On the Categorification problem for Motivic Donaldson-Thomas invariants

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Motivation from Physics

- **General Relativity**:

$$\text{Gravity} = \text{Geometry of a 4-dim. spacetime } M^4$$

- **Other Forces?**: Kaluza-Klein (1920's)

$$\text{Gravity} + \text{Electromagnetic force in 4-dim} = \text{Geometry of a 5-dim. spacetime } M^4 \times S^1$$

- **Other Forces? Candelas-Horowitz-Strominger-Witten**

  In string theory, spacetime $= M^4 \times Y$ where Y is a Calabi-Yau-manifold of real dim. 6 (complex dimension 3).

Principle

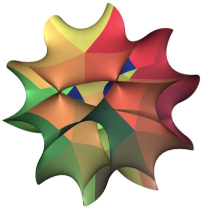
Physical forces in 4-dim are consequence of the geometric and topological properties of the extra dimensions in Y.

CY varieties

- Y a Calabi-Yau variety of dimension 3 over $\mathbb{C}$, ie, $\omega_Y \simeq \mathcal{O}_Y$.

- Example: The Fermat quintic

$$X_5 = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subseteq \mathbb{P}^4_{\mathbb{C}}$$



or more generally, any smooth quintic in $\mathbb{P}^4_{\mathbb{C}}$, $\omega_Y \simeq \mathcal{O}(5-4-1) \simeq \mathcal{O}_Y$

Worldsheets

Paths/interactions of string-particles through a spacetime $M^4 \times Y$, define 2-dimensional real surfaces (1-dimensional algebraic curves) of genus g in Y.

Path-integrals $\rightsquigarrow$ summing over all possible such curves.

Counting algebraic curves in a Calabi-Yau

- Counting **parametrized** curves $f: C \to Y$ (GW-invariants)

$$\underbrace{\overline{\mathcal{M}}_{g,n}(Y,\beta)}_{\text{moduli space of stable maps}} \qquad \text{quasi-smooth}, \ Vol = \int_{[\overline{\mathcal{M}}_{g,n}(Y,\beta)]} \in \mathbb{Q} \ \checkmark$$

- Counting **embedded** curves $C \subseteq Y$:

$$\mathrm{Hilb}_{codim\,2}(Y) \qquad \text{not quasi-smooth}, \qquad Vol \ \text{✗}$$

- Counting **ideal sheaves** $I_C \in \mathrm{Coh}(Y)$ (DT-invariants)

$$\underbrace{\mathcal{M}Coh(Y)^{st}}_{\text{Moduli of coherent sheaves}} \qquad \underbrace{\text{quasi-smooth}}_{\text{CY + Serre duality+ stability}} \quad , \ Vol = \int_{virt.\,class} \in \mathbb{Z} \ \checkmark$$

Behrend approach to DT-invariants

Observation(Thomas): Serre duality + CY condition imposes a symmetry on the obstruction theory of $\mathcal{M}Coh(Y)^{st}$:

$$\{1^{st\, order}\text{def. of } E \in Coh(Y)\} \simeq \{\text{Obstructions to def. of } E \in Coh(Y)\}^{\vee}$$

Theorem (K. Behrend)

There is a uniquely defined function $\nu_{Behrend} : \mathcal{M}Coh(Y)^{st} \to \mathbb{Z}$ such that

$$Vol(\mathcal{M}Coh(Y)^{st}) := \int_{[\mathcal{M}Coh(Y)^{st}]^{vir}} = \sum_n n.\chi(\nu_{Behrend} = n)$$

Behind the scenes: This extra symmetry is a shadow of a *(-1)-shifted symplectic form* on $\mathcal{M}Coh(Y)^{st}$ [Pantev-Toën-Vaquié-Vezzosi].

In this talk: DT-theory $\leftrightarrow$ (-1)-shifted symplectic derived geometry

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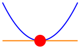
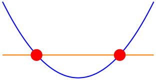
Why Derived Geometry?

Example with multiplicities:

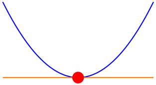
$$(y = x^2) \cap (y = 0)$$



Multiplicity 2



$$\rightsquigarrow$$



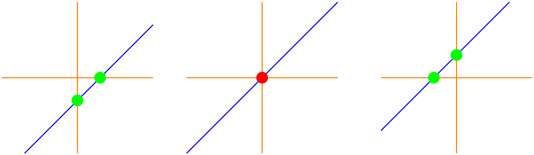
$$dim_k[k[X,Y]/(Y - X^2) \otimes_{k[X,Y]} k[X,Y]/(Y-0)] \simeq dim_k k[X]/(X^2) = 2$$

This computation fails in higher dimensions:

Why Derived Geometry?

Example: Intersect the axis in 4-dimensions (x, y, z, w), with the diagonal

$$\text{Axis} := \begin{cases} xz = 0 \\ xw = 0 \\ yz = 0 \\ yw = 0 \end{cases} \qquad \text{Diag} := \begin{cases} x - z = 0 \\ y - w = 0 \end{cases}$$



The intersection • should have multp. 2 but the naive computation gives:

$$k[x, y, z, w]/(xz, xw, yz, yw) \otimes_{k[x,y,z,w]} k[x, y, z, w]/(x - z, y - w) \simeq k[x, y]/(x^2, xy, y^2), \ dim_k = 3 \neq 2$$

Why Derived Geometry?

Problem: Our strategy for solving equations (tensor product) does not discount redundancies!

Behind the scenes: $f := xw - yz = w(x-z) - z(y-w)$ vanishes for <u>two</u> reasons:

$$f \in \underbrace{(x-z, y-w)}_{I_{Diag}} \cap \underbrace{(xz, xz, yz, yw)}_{I_{Axis}} = (wy[w-y],\ yz[y-w],\ f,\ [x-z]yz,\ xz[x-z])$$

but f does not belong to the product ideal $I_{Diag}.I_{Axis}$, ie, the two reasons why f vanish are <u>different</u>.

$$< f > = \frac{I_{Diag} \cap I_{Axis}}{I_{Diag}.I_{Axis}} \qquad \simeq \mathrm{Tor}^1_{R=k[x,y,z,w]}(R/I_{Diag}, R/I_{Axis})$$

Corrected formula: $\underbrace{\text{Multiplicity}}_{2} = \underbrace{\text{Algebraic multiplicity}}_{3} - \underbrace{\text{redundancies}}_{dim\mathrm{Tor}^1=1}$

Derived geometry

Lesson: functions in higher cohomological degree contain geometric information

Derived geometry: Systematically resolve/replace in every computation, the naive tensor ring $\otimes$ by the resolved/derived ring

$$\underbrace{R/I_{Diag} \otimes_R^{\mathbb{L}} R/I_{Axis}}_{cdga^{\leq 0}}$$

Derived geometry

Example: Landau-Ginzburg model: U a smooth k-scheme (space of fields) with a function $f : U \to \mathbb{A}^1_k$ (the action).

The derived critical locus $X = dCrit(f)$ is the derived intersection

$$\begin{tikzcd}
X := dCrit(f) = \{(u \in U, \lambda) : df_{|_u} \underset{\lambda}{\sim} 0\} \arrow[r, "i"] \arrow[d, "i"] & U \arrow[d, "df"] \\
U \arrow[r, "0"] & T^*U
\end{tikzcd}$$

$$\mathcal{O}_X := \mathcal{O}_U \otimes^{\mathbb{L}}_{\mathcal{O}_{T^*U}} \mathcal{O}_U$$

Tangent information distributed through multiple cohomological degrees:

$$\begin{array}{ccccc}
coh.deg & -1 & 0 & 1 & & \\
 & [0 \longrightarrow & \mathbb{T}_U & \xrightarrow{\text{Hess(f)}} \mathbb{L}_U] & = & \mathbb{T}_X
\end{array}$$

Derived Geometry

Example: In this talk we care about the natural derived structure on the moduli of coherent sheaves on a Calabi-Yau 3-fold,

$$X = \mathcal{M}Coh(Y)^{st}$$

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Shifted Symplectic Geometry

- Deformation theory of a stack/derived stack X is controlled by its *cotangent complex* $\mathbb{L}_X \in \mathrm{DQcoh}_\infty(X)$. When X is locally of finite presentation, $\mathbb{L}_X$ is a *perfect complex*, with dual $\mathbb{T}_X = \mathbb{L}_X^\vee$ the *tangent complex*.
- X Smooth, $\mathbb{L}_X = \Omega^1_X =$ Classical 1-forms.
- n-shifted 2-forms $= \{\mathcal{O}_X \to \mathbb{L}_X \wedge \mathbb{L}_X[n]\} = \{\mathbb{T}_X \wedge \mathbb{T}_X \to \mathcal{O}_X[n]\}$
- de Rham diff. $DR(X) := [\mathcal{O}_X \xrightarrow{d_R} \mathbb{L}_X \xrightarrow{d_R} \mathbb{L}_X \wedge \mathbb{L}_X \longrightarrow \ldots]$

  [Connes, Toën-Vezzosi]: d_R should not be understood as an internal differential but rather as the action of an extra operator ϵ of degree 1
- n-shifted closed 2-forms: Need homotopy $d_R(\omega) \sim 0$

$$\{\text{maps} \quad \underbrace{k(2)[-2n-1]}_{\epsilon=0 \ \textit{action}} \to \underbrace{DR(X)}_{\epsilon=d_R}\}$$

Shifted Symplectic Geometry

Definition (Pantev-Toën-Vaquié-Vezzosi)

An n-shifted symplectic form on X is a n-shifted closed 2-form such that its underlying 2-form $\mathbb{T}_X \wedge \mathbb{T}_X \to \mathcal{O}_X[n]$ is non-degenerate , ie, induces an equivalence

$$\mathbb{T}_X \simeq \mathbb{L}_X[n]$$

- $X = T^*\mathbb{A}^1 = \mathbb{A}^2$ has 0-shifted symplectic form given by $\omega = dx \wedge dy$.
- $X = Perf$ the derived stack classifying perfect complexes has a 2-shifted symplectic form.

$$\mathbb{T}_{E,Perf} = REnd(E)[1] \simeq E \otimes E^{\vee}[1]$$

$$\mathbb{T}_{E,Perf} \wedge \mathbb{T}_{E,Perf} \simeq E \otimes E^{\vee}[1] \otimes E \otimes E^{\vee}[1] \to \mathcal{O}[2] \quad \text{evaluation map}$$

- (PTVV) Y a CY of dimension 3 over k. Then $X := \mathrm{Map}(\underbrace{Y}_{3}, \underbrace{Perf}_{2})$

  is (2-3=-1)-symplectic. In particular,

  $\mathcal{M}Coh^{st}(Y) \subseteq \mathrm{Map}(Y, Perf)$ is -1-symplectic $\quad$ ($\Rightarrow$ Behrend Symmetry)

Shifted Symplectic Geometry

Theorem (Pantev-Toen-Vaquié-Vezzosi)

If M is a classical symplectic manifold (0-shifted) and L_1 and L_2 are Lagrangians, then the derived intersection

$$L_1 \times_M^{\mathbb{L}} L_2$$

is (-1)-shifted symplectic.

Shifted Symplectic Geometry

Example: Landau-Ginzburg model. The derived critical locus $X = dCrit(f)$ is defined as

$$\begin{tikzcd} X := dCrit(f) = \arrow[r, "i"] \arrow[d, "i"] & U \arrow[d, "df"] \\ U \arrow[r, "0"] & T^*U \end{tikzcd}$$

$$\begin{array}{lccccc} coh.deg & -1 & 0 & 1 & & \\ & & \mathbb{T}_U & \xrightarrow{\text{Hess(f)}} \mathbb{L}_U & = & \mathbb{T}_X \\ & \mathbb{T}_U & \xrightarrow{\text{H(f)}^\vee} \mathbb{L}_U & & = & \mathbb{L}_X \end{array}$$

symmetry of the Hessian $\Rightarrow$ $\mathbb{T}_X \simeq \mathbb{L}_X[-1]$ is a (-1)-shifted symplectic structure on X.

Example: $(U, f) = (\mathbb{A}^1, x^3)$ $\quad dCrit = Spec\, k[x]/(f' = 3x^2)$

Joyce's approach to DT-invariants

All examples are locally of this form:

Theorem (Brav-Bussi-Joyce (Darboux Lemma))

Let X be a (-1)-symplectic derived scheme. Then Zariski locally X is symplectomorphic to a derived critical locus $dCrit(U, f)$ with U smooth.

Consequence: Locally on X it makes sense to analyse the singularities of the function f on U via the perverse sheaf of vanishing cycles

$$P_{U,f} \in \mathrm{Perv}_{dCrit(f)}(U) = \mathrm{Perv}(dCrit(f)) = \mathrm{Perv}(Crit(f))$$

Problem: Ambiguity in the choice of local presentations:

$$dCrit(\mathbb{A}^1, x^3) = \mathrm{Spec}\, k[x]/(3x^2) \simeq \mathrm{Spec}\, k[x,y]/(3x^2, 2y) = dCrit(\mathbb{A}^2, x^3 + y^2)$$

$P_{(\mathbb{A}^1, x^3)}$ and $P_{(\mathbb{A}^2, x^3+y^2)}$ non-canonically isomorphic.

Joyce's approach to DT-invariants

Theorem (Brav-Bussi-Dupont-Joyce-Szendroi (BBDJS))

Let X be a (-1)-symplectic derived scheme. Assume that there exists a line bundle L together with an equivalence $L \otimes L \simeq \det(\mathbb{T}_X)$ (aka orientation data). Then:

- *The locally defined perverse sheaves of vanishing cycles $P_{U,f}$ glue to a globally defined perverse sheaf $P \in Perv(X)$.*
- *$\chi(P) = \nu_{Behrend}$ computing locally the Euler characteristic of vanishing cycles. Gives back DT-counting.*

Proof: Glue by hand using local presentations of the underlying classical scheme as classical critical loci.

- method does not see the full derived structure.
- strategy works for perverse sheaves because:
  - they form a **1-category** (no higher homotopies needed to glue).
  - they have the $\mathbb{A}^1$-**homotopy invariance** property.

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Motivic DT and categorification

Different invariants capture vanishing cycles of f on U:

$$\begin{tikzcd}
\underset{\text{Ayoub Motivic Van. Cycles}}{\mathcal{V}_{U,f}} \arrow[d, "Realization"] & \in \text{Motives} & \text{dgcats}^{2\text{per}} \arrow[l, dashed, "\text{Blanc-Toën-R.-Vezzosi, Pippi}"'] \ni & \underset{\text{Orlov/Eisenbud cat. of singularities. B-type D-branes (Kapustin-Li)}}{MF(U,f)} \arrow[dlll, "HP"', "Efimov{,}Sabbah"] \\
P_{U,f} \in Perv & & &
\end{tikzcd}$$

MF: $U_0 := f^{-1}(0)$, $M \in Coh(U_0)$, infinite resolution by projective modules becomes eventually 2-periodic [Serre-Auslander-Buchsbaum-Eisenbud]

$$\underbrace{... \to F \to Q \to F \to Q}_{\in MF(U,f)} \to \underbrace{P_n \to ... \to P_2 \to P_1 \to P_0}_{\in Perf(U_0)} \to M$$

Motivic DT and categorification

Gluing Problem: Given a (-1)-symplectic derived scheme X, can we glue the Darboux locally defined dg-categories $MF(U, f)$ as a sheaf of dg-categories on X? Is Joyce's orientation data enough?

Rmk: Version of the gluing problem for the Fukaya category (Seidel, Kontsevich, Nadler, Shende, Ganatra, Pardon,...).

Complications: The gluing no longer takes place in a 1-category but in an ∞-category. Complicated coherences are required. Need a gluing mechanism.

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Moduli of Darboux Coordinates

Classical Picture: X a classical symplectic manifold, then locally X is of symplectomorphic to some T^*M (Darboux's lemma). We can analyse the moduli of such Darboux parametrizations:

$$Darb_X : S \subseteq X \text{ open} \mapsto \{M \text{ smooth manifold}, S \simeq T^*M \text{ symplectic}\}$$

The data of a symplectomorphism $S \simeq T^*M$ in particular implies:

- The fibers of the projection $S \simeq T^*M \to M$ define a smooth Lagrangian foliation $\mathcal{F}$ on S (ie, $\omega_{|_{\text{fibers}}} = 0$).
- The symplectic form on S is exact ie, there exists a 1-form α (Liouville form on T^*M) with $d_R(\alpha) = \omega$.

We call such $(\mathcal{F}, \alpha)$ a Darboux datum on S.

Moduli of Darboux Coordinates

(-1)-shifted geometry: These notions make sense thanks to the work of Toën-Vezzosi on derived foliations.

Theorem (Pantev-Toën)

S a (-1)-symplectic derived scheme. Then the following data are equivalent:

- *Darboux data on S, ie a globally defined smooth derived Lagrangian foliation $\mathcal{F}$ on S + an exact structure α.*
- *the data of a smooth formal scheme $\mathcal{U}$, a function f on $\mathcal{U}$ and a symplectomorphism $S \simeq dCrit(\mathcal{U}, f)$*

Classical Picture: Darboux data on $S \Leftrightarrow [S \subseteq T^*M \to M]$.

(-1)- **picture :** Darboux data on $S \Leftrightarrow [S \simeq dCrit(\mathcal{U}, f) \hookrightarrow \mathcal{U}]$.

Idea: $\mathcal{U} := S/\mathcal{F}$ the formal leaf space. f = exact struct. - isotropic struct.

The Darboux Stack

Example: $(\widehat{\mathbb{A}^1}, x^3)$ gives Darboux data

$$dCrit(x^3) = Spec(k[x]/(3x^2)) \hookrightarrow \widehat{\mathbb{A}^1}$$

Construction (Gluing Moduli of Darboux coordinates)

The assignment:

$$S \to X \text{ étale} \mapsto \{(\alpha, \mathcal{F}) : \textit{Exact structure } \alpha + \textit{smooth Lag. fol. } \mathcal{F} \textit{ on } S\}$$

defines a hypercomplete stack on the small étale site of a n-shifted symplectic derived scheme X. We call it the Darboux stack $Darb_X$.

Remark: $Darb_X := Exact_X \times LagFol_X^{sm}$

Comment: In the case where X is (-2)-symplectic, this recovers the local data used by Borisov-Joyce and Oh-Thomas to glue DT-invariants for Calabi-Yau 4-folds.

The Darboux Stack

Construction

The Behrend's function, MF and Joyce's construction have $Darb_X$ *as a natural domain, and define natural transformations of sheaves on the small étale site of* X*:*

$$\nu : Darb_X(S) \ni (\mathcal{U}, f) \to dim(vanishing\ cycles\ of\ \ f) \in \mathbb{Z}_X(S) := \mathbb{Z}$$

$$\boldsymbol{P} : Darb_X(S) \ni (\mathcal{U}, f) \to P_{\mathcal{U},f} \in Perv_X(S) := Perv(S)^{\simeq}$$

$$\boldsymbol{MF} : Darb_X(S) \ni (\mathcal{U}, f) \to MF(\mathcal{U}, f) \in dgcat^{2per}_{X_{dR}}(S) := \underbrace{(dgcat^{2per}_{S_{dR}})^{\simeq}}_{categorical\ \ crystals}$$

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Action of Quadratic Bundles

Ambiguity Problem: in the choice of local presentations:

$$dCrit(\mathbb{A}^1, x^3) = \mathrm{Spec}\, k[x]/(3x^2) \simeq \mathrm{Spec}\, k[x,y]/(3x^2, 2y) = dCrit(\mathbb{A}^2, x^3 + y^2)$$

Definition

$Quad_{dR}(S) := \{(Q, q) :$ (loc. trivial) quadratic vector bundles on $S_{dR}\}$

Construction (Moduli of Quadratic bundles)

X a (-1)-symplectic derived scheme. Then:

- *The assignment S/X étale $\mapsto Quad_{dR}(S)$ defines a sheaf of monoids $Quad_{X_{dR}}$ on X_{et} for the sum of quadratic bundles.*
- *$Quad_{X_{dR}}(S)$ acts on $Darb_X(S)$,*

$$dCrit(\mathcal{U}, f) \simeq S \simeq dCrit(\mathcal{U} \underset{S_{dR}}{\times} Q, f + q)$$

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Recovering the perverse gluing of BBDJS:

Fact: $(M, q) \in Quad_{X_{dR}}(S)$ then $det(M)$ is a 2-torsion line bundle over S, ie, $det(M)^2 \simeq \mathcal{O}_S$. This follows from the non-degeneracy of the Hessian.

Construction

X a (-1)-symplectic derived scheme. Then:

- ***det*** : $Quad_{X_{dR}} \to B\mu_{2,X} = Ker\left(B\mathbb{G}_{m,X} \underset{2}{\to} B\mathbb{G}_{m,X}\right)$ *is a map of monoids.*
- $\boldsymbol{P}$: $Darb_X \to Perv_X$ *comes with* ***homotopy coherent data*** *rendering the actions compatible on both sides*

$$Quad_{X_{dR}} \circlearrowright Darb_X \to Perv_X \circlearrowleft B\mu_2$$

(on the right the action of $B\mu_2$ is defined by BBDJS).

Recovering the perverse gluing of BBDJS:

Corollary (Hennion-Holstein-R. as a reformulation of BBDJS)

Let X be a (-1)-shifted symplectic derived scheme with a fixed exact structure α (always exists by a theorem of Deligne).

Then there exists a canonical factorization

$$\begin{tikzcd} Darb_X/Quad_{X_{dR}} \arrow[r, "\overline{P}"] \arrow[d] & Perv_X/B\mu_{2,X} \\ X \arrow[ur, dashed] & \end{tikzcd}$$

Here X is the final object of the étale topos of X. In other words, the gluing of the perverse sheaves $P_{U,f}$ is always well-defined in the quotient:

$$\overline{P} : X \to Perv_X/B\mu_{2,X}$$

Recovering the perverse gluing of BBDJS:

Remark

The composition

$$X \to Perv_X / B\mu_{2,X} \to * / B\mu_{2,X} = BB\mu_{2,X}$$

is the class in $H^2(X, \mathbb{Z}/2\mathbb{Z})$ of the bundle classifying square roots of $det(\mathbb{T}_X)$.

An orientation data of BBDJS corresponds precisely to the choice of a ***null-homotopy*** *of this composition*

$$\begin{tikzcd}
 & Perv_X \arrow[r] \arrow[d] \arrow[dr, phantom, "pullback"] & * \arrow[d] \\
X \arrow[r, "\overline{P}"'] \arrow[ur, dashed] \arrow[urr, bend left, red] & Perv_X / B\mu_{2,X} \arrow[r] & BB\mu_{2,X}
\end{tikzcd}$$

Such a null-homotopy provides a lifting through the fiber product and defines a well-defined glued perverse sheaf $P_{Joyce} : X \to Perv_X$.

Gluing MF:

Fact: $(M, q) \in Quad_{X_{dR}}(S)$ then $MF(M, q)$ has a structure of 2-torsion 2-periodic derived Azumaya algebra over S_{dR}. This is a consequence of Preygel-Thom-Sebastiani followed by Knörrer periodicity

$$MF(M, q) \otimes MF(M, q) \simeq MF(M \times M, q \boxplus -q) \simeq MF(S_{dR}, 0)$$

Construction

X a (-1)-symplectic derived scheme. Then:

- ***MF*** : $Quad_{X_{dR}} \to Az^{2per, 2-tor}_{X_{dR}}$ *is a map of monoids.*
- ***MF*** : $Darb_X \to dgcat^{2per}_{X_{dR}}$ *comes with **homotopy coherent data** rendering the actions compatible on both sides*

$$Quad_{X_{dR}} \circlearrowright Darb_X \to dgcat^{2per}_{X_{dR}} \circlearrowleft Az^{2per, 2-tor}_{X_{dR}}$$

On the right the action of $Az^{2per, 2-tor}_{X_{dR}}$ is given by tensor products of dg-categories.

Gluing MF:

Work in progress (Hennion-Holstein-R.)

X a (-1)-shifted symplectic derived scheme with an exact structure. There exists a factorization of morphisms of étale sheaves:

$$\begin{tikzcd} Darb_X/Quad_{X_{dR}} \arrow[r, "\overline{MF}"] \arrow[d, "final"'] & dgcat^{2per}_{X_{dR}}/Az^{2per,2-tor}_{X_{dR}} \\ X \arrow[ur, dashed] & \end{tikzcd}$$

Definition

A categorical orientation data is a trivialization of the composition

$$X \to Darb_X/Quad_{X_{dR}} \to dgcat^{2per}_{X_{dR}}/Az^{2per,2-tor}_{X_{dR}} \to B\,Az^{2per,2-tor}_{X_{dR}}$$

Gluing MF:

Corollary

Let X be a (-1)-shifted symplectic derived scheme. Assume X is equipped categorical orientation data. Then the locally defined categories $MF(\mathcal{U}, f)$ glue as a sheaf of 2-periodic dg-categories on X as a result of

$$\begin{tikzcd}
 & dgcat^{2per}_{X_{dR}} \arrow[r] \arrow[d] \arrow[dr, phantom, "pullback"] & * \arrow[d] \\
X \arrow[r, "\overline{P}"'] \arrow[ur, dashed] \arrow[urr, bend left] & dgcat^{2per}_{X_{dR}} / Az^{2per,2-tor}_{X_{dR}} \arrow[r] & B\,Az^{2per,2-tor}_{X_{dR}}
\end{tikzcd}$$

New Orientation data

The orientation data of BBDJS is (a priori) not enough to glue MF. A categorical orientation provides new obstruction classes coming from the fibration sequence

$$Az_{X_{dR}}^{2per,2-tor} \to Az_{X_{dR}}^{2per} \underset{2}{\to} Az_{X_{dR}}^{2per}$$

- $\pi_0(Az_{X_{dR}}^{2per,2-tor}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \{MF(*,0), MF(\mathbb{A}^1, x^2)\}\}.$
- $\pi_1(Az_{X_{dR}}^{2per,2-tor}) \simeq \underbrace{\mathbb{Z}/2\mathbb{Z}}_{BBDJS} \simeq \{Id, [1]\}$
- $\pi_2(Az_{X_{dR}}^{2per,2-tor}) = \mathbb{Z}/2\mathbb{Z} \simeq \mathrm{Ker}(z^2 : \mathbb{C}^* \to \mathbb{C}^*)$
- $\pi_n(Az_{X_{dR}}^{2per,2-tor}) = 0 \ \ n \geq 3,$

Thank you for your time.