Mock Modularity from Supergravity?

Abhiram Kidambi
Kavli IPMU, U.Tokyo &
Riemann Center for Geometry and Physics, Leibniz Universität Hannover

LPTHE seminar, 16.06.2022

Based on

- Past work with Abhishek Chowdhury, Sameer Murthy, Valentin Reys & Timm Wrase
  "Dyonic black hole degeneracies in N = 4 string theory from Dabholkar-Harvey degeneracies"

- Works in progress with:
  Suresh Nampuri, Gabriel Cardoso Lopes, Martí Rossello & Valentin Reys

Part 1: Motivation of the problem for non-experts/non-string theorists

Black hole partition functions, & how to construct them

Part 2: Some background on automorphic forms

Part 3: Constructing 4d black hole partition functions in supergravity/string theory with 16 supercharges and study their automorphic symmetries.

Complex analytic structure of the partition function leads to non-trivial phenomena which leads to partition functions with "richer" automorphic symmetry (mock-modularity).

Part 4: How do we reconcile this mock modular symmetry with gravity?

For the physically oriented:
In the context of this talk, we will motivate how and possible ways of seeing how supergravity might see this symmetry.

For the mathematically oriented:
Re-casting AdS/CFT as a dictionary between geometry and number theory.

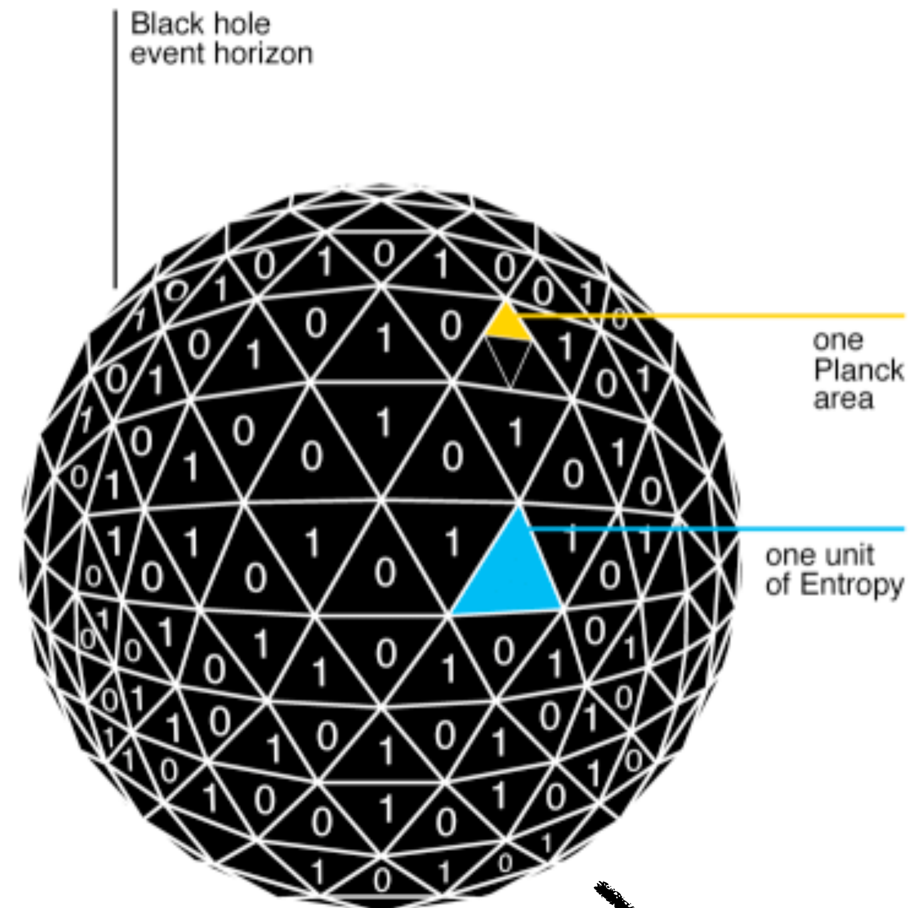
Part 1: Motivation

Classically, black holes have no entropy.
Semi-classically, black holes have entropy. [Bekenstein; Hawking]
What is the quantum origin of this entropy?

Semiclassical black hole entropy:

$$S_{\text{Bekenstein-Hawking}} = \frac{A}{4G_N}$$

Known as the "area law" for black holes. It is universal.



Q: Is there a partition function (PF)/microstate counting function Z such that

$$\ln Z = S_{\text{Bekenstein-Hawking}} = \frac{A}{4G_N} \; ?$$

Image taken from Sean Carroll's blog.

This paradigm is not entirely sufficient.

The Bekenstein-Hawking entropy is a semi-classical entropy: No quantum corrections
Works when the radius of the black hole $\gg 1$.
(In string theory it works in the limit of large charges, no $O(R^2)$ corrections,..)

Quantum entropy of a black hole: $S_{BH} = \frac{A}{4G} + \underbrace{c_0 \ln A + c_1 \frac{1}{A} + c_2 \frac{1}{A}^2 + \cdots + c_n e^{-A}}_{\substack{\text{Sub-leading/quantum corrections} \\ \text{Model dependent}}}$

More pertinent Qs:

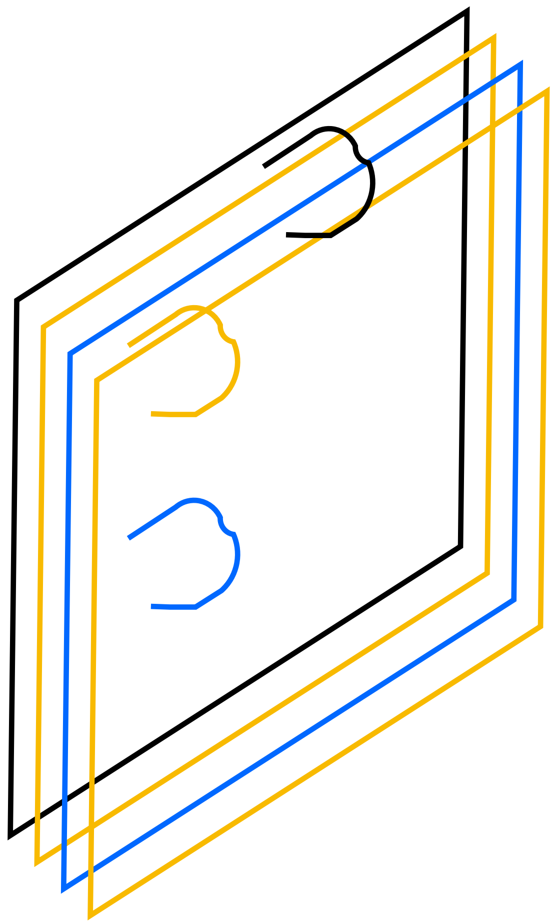
a) Is there a function Z_{BH} such that $S_{BH} = \ln Z_{BH}$? What is the quantum PF of a black hole?

b) What is the corresponding “geometry” of a given state counted by the PF?

Computing the quantum entropy requires ability to see all the saddle points of the Euclidean path integral. Such saddles are difficult to evaluate semi-classically.

However, in string theory, one can evaluate all these saddle contributions exactly.

Before we proceed, it usually helps to understand how black holes arise in string theory.

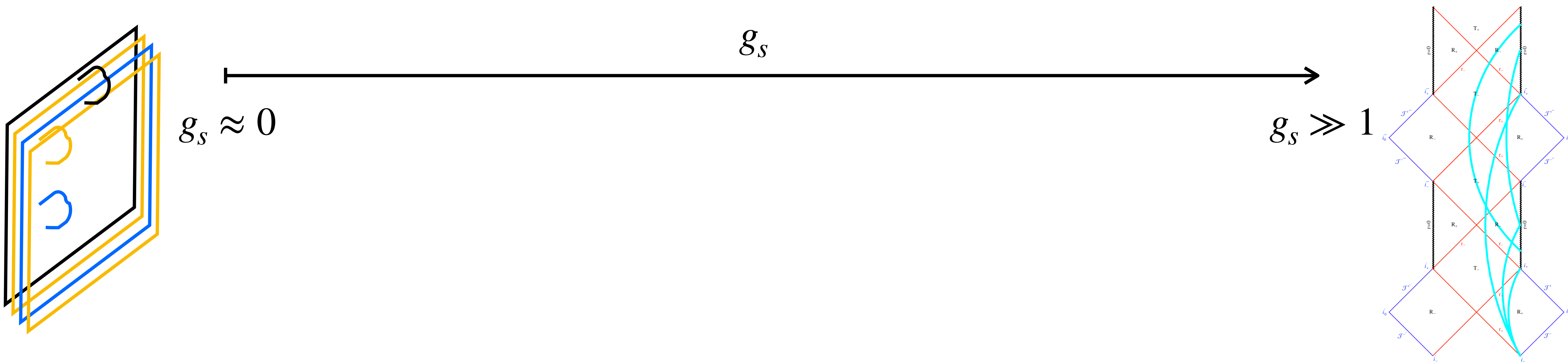


Key ingredients are

a. Strings
b. Extended "solitonic" objects called (**D**)(NS)Branes

At low string coupling (g_s), the D-branes have no back reaction with gravity. Increase g_s, the D-branes back react and strongly couple to gravity/background spacetime. Spacetime eventually becomes singular.

Break down of the problem based on dualities. At different values of g_s the brane dynamics is can be obtained from different "theories".



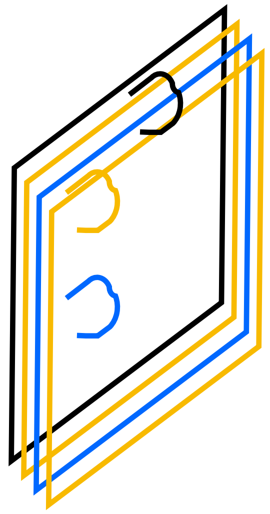
World volume theory of the D branes
Superconformal field theory

Supergravity

In string theory, the number of string states grows as e^M (M is mass), while the number of black hole states in supergravity grows as e^{M^2}. Expect some sort of renormalization to account for the scaling of number of states. [Susskind ’93, Susskind-Uglum ’94, Russo-Susskind ‘95]

The existence/proof of concept of such a black hole mass renormalization is *still unknown*.

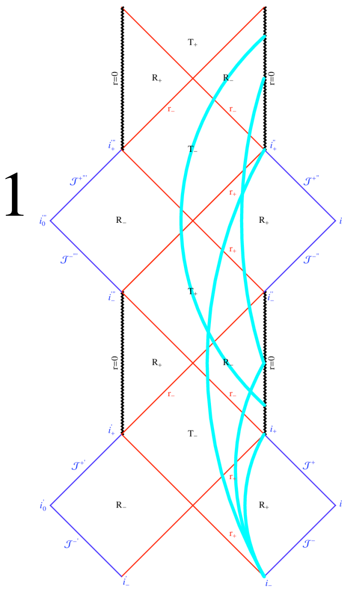
Could restrict to special Hilbert subspaces in string theory with extended supersymmetry (BPS states). Protected by non-renormalization theorems, degeneracy/spectrum is independent of g_s.



g_s

$g_s \approx 0$ $\longrightarrow$ $g_s \gg 1$

Holography!
AdS/CFT



World volume theory of the D branes
Superconformal field theory

Supergravity

$$\ln Z^{d-1}_{SCFT} = \ln Z^{d}_{grav}$$

$$= \frac{A}{4G} + c_0 \ln A + c_1 \frac{1}{A} + c_2 \frac{1}{A}^2 + \cdots + c_n e^{-A}$$

Derived for such BPS solutions as the leading order growth of density of states in the SCFT aka "Cardy Formula" [Strominger-Vafa '95]

How does one obtain all quantum corrections to the entropy in the partition function? How does one ensure that it matches with supergravitational quantum BH entropy? Does the BH quantum PF yield integer degeneracies?

We will mostly consider the case of maximal and half-maximal supersymmetry in *4d*.

For $\mathcal{N} = 8$: D1-D5-p-KK on $T^6 = T^4 \times S^1 \times \tilde{S}^1$

- a D1 brane on S^1
- a D5 brane on $\tilde{S}^1 \times T^4$
- n units of momentum along S^1
- ℓ units of momentum along $\tilde{S}^1$
- KK monopole along $\tilde{S}^1$

The charge invariant $\Delta_{\mathcal{N}=8} = 4n - \ell^2$

For $\mathcal{N} = 4$: D1-D5-p-KK on $K3 \times T^2 = K3 \times S^1 \times \tilde{S}^1$

- Q_1 D1 brane on S^1
- Q_5 D5 brane on $\tilde{S}^1 \times K3$
- n units of momentum along S^1
- ℓ units of momentum along $\tilde{S}^1$
- KK monopole along $\tilde{S}^1$

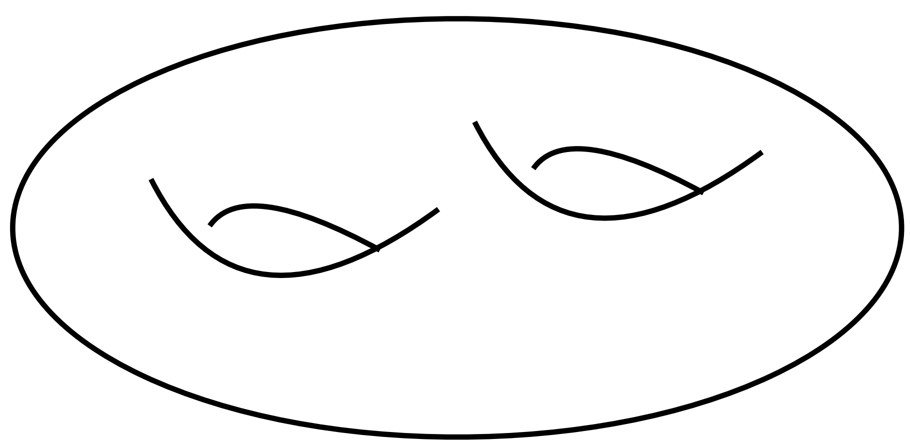
The charge invariant $\Delta_{\mathcal{N}=4} = 4mn - \ell^2$, where $m = Q_1 Q_5$

To study 4d BPS black holes, you need to compactly pack away (compactify) the extra dimensions by wrapping the D branes on “cycles” of Calabi-Yau manifolds (complex, Ricci flat manifolds) as seen on the previous slide.

In this compactified picture, the “gravitational” part of the black hole spacetime is AdS_3, and the world sheet part is a two dimensional superconformal field theory.

Computing the quantum PF of such 4d BPS black holes is therefore the same as computing the BPS PF of the 2d CFT.

Σ_g



PFs have to behave a certain way under the deformation of such genus g Riemann surfaces

Part 2: Automorphic forms

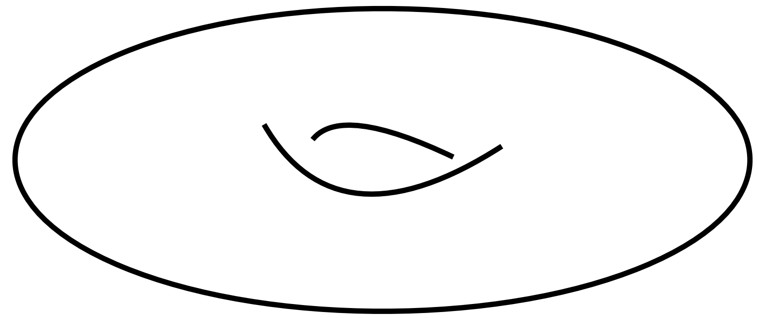
An automorphic form is a complex function $f : G \longrightarrow \mathbb{C}$, for G a topological group, such that three properties are satisfied:

1. If Γ is a discrete subgroup of G, f has nice transformation properties:
$$f(\gamma \cdot g) = j(\gamma)f(g),\ g \in G,\ \gamma \in \Gamma,\ j \text{ is a "factor of automorphy"}$$

2. f is an eigenfunction of certain Laplacians on G

3. f satisfies some growth condition as g approaches the boundary of the domain of G.

$\Sigma_{g=1}$



If $G = SL(2,\mathbb{R})$, $\Gamma = SL(2,\mathbb{Z})$, the automorphic forms are called "modular forms".

Let $\mathbb{H} = \left\{\tau \in \mathbb{C} \,|\, \Im\tau > 0\right\} \simeq SL(2,\mathbb{R})/SO(2,\mathbb{R})$.

A holomorphic modular form f of weight $k \in 2\mathbb{Z}_+$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, such that

a. $f(\tau) \mapsto f\left(\dfrac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau),\ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

b. f is bounded as $\tau \to i\infty$

I won't go into the Laplacian/Quadratic Casimir here. It does become important when one considers spectral theory of automorphic forms.

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z})$, $f(\tau) \simeq f(\tau + 1)$: $f(\tau) = \sum_{n=0}^{\infty} c(n) q^n$, $q := e^{2\pi i \tau}$ (Fourier expansion)

Holomorphic modular on $SL(2,\mathbb{Z})$ form generated by $E_4(\tau)$ and $E_6(\tau)$ (Weight 4 and 6 Eisenstein series on $SL(2,\mathbb{Z})$, respectively)

$$\{f_k\} =: M_k, \quad f_k = \sum_{\substack{i,j \\ 4i+6j=k}} \alpha E_4(\tau)^i E_6(\tau)^j, \quad M = \bigoplus_{k=4}^{\infty} M_k, \text{ where } \alpha \in \mathbb{R}$$

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} q^n \sigma_3(n) \qquad E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} q^n \sigma_5(n)$$

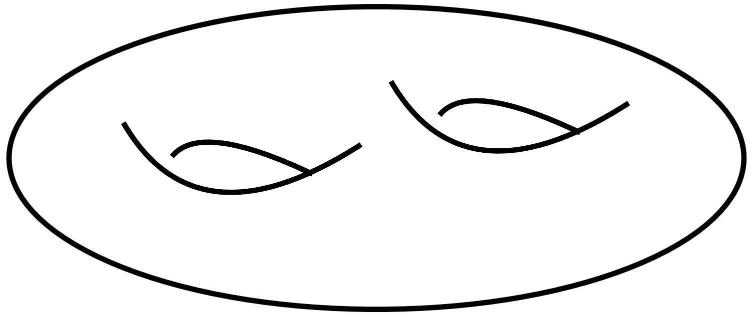
Eisenstein Series

$$\sigma_d(n) = \sum_{x|n} x^d : \text{Divisor Sigma Function} \quad (\text{Ex: } \sigma_3(6) = \sum_{x|6} x^3 = 1^3 + 2^3 + 3^3 + 6^3 = 251)$$

Example: $\eta(\tau)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \dfrac{E_4(\tau)^3 - E_6(\tau)^2}{1728}$ is a weight 12 modular form

$$\frac{1}{\eta(\tau)^{24}} = \frac{1}{q} + 24 + 324q + O(q^2) : \text{PF of 24 free bosons on a torus (weakly holomorphic)}$$

Also the indexed PF of perturbative 1/2-BPS states in the Heterotic frame ($4d$, $\mathcal{N} = 4$)

$\Sigma_{g=2}$ 

If $G = Sp(2,\mathbb{R})$, $\Gamma = Sp(2,\mathbb{Z})$, automorphic forms are "Siegel modular forms"

$$Sp(2,\mathbb{Z}) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Mat_{4\times4}(\mathbb{Z}) \,\middle|\, M^T J M = J, \text{ for } J = \begin{pmatrix} 0 & -\mathbf{I}_2 \\ \mathbf{I}_2 & 0 \end{pmatrix} \right\}$$

Let $\mathbb{H}_2 = \left\{ \Omega = \begin{pmatrix} \tau & \sigma \\ \sigma & \rho \end{pmatrix} \in \mathbb{C} \,|\, \Im\tau, \Im\sigma > 0, \det(\Omega) > 0 \right\}$.

A Siegel modular form f of weight k is a function $f : \mathbb{H}_2 \to \mathbb{C}$, such that

$$f(\Omega) \mapsto f(A\Omega + B)(C\Omega + D)^{-1} = \det(C\Omega + D)^k f(\Omega), \ \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbb{Z})$$

Siegel Modular Forms also admit Fourier expansions, as well as Fourier-Jacobi expansions

$$f_k(\Omega) = \sum_{n,m,\ell} d(m,n,\ell)\ q^m p^m y^l, \text{ where } q := e^{2\pi i\tau},\ p := e^{2\pi i\sigma},\ y := e^{2\pi i z}$$

$$f_k(\Omega) = \sum_m \psi_m(q,\ell) q^n y^l$$

Jacobi form of weight k and ***index*** m

It is also key to motivate a "Jacobi form" which in some sense are "elliptic" versions of modular forms. Physically, they are "refinements" of a modular form by the supersymmetric R-charge.

Indexed PFs refined under R-charge: $Z = \mathrm{Tr}_{\mathscr{H}}\left((-1)^F q^{L_0-\frac{c}{24}} y^{2J_0}\right)$, $y = e^{2\pi i z}$ is the "elliptic" variable

Jacobi forms are automorphic forms on $SL(2,\mathbb{Z}) \rtimes \mathrm{H}(\mathbb{Z}_2)$* (*A particular parametrization choice due to [Eichler-Zagier]. I will tell you what this is later since it does play a key role.)

A Jacobi form of weight k and index m is a function $f : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ such that, for $\tau \in \mathbb{H}, z \in \mathbb{C}$,

a. $f(\tau, z) \to f\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}} f(\tau, z),\ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

b. $f(\tau, z) \to f(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} f(\tau, z),\ \forall\ \lambda, \mu \in \mathbb{Z}$

Jacobi forms are periodic in both their arguments and admit a Fourier expansion

$$f(\tau, z) = \sum_{n,\ell} c(n, \ell) q^n y^\ell$$

We will be interested in objects known as weak Jacobi forms.

- Weak Jacobi forms exhibit exponential growth of degeneracy (just like weakly holomorphic modular forms)

  They are generated by the ring:

$$\left\langle \phi_{-2,1}(\tau,z) = \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}, \phi_{0,1}(\tau,z) = 4\sum_{i=2}^{4} \frac{\vartheta_i(\tau,z)^2}{\vartheta_i(\tau,0)^2}, E_4(\tau), E_6(\tau) \right\rangle$$

  [Eichler, Zagier, Feingold, Frenkel]

- Examples include the elliptic genus of K3 which is $2\phi_{0,1}(\tau,z)$

Certain types of these automorphic forms (modular, Siegel, Jacobi) correspond to examples of physical PF's.

Modular forms: Black holes with electric **or** magnetic charges (zero area)

Jacobi forms: Black holes with electric **or** magnetic charges + angular momentum

Siegel modular forms: Black holes with electric **and** magnetic charges + angular momentum

Fourier expansion coefficients: Degeneracies of black holes

Part 3: Constructing partitions

Warm up example: Black holes in $\mathcal{N} = 8$ 4d compactification of string theory

Type II String Theory on T^6: Theory with $\dfrac{1}{8}$$-$BPS dyonic black holes

The indexed partition(*) of these black holes are the Jacobi form $\phi_{-2,1}(\tau, z) = \dfrac{\vartheta_1^2(\tau, z)}{\eta(\tau)^6}$

[Maldacena-Moore-Strominger]

The Fourier expansion: $\phi_{-2,1}(\tau, z) = \sum_{(n,\ell)} c(n, \ell) q^n y^\ell = \sum_{(n,\ell)} C(\Delta = 4n - \ell^2) q^n y^\ell$

give you the BH degeneracies $d_{BH}(n, \ell) = (-1)^\ell c(n, \ell) = (-1)^{\Delta+1} C(\Delta)$

Instead of computing the Fourier expansion term by term, there is an analytic way to derive all $C(\Delta > 0)$ given $C(\Delta < 0)$ (*and information about modular properties and cusps) [Rademacher-Zuckermann] [G. Andrews]

Known as the Hardy-Ramanujan-Rademacher circle method

$$C(\Delta) = (2\pi)^{2-w} \sum_{\tilde{l} \in \mathbb{Z}/2m\mathbb{Z}} \sum_{\tilde{\Delta}<0} \sum_{c=1}^{\infty} C_{\tilde{l}}(\tilde{\Delta})\, Kl(\Delta, l, \tilde{\Delta}, \tilde{l}; c) \left|\frac{\tilde{\Delta}}{4m}\right|^{1-w} \times \dots$$

$$\dots \times \tilde{I}_{1-w}\left[\frac{\pi}{c}\sqrt{|\tilde{\Delta}|\,\Delta}\right]$$

$$\Delta = 4mn - l^2$$

$$\tilde{I}_w(x) = (x/2)^{-w} I_w(x) \Rightarrow \text{Modified I-Bessel fcn}$$

On the supergravity side: Difficult because the path integral is complicated (infinite dim integral)

$$Z_{BH}(q) = \left\langle \exp\left(-iq\oint A_I\right)\right\rangle^{reg.}_{AdS_2}$$

[Sen]

The leading term of this expression is the area law. But to evaluate the full PI with saddle contributions is hard. Localize this path integral (in the sense of Duistermaat-Heckman, Atiyah-Bott)

$Q \longrightarrow$ constrain field configurations that contribute to QEF

Supercharge

$\mathcal{M}_Q \longrightarrow$ localization manifold.

* Q ill defined when $g_{\mu\nu}$ fluctuates

* Equivariant S_{eq} from BRST operator

[de Wit, Murthy, Reys]

Find M_Q :

grav. sector: $\delta_{eq} \psi_\mu = 0$ [Attractor values] $+ AdS_2 \times S^2$

[Gupta, Murthy]

matter sector: Fluctuations of ϕ_I

Now, we can in principle compute Z_{BH}.

The path integral can be re-written in the form

$$Z_{BH}(p,q) = \left\langle \exp\left[-i q_I \oint d\tau\, A^I_\tau\right] \right\rangle^{\text{fin.}}_{AdS_2}$$

$$= \int_{M_Q} d\phi^I \; e^{-\pi q_I \phi^I + S(\phi + ip)} \; Z_{\text{1-loop}}(\phi)\, Z_{\underbrace{\text{measure}}}(\phi)$$

gaussian integration on M_Q

tricky to compute.

Action and measure :

$\mathcal{N}=2$ sugra action → F-terms (chiral superspace integrals)

→ D-terms (full superspace integrals)

D-terms ⟶ constructed by [de Wit, Katmadas, van Zalk]

vanish on M_Q

Conjecture : D-terms are Q-exact ?

F-terms + measure ⟶ Prepotential of $\mathcal{N}=2$ theory

$\mathcal{N} = 8$ Localization

$$Z_{1\text{-loop}} = K(\phi)^{-4}$$

$$K(y) := i\left(\bar{y}^1 F_1 - y^1 \bar{F}_1\right)$$

$$S = 4\pi \, \mathrm{Im}\, F(\phi + ip), \quad F(y) = -\frac{y^1}{y^0}\, y^a C_{ab} y^b$$

Intersection matrix of 6-cycles on $T^4 \subset T^6$.

Localization

$$Z_{\text{measure}} = \sqrt{\left| \det \left[\mathrm{Im}\, F_{IJ}(\phi + ip) \right] \right|}$$

[Cardoso, de Wit, Mahapatra +
Dabholkar, Murthy, Gomes]

Put everything together:

$$Z_{1/8} \overset{\text{l.o}}{\sim} \sqrt{2}\,\pi \left(\frac{1}{\Delta}\right)^{7/4} I_{7/2}\left(\pi\sqrt{\Delta}\right)$$

exp behaviour @ large $\pi\Delta$.

$\Rightarrow$ $K = 1$ term in the Rademacher expansion

Higher K : orbifolds of AdS_2

Rademacher expansion of a Jacobi form v/s Supersymmetric Localization

Consider Rademacher for $\phi_{-2,1} = Z_{1/8}(\tau, z)$

$$d(\Delta) = (-1)^{\Delta+1}\, 2\pi \left(\frac{\pi}{2}\right)^{7/2} \sum_{c=1}^{\infty} c^{-9/2}\, Kl_c(\Delta)\, \tilde{I}_{7/2}\left[\frac{\pi}{c}\sqrt{\Delta}\right]$$

only one term with $\Delta < 0$.

Rademacher

Δ	-1	0	3	4	7	8	11	12	15
$d(\Delta)$	1	2	8	12	39	56	152	208	513
$W_1(\Delta)$	1.040	1.855	7.972	12.201	38.986	55.721	152.041	208.455	512.958
$\exp(\pi\sqrt{\Delta})$	-	1	230.765	535.492	4071.93	7228.35	33506	53252	192401

Localization

[Dabholkar - Murthy- Gomes]

A dictionary

Microscopic		Macroscopic
Number theory		Geometry
"Modular" forms	$\longleftrightarrow$	Quantum Entropy fcn
BH deg: Fourier exp. coeffs of MF's	$\longleftrightarrow$	Coeffs of Bessels in localization of QEF
Rademacher techique	$\longleftrightarrow$	Localization of QEF

$\mathcal{N} = 4$, $d = 4$ black holes

* More complicated than $\mathcal{N} = 8$.

* Type $\overline{\underline{\text{II}}}$ on $K3 \times T^2 \longleftrightarrow$ Het on T^6

[Aspinwall]

- $\frac{1}{2}$ BPS black holes $\rightarrow$ small i.e $A = 0$
- $\frac{1}{4}$ BPS black holes $\rightarrow$ large i.e $A \neq 0^*$

Zero area black holes

$\frac{1}{2}$ BPS states:

* $$Z_{1/2}(\tau) = \frac{1}{\eta(\tau)^{24}} = q^{-1} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{24}}$$

[Dabholkar, Harvey]

* Purely Electric | magnetic

* No wall crossing. → Sugra analysis is simple

  $A = 0$, attractor mechanism

[Cardoso, de Wit, Mohaupt, Kapelli]

Large black holes

$\frac{1}{4}$ BPS states : [Dijkgraaf, Verlinde^2]

$$EG_{K3}[\tau,z] = 2\phi_{0,1}(\tau,z) = 8\sum_{i=2}^{4} \frac{\vartheta_i(\tau,z)^2}{\vartheta_i(\tau,0)^2}$$

$$= \sum_{n,L} c(n,L)_{K3}\, q^n y^L$$

LIFT

$$Z_{1/4}(\tau,\sigma,z) = \frac{1}{\Phi_{10}(\tau,\sigma,z)} = (qyp)^{-1} \prod_{(m,n,L)>0} \left(1-q^m p^n y^L\right)^{2C_{K3}(4mn-L^2)}$$

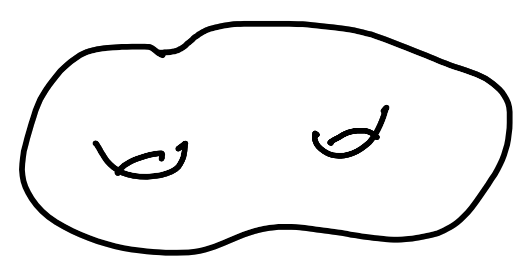
"Igusa cusp form"

$e(\tau) = q$, $e(z) = y$, $e(\sigma) = p$ [Kawai, DVV, Nikulin]

Large black holes & Siegel Modular Forms

Conjecture: $\frac{1}{\Phi_{10}}$ counts ALL $\frac{1}{4}$-BPS states in $\mathcal{N}=4$, $d=4$

$$d_{1/4} = (-1)^{L+1} \int_{C} \frac{q^n y^L p^m}{\Phi_{10}(\Omega)}\, d\Omega \qquad \Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}$$

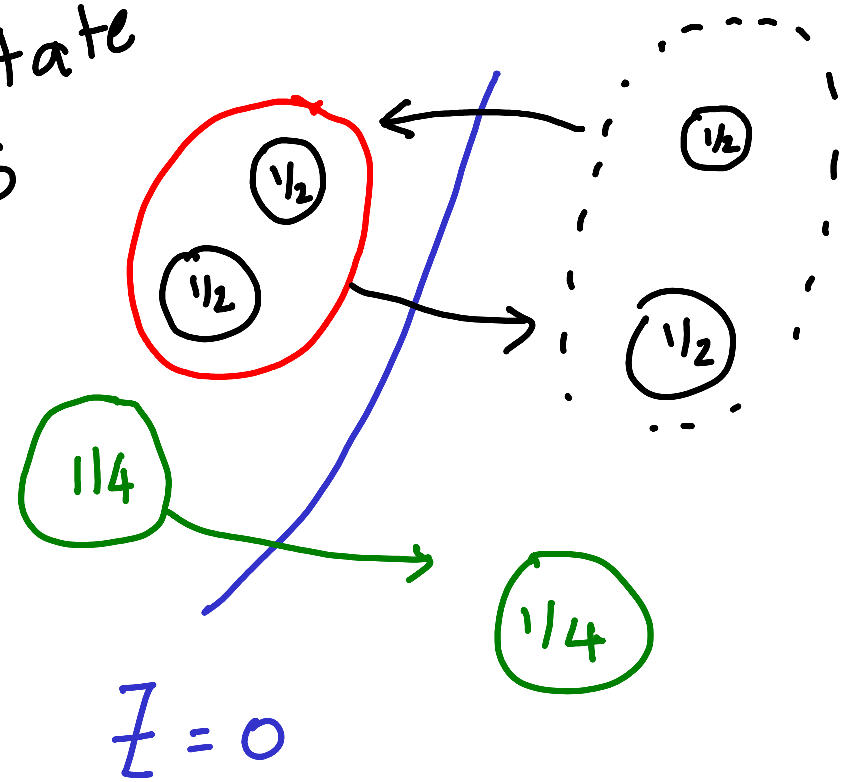


Φ_{10}^{-1} is doubly meromorphic

[Dijkgraaf-Verlinde-Verlinde]

Wall Crossing of Dyonic Black Holes

Bound state
$\sim 1/4$ BPS



No more $\frac{1}{4}$ BPS state.

[Denef]
[Cheng, Verlinde]

Wall crossing of $\frac{1}{4}$-BPS states

So, we're over counting 1/4 BPS states.

Donald J. Trump
@realDonaldTrump

STOP THE COUNT!

6:12 AM · 11/5/20 · Twitter for iPhone

DMZ theory

[Dabholkar-Murthy-Zagier]

$$\frac{1}{\Phi_{10}(\Omega)} = \sum_{n,m,L} C(m,n,L)\, q^n p^m y^L$$

$$\to \sum_{m=-1}^{\infty} \psi_m(\tau,z)\, p^m$$

$$\psi_m(\tau,z) = \underbrace{\psi_m^F(\tau,z)}_{Z_{1/4}^{\text{single center.}}} + \psi_m^P(\tau,z)$$

"Mock modular form"

“Polar” mock Jacobi forms

$p_{24}(m+1)$ is the m-th coefficient of $\eta(\tau)^{-24}$

$$\psi_p(\tau, z) = \frac{p_{24}(m+1)}{\eta(\tau)^{24}} \sum_{a \in \mathbb{Z}} \frac{q^{ma^2+a} y^{2ma+1}}{(1-q^a y)^2}$$

“Appell-Lerch sum”;
Derived from the averaging function of all double poles of $\Phi_{-10}(\tau, \sigma, z)$

The holomorphic part of the Jacobi forms still admit a Fourier expansion

$$d^{*}_{1/4} = (-1)^{L+1} \, C^{F}_{m}(n, L)$$

$$\downarrow$$

$$\psi^{F}_{m} = \sum_{n, L} C^{F}_{m}(n, L) \, q^{n} y^{L}$$

$d^{*}_{1/4}$ should be captured by the attractor mechanism for $\Delta > 0$ states.

$\Delta < 0$ states "seed" the MMJF via Rademacher.

The essence of holography for negative discriminant states boils down to equivalence of contours: Region in $\mathbb{H}$ where one can compute the degeneracies of negative discriminant states.

This region corresponds to a contour in supergravity over which the path integral is finally localized i.e. expressed as a convergent series of I-Bessel functions.

Special attractor contour :

$$C_* = \lim_{\epsilon \to 0^+} \left\{ \operatorname{Im} \tau = \frac{2m}{\epsilon}, \operatorname{Im} \sigma = \frac{2n}{\epsilon}, \operatorname{Im} z = -\frac{L}{\epsilon} \right\}$$

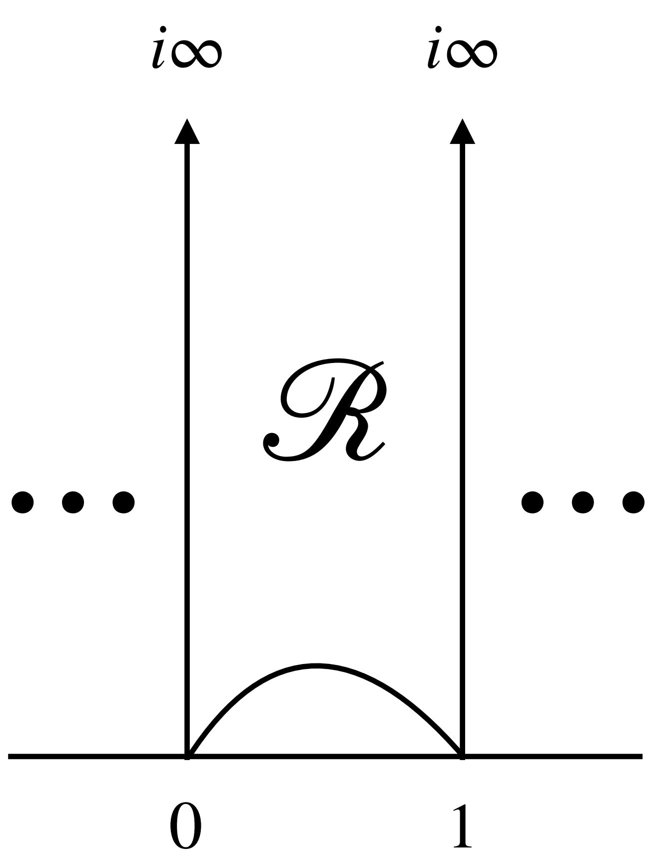
[Cheng - Verlinde]

[CMKRW]

$\psi_m^F(\tau, z)$ computable via Rademacher in this contour region.

The region bounded by the semicircle ("Farey circle at level 1") of radius 1/2 with end points (0,1), the rays $(0,i\infty)$ and $(1,i\infty)$ corresponds to the region where one can compute single center/Large black hole degeneracies.

So we wish to compute the degeneracies of negative discriminant states here.



The degeneracies of these $\Delta < 0$ states are in fact governed by finite sums of coefficients of $\eta(\tau)^{-24}\eta(\tau)^{-24}$.

There is a very nice "particle physics decay channel approach that one can take here to show that set.

[AK, Chowdhury, Murthy, Reys, Wrase]

There is also a more "topological way" of writing these degeneracies in terms of "continued fractions".

[Cardoso, Nampuri, Rossello]

Once you know what all the negative discriminant degeneracies are, feed into Rademacher*

*for a mixed mock modular Jacobi form on $SL(2,\mathbb{Z})$

Modular

$$c_m^{\mathrm{F}}(n,\ell) \sim \sum_{k=1}^{+\infty} \Bigg[\sum_{\substack{\tilde\ell \in \mathbb{Z}/2m\mathbb{Z} \\ \tilde\Delta<0}} c_m^{\mathrm{F}}(\tilde n, \tilde\ell)\, \frac{\mathrm{Kl}(k,\tilde\Delta,\Delta)}{k} \Big(\frac{|\tilde\Delta|}{\Delta}\Big)^{23/4} I_{23/2}\Big(\frac{\pi\sqrt{|\tilde\Delta|\Delta}}{mk}\Big) + \frac{\mathrm{Kl}(k,m,\Delta)}{\sqrt{k}} \Big(\frac{4m}{\Delta}\Big)^{6} I_{12}\Big(\frac{2\pi\sqrt{\Delta}}{\sqrt{m}k}\Big) + \mathcal{I}(k,m,\Delta) \Bigg].$$

Mock pieces

[Ferrari-Reys]

It is therefore possible to construct the entire Siegel modular form $\Phi_{10}^{-1}(\Omega)$ from just Dedekind eta's.

$$\text{Use } \left. C_m^F \right|_{\tilde{\Delta} < 0} \longrightarrow \text{Rademacher}$$

$$\big\downarrow$$

$$\Phi_{10}^{-1} \xleftarrow{+\psi_m^P} \psi_m^F(\tau, z)$$

$\left. C_m^F \right|_{\Delta < 0}$ is the smallest amount of data needed to construct $\frac{1}{4}$ BPS entropy.

Can the microscopic formulation help understand localization better?

Where/Why/How does the mock-modular nature of black hole entropy arise in supergravity?

Localizing 1/4- BPS states in 4d $\mathcal{N} = 4$

→ Measure

→ $Z_{1\text{-loop}} \longrightarrow = 1$ (cancellations b/w various multiplets)

→ $S(F)$

$$\longrightarrow F = -\frac{Y^1}{Y^0} Y^a \underbrace{C_{ab}}_{\downarrow\ K3} Y^b + \underbrace{\ldots\ldots\ldots}_{\substack{\text{Higher derivative} \\ \text{Non-hol.}}}$$

Correcting F

$$F = \frac{-Y^1}{Y^0} Y^a C_{ab} Y^b + 2i\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$$

$$N=4: \quad S = -i\frac{Y^1}{Y^0}$$

$$\Omega = \frac{1}{256\pi}\left[\Upsilon \log \eta(S)^{24} + \bar{\Upsilon} \log \eta(\bar{S})^{24} + \frac{1}{2}(\Upsilon + \bar{\Upsilon}) \log (S+\bar{S})^{12}\right]$$

$$\downarrow$$

$\mathbb{R}$, homogeneous

Correct the measure

Non-holomorphy of Ω $\Rightarrow$ derivative = MF of wt 2.

[Cardoso, de Wit, Mahapatra]

* Since Ω is corrected, so is measure

$$Z_{\text{measure}} = \sqrt{|\det \operatorname{Im}(F_{IJ} - \bar{F}_{\bar{I}\bar{J}})|}$$

* Gives $K=1$ leading Bessel terms

* OSV conjecture?

Accounting for all contributions from negative discriminant states

The uncorrected prepotential reproduced only the leading decay channel in W.

↳ only $K=1$ term in I_{23}.

Goal & WIP:

→ Supergravity localization that gives all $\gamma \in W$ that contributes

→ Understand "Mock" pieces of Rademacher from SUGRA. (Non-hol. F) $\sim I_{12}$

Mock modularity from supergravity

$$c^{\mathrm{F}}_m(n,\ell) \sim \sum_{k=1}^{+\infty} \Bigg[\sum_{\substack{\widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ \widetilde{\Delta}<0}} c^{\mathrm{F}}_m(\widetilde{n},\widetilde{\ell})\, \frac{\mathrm{Kl}(k,\widetilde{\Delta},\Delta)}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi\sqrt{|\widetilde{\Delta}|\Delta}}{mk}\right)$$

$$+ \frac{\mathrm{Kl}(k,m,\Delta)}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi\sqrt{\Delta}}{\sqrt{mk}}\right) + \mathcal{I}(k,m,\Delta) \Bigg].$$

$$= 2\pi \sum_{\substack{\widetilde{n}\geq\widetilde{n}_0 \\ \widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ \widetilde{\Delta}<0}} \alpha_m(\widetilde{n},\widetilde{\ell}) \sum_{k=1}^{\infty} \frac{Kl\left(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m};k,\psi\right)_{\ell\widetilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\right)$$

$$+ \frac{\sqrt{2m}}{\kappa} \sum_{k=1}^{\infty} \frac{Kl\left(\frac{\Delta}{4m},-1;k,\psi\right)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right)$$

$$- \frac{1}{2\pi\kappa} \sum_{k=1}^{\infty} \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z} \\ g(2mk) \\ g\equiv j(2m)}} \frac{Kl\left(\frac{\Delta}{4m},-1-\frac{g^2}{4m};k,\psi\right)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times$$

$$\times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right) (1-mu^2)^{25/4}\, du\,.$$

?

Conclusions and Future Directions

- Hidden number theory in supergravity/localization?
- Microscopics as a guide for gravity
- Mock modularity should be apparent in supergravity. But how/why is still WIP.