_i)\,.$$

- mass of bulk scalar: $m^2 = \Delta(d-\Delta)$

Witten Diagrams

- Bulk-to-boundary prop:

$$\mathcal{K}_\nu(k,\eta) = \mathcal{N} k^\nu \eta^{d/2} H_\nu(-k\eta), \;\; \nu = \Delta - d/2, \; k = |\vec{k}|$$

- Contact diagram:

$$\mathcal{C}_n^\Delta \equiv \int \frac{d\eta}{\eta^{d+1}} \prod_{a=1}^{n} \mathcal{K}_\nu(k_a,\eta)$$

- Bulk-to-bulk prop:

$$\left(\eta^2\partial_\eta^2 + (1-d)\eta\partial_\eta + \eta^2 k^2 + m^2\right) G_\nu(k,\eta,\bar{\eta}) = \eta^{d+1}\delta(\eta-\bar{\eta})$$

(Maldacena,Pimentel,Raju)

Scattering Equations in dS

- How do we lift the SE to dS momentum space?

$$k_a \cdot k_b \to \mathcal{D}_a \cdot \mathcal{D}_b$$

$$\mathcal{D}_a \cdot \mathcal{D}_b = \frac{1}{2}(P_a^i K_{bi} + K_{ai} P_b^i - M_{a,ij} M_b^{ij}) + D_a D_b$$

$$\delta^{d+1}\left(\sum_{a=1}^{n} k_a^\mu\right) \to \delta^d\left(\sum_{a=1}^{n} \vec{k}_a\right) \mathcal{C}_n^\Delta$$

- Inspired by scattering equations in AdS embedding coordinates (Eberhardt,Komatsu,Mizera,Roehrig,Skinner)

Flat Space Limit

- Differential ops act in a simple way on bulk-to-boundary propagators:

$$(\mathcal{D}_a \cdot \mathcal{D}_b)\mathcal{K}_\nu^a \mathcal{K}_\nu^b = \eta^2[\partial_\eta \mathcal{K}_\nu^a \partial_\eta \mathcal{K}_\nu^b + (\vec{k}_a \cdot \vec{k}_b)\mathcal{K}_\nu^a \mathcal{K}_\nu^b]$$

- Flat space limit can be read off from $\eta \to -\infty$ limit:

$$\lim_{\eta \to -\infty} \mathcal{K}_\nu(k,\eta) \propto k_i^{\nu-1/2} \eta^{(d-1)/2} e^{ik\eta}$$

$$\lim_{\eta \to -\infty} \mathcal{D}_a \cdot \mathcal{D}_b \left(\mathcal{K}_\nu^a \mathcal{K}_\nu^b\right) = \eta^2 (k_a \cdot k_b) \mathcal{K}_\nu^a \mathcal{K}_\nu^b$$

- Hence, we obtain flat space SE

Worldsheet formula

- Tree-level wavefunction for massive ϕ^4 in dS (toy model for inflation):

$$\Psi_n = \frac{\delta^d(\vec{k}_T)}{(3!)^{p-1}} \sum_{\rho \in S_{n-1}} \text{sgn}_\rho \, \mathcal{A}(\rho(1,2,\ldots,n-1),n)\, \mathcal{C}_n^{\Delta}$$

$$\mathcal{A}(\mathbb{1}_n) = \int_\gamma \prod_{a \neq b,c,d}^{n} d\sigma_a \, S_a^{-1} (\sigma_{bc}\sigma_{cd}\sigma_{db})^2 \, \mathcal{I}(\mathbb{1}_n), \quad S_a = \sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{2\,(\mathcal{D}_a \cdot \mathcal{D}_b) + \mu_{ab}}{\sigma_{ab}} \equiv \sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{\alpha_{ab}}{\sigma_{ab}}$$

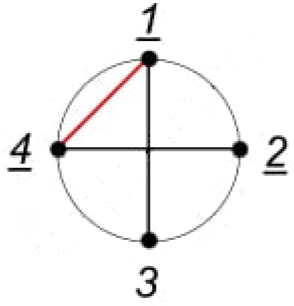
$$\mathcal{I}(\mathbb{1}_n) = (\sigma_{12}\sigma_{23}\ldots\sigma_{n1})^{-1}\, \text{Pf}' A \times \sum_{\{a,b\} \in cp(\mathbb{1}_n)} \frac{\text{sgn}(\{a,b\})}{\sigma_{a_1 b_1} \cdots \sigma_{a_p b_p}}, \quad A_{rs} = \begin{cases} \frac{\alpha_{rs}}{\sigma_{rs}}, & r \neq s \\ 0, & r = s \end{cases}$$

- Mass deformation $\mu_{a\,a\pm 1} = -m^2$ found by Dolan, Goddard in flat space
- We have checked this up to 8 points. No ambiguities!

Example: 4 points

- Fix legs 1,2,4 and removing rows/columns 1,4 from A-matrix:

$$\Psi_4 = \delta^3\left(\vec{k}_T\right)\int_{\hat{\gamma}_3} d\sigma_3 \frac{\sigma_{12}\sigma_{34}}{\sigma_{23}}\frac{1}{\hat{S}_3}\alpha_{23}\mathcal{C}_4^{\Delta}$$



where contour encircles pole at

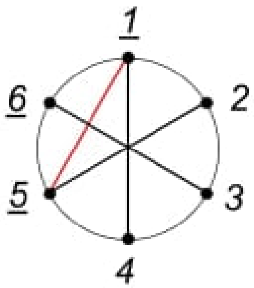
$$\hat{S}_3 = \sigma_{13}\sigma_{23}\sigma_{43}S_3 = \alpha_{13}\sigma_{23}\sigma_{43} + \alpha_{23}\sigma_{13}\sigma_{43} + \alpha_{43}\sigma_{13}\sigma_{23}$$

- Wrap contour around pole at σ_{23} and evaluate residue:

$$\Psi_4 = \delta^3\left(\vec{k}_T\right)\left.\frac{\sigma_{12}\sigma_{34}}{\hat{S}_3}\right|_{\sigma_{23}=0}\alpha_{23}\mathcal{C}_4^{\Delta} = \mathcal{C}_4^{\Delta}$$

Higher Points

- At 6-points, we have the following perfect matchings:



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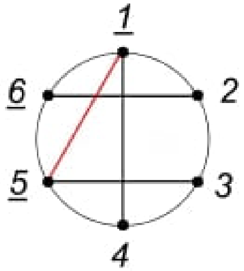
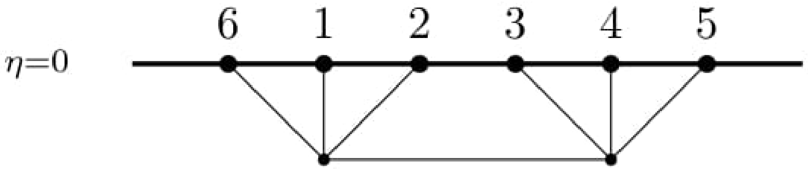


+ perms

- Using global residue theorem, first one vanishes and second one is

$$[(\mathcal{D}_3+\mathcal{D}_4+\mathcal{D}_5)^2+m^2]^{-1}\mathcal{C}_6^{\Delta} \rightarrow$$



- Key identity: $[(\mathcal{D}_1+\ldots+\mathcal{D}_p)^2+m^2]^{-1}\mathcal{C}_n^{\Delta} = \int \frac{d\eta}{\eta^{d+1}}\frac{d\tilde{\eta}}{\tilde{\eta}^{d+1}} U_{p+1,n}(\eta) G_\nu(k_{1\ldots p},\eta,\tilde{\eta}) U_{1,p}(\tilde{\eta})$

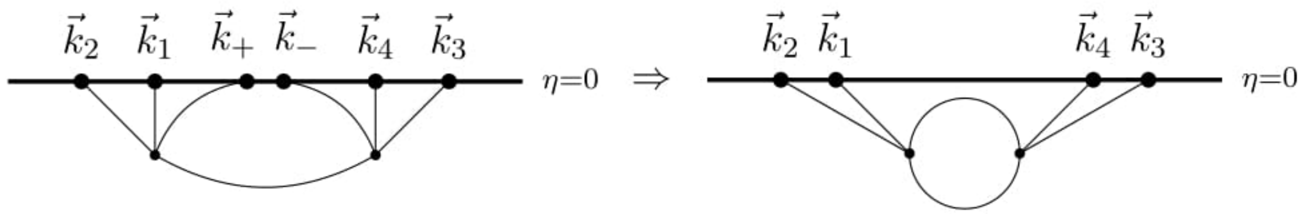
Loops

- 1-loop n-point correlator from deformed tree-level (n+2)-point:

$$\Psi_n^{1-\text{loop}} = -\frac{1}{\pi}\int \mathrm{d}^d\ell \int_{-\infty}^{\infty} \frac{\omega^2 \mathrm{d}\omega}{\omega^2+\nu^2} \lim_{\vec{k}_\pm \to \pm\vec{\ell}} \tilde{\Psi}_{n+2}\left(\left(\nu_+,\vec{k}_+\right),\left(\nu_1,\vec{k}_1\right),\ldots,\left(\nu_n,\vec{k}_n\right),\left(\nu_-,\vec{k}_-\right)\right)$$

where $\nu_\pm = \pm i\omega,\ \nu_1 = \ldots. = \nu_n = \nu = \Delta - d/2$

- Integral over ω pastes together auxiliary legs:



- Similar structure found in position space (Herderschee)

Next Steps:

- More general scalar theories
- Break boost symmetry to connect with observations
- Spinning correlators
- Systematic formulation of double copy in dS
- Solve the CSE

Double Copy in Flat Space

- 3-point gluon amplitudes square into 3-point graviton amplitudes
- Color/kinematics duality can be used to extend double copy beyond three points. Take a 4-point gluon amplitude:

$$\mathcal{A}_4 = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}, \quad c_s + c_u + c_t = 0$$

It is possible to choose $n_s + n_t + n_u = 0$

- Squaring these numerators gives 4-point graviton amplitude:

$$\mathcal{M}_4 = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

(Bern,Carrasco,Johansson)

Double Copy in (A)dS

- 3-point correlators inherit double copy from flat space limit, but there are additional simplifications and extensions beyond the flat space limit in d=3. Farrow,Lipstein,McFadden

- 4-point gluon correlators also have color/kinematics duality! Armstrong,Mei,Lipstein,Albayrak,Kharel,Meltzer,Alday,Behan,Ferrero,Zhou, Diwakar,Herderschee,Roiban,Teng,Drummond,Glew,Santagata

- Also holds for NLSM (Diwakar,Herderschee,Roibana,Teng,Cheung,Parra-Martinez,Sivaramakrishnan)

- Naively squaring numerators only works for supersymmetric theories in AdS5 (Zhou) but I am optimistic that it can be generalised!

Effective Field Theories

- NLSM: $\mathcal{L}_{\text{NLSM}} = \frac{1}{8\lambda^2}\text{Tr}(\partial_\mu U^\dagger \partial^\mu U), \quad U = (\mathbb{I} + \lambda\Phi)(\mathbb{I} - \lambda\Phi)^{-1}$

- DBI: $\mathcal{L}_{\text{DBI}} = \frac{1}{\lambda}\left(\sqrt{1 - \lambda\,(\partial\phi)^2} - 1\right)$

- sGal: $\mathcal{L}_{\text{sGal}} = -\frac{1}{2}\,(\partial\phi)^2 - \frac{\lambda}{8}\,(\partial_\mu\partial_\nu\phi)^2\,(\partial\phi)^2$

CHY Formulae

Theory	Integrand
NLSM	$\mathrm{PT}(\mathrm{Pf}'A)^2$
DBI	$\mathrm{Pf}X(\mathrm{Pf}'A)^3$
sGal	$(\mathrm{Pf}'A)^4$

$$\mathrm{PT} = (\sigma_{12}\sigma_{23}...\sigma_{n1})^{-1} \qquad \mathrm{Pf}'A = \frac{(-1)^{c+d}}{\sigma_{cd}}\mathrm{Pf}A^{cd}_{cd}$$

$$A_{rs} = \begin{cases} \frac{2k_r \cdot k_s}{\sigma_{rs}}, & r \neq s, \\ 0, & r = s, \end{cases} \qquad X_{rs} = \begin{cases} \frac{1}{\sigma_{rs}}, & r \neq s, \\ 0, & r = s, \end{cases}$$

- Double copy: $\mathrm{PT} \rightarrow \mathrm{Pf}X(\mathrm{Pf}'A),$ DBI = NLSM x YM

  $\mathrm{PT} \rightarrow (\mathrm{Pf}'A)^2,$ sGal= NLSM^2

Curvature Corrections

- When lifting to curved background, new terms can arise. Restricting to 4-point vertices: (Heemskerk,Penedones,Polchinski,Sully)

$$S_4^{NLSM} = -\int d^4x\sqrt{-g}\{\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi + \tfrac{1}{2}m^2\Phi^2 + \lambda^2\Phi^2\nabla\Phi\cdot\nabla\Phi + \tfrac{1}{4}C\Phi^4\}$$

$$S_4^{(6)} = -\int d^4x\sqrt{-g}\{\tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{8}A(\nabla_\mu\nabla_\nu\phi)^2\nabla\phi\cdot\nabla\phi + \tfrac{1}{8}B(\nabla\phi\cdot\nabla\phi)^2 + \tfrac{1}{4!}C\phi^4\}$$

For sGal, A≠0 and B,C curvature corrections.

for DBI, A=0 and C is curvature correction.

4-point Correlators

- Using Witten diagrams, we obtain

$$\Psi_4^{NLSM} = -\delta^3\left(\vec{k}_T\right)\left(2\lambda^2\hat{u} + C\right)\mathcal{C}_4^{\Delta}$$

$$\Psi_4^{(6)} = \delta^3\left(\vec{k}_T\right)[A(\hat{s}^3 + \hat{t}^3 + \hat{u}^3) + (dA - B)(\hat{s}^2 + \hat{t}^2 + \hat{u}^2) - C]\mathcal{C}_4^{\Delta}$$

where $\hat{s} = \mathcal{D}_1 \cdot \mathcal{D}_2, \qquad \hat{t} = \mathcal{D}_1 \cdot \mathcal{D}_4, \qquad \hat{u} = \mathcal{D}_1 \cdot \mathcal{D}_3$

- Using CSE, this can be obtained from simple building blocks

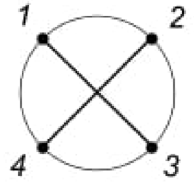
Building Blocks

- Integrand constructed from:

$$\mathrm{PT}\left(\mathrm{Pf}'A\right)^2 \qquad \mathrm{Pf}X\left(\mathrm{Pf}'A\right)^3 \qquad \mathrm{PT}\,\mathrm{Pf}X|_{\mathrm{conn}}\,\mathrm{Pf}'A \qquad (\mathrm{Pf}'A)^3\,\mathrm{Pf}X|_{\mathrm{conn}}$$

$$(\mathrm{Pf}'A)^4 = \frac{1}{3}\left\{\frac{1}{\sigma_{34}^2}(\mathrm{Pf}A_{34}^{34})^2\,\frac{(-1)}{\sigma_{23}}(\mathrm{Pf}A_{23}^{23})\frac{1}{\sigma_{24}}\,(\mathrm{Pf}A_{24}^{24}) + \mathrm{cyclic}(2,3,4)\right\}$$

where $\mathrm{Pf}\,X = \frac{1}{\sigma_{12}\sigma_{34}} - \frac{1}{\sigma_{13}\sigma_{24}} + \frac{1}{\sigma_{23}\sigma_{14}}, \quad \mathrm{Pf}\,X|_{\mathrm{conn}} = -\frac{1}{\sigma_{13}\sigma_{24}}$



- Choose Pfaffians to obtain permutation-invariant result

Generalised Double Copy

- 4-point correlators obtained from the following integrands:

$$\mathcal{I}_4^{NLSM} = \lambda^2 \mathrm{PT} \left(\mathrm{Pf}' A\right)^2 + c \mathrm{PT}\, \mathrm{Pf} X|_{\mathrm{conn}} \mathrm{Pf}' A$$

$$\mathcal{I}_4^{(6)} = a(\mathrm{Pf}' A)^3 (\mathrm{Pf}' A + m^2\, \mathrm{Pf} X|_{\mathrm{conn}}) + b(\mathrm{Pf}' A)^2 (\mathrm{Pf}' A \mathrm{Pf} X + m^2 \mathrm{PT}) + c(\mathrm{PT}\, \mathrm{Pf} X|_{\mathrm{conn}} \mathrm{Pf}' A)$$

where $A = \frac{8}{3}a, \;\; B = 2a\left(m^2 + \frac{4}{3}d\right) - 2b, \;\; C = -\frac{1}{3}am^6 + bm^4 - c$

- Second integrand can be obtained from first via

$$\lambda^2 \mathrm{PT} \to a \mathrm{Pf}' A \left(\mathrm{Pf}' A + m^2\, \mathrm{Pf} X|_{\mathrm{conn}}\right) + b\left(\mathrm{Pf}' A \mathrm{Pf} X + m^2 \mathrm{PT}\right)$$

Soft Limits

- Inflationary 3-point can be obtained from soft limit of dS 4-point (Creminelli,Kundu,Shukla,Trivedi,Assassi,Baumann,Greene)
- Soft limits also can be written in terms of boundary conformal generators acting on contact diagrams!
- Example: conformally coupled scalar

$$\lim_{\vec{k}_1\to 0} C_4^{\Delta=2} = \mathcal{C}_{3,\eta}^{\Delta=2} \qquad \lim_{\vec{k}_1\to 0} \hat{u}\mathcal{C}_4^{\Delta=2} = D_3\,\mathcal{C}_{3,\eta}^{\Delta=2} \qquad \lim_{\vec{k}_1\to 0} (\hat{s}^2+\hat{t}^2+\hat{u}^2)\mathcal{C}_4^{\Delta=2} = 2(D_2^2+D_3^2+D_4^2)\mathcal{C}_{3,\eta}^{\Delta=2}$$

$$\lim_{\vec{k}_1\to 0} (\hat{s}^3+\hat{t}^3+\hat{u}^3)\mathcal{C}_4^{\Delta=2} = \left(6(D_2^3+D_3^3+D_4^3) - 22(D_2^2+D_3^2+D_4^2) + 20\right)\mathcal{C}_{3,\eta}^{\Delta=2}$$

where $\mathcal{C}_{3,\eta}^{\Delta=2} = \int \frac{d\eta}{\eta^4}\left(\eta \prod_{i=2}^{4} \mathcal{K}_{1/2}^i\right) = \frac{1}{E}$

Hidden Symmetries?

- Can choose coefficients to obtain simple soft limits:

$$\{A, B, C\} = \{1, -8, -20\} \longrightarrow \lim_{\vec{k}_1 \to 0} \Psi_4^{(6)} = 6(D_2^3 + D_3^3 + D_4^3)\mathcal{C}_{3,\eta}^{\Delta=2}$$

- In flat space, soft limits encode hidden symmetries (Adler,Low,Cheung,Kampf,Novotny,Trnka,Hinterbichler,Joyce,Padilla, Stefanzyn,Wilson)

- Lagrangians for DBI and sGal with hidden symmetries in dS4 recently proposed by Bonifacio,Hinterbichler,Joyce,Roest

- Do their correlators have simple soft limits? Does this provide a principle for fixing coefficients of generalised double copy?

Thanks!