Around Donaldson-Thomas invariants of symmetric quivers

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Motivation

Let Q be a quiver (directed graph). A *representation* of Q: assignment of vector spaces to vertices of Q and of linear maps to the arrows of Q (Gabriel 1972). For example $\bullet \to \bullet$ is represented by two vector spaces and a linear map between them.

For a given "dimension vector" $\mathbf{d} = (\dim(V_i))_{i \in Q_0}$, we can define the *moduli space of representations* $M_{\mathbf{d}}$: the quotient of the space of representations $R_{\mathbf{d}}$ by the group of change of basis $G_{\mathbf{d}}$. (Finite type classified by Gabriel, Gelfand–Ponomarev, Ringel etc. using Dynkin diagrams, started a really active research area.)

In 1990s quivers became prominent in geometric representation theory (Lusztig, Ringel–Hall, Nakajima etc.): construction of representations of Kac–Moody algebras and quantum groups using geometry of certain moduli spaces of quiver representations. (Actions were defined on cohomology and/or constructible functions on those — generally very complicated — spaces.)

Motivation

More recenty (Kontsevich–Soibelman 2010): the *cohomological Hall algebra* (CoHA) $\mathcal{H}_Q$ made of equivariant cohomologies $H^\bullet_{G_\mathbf{d}}(R_\mathbf{d})$ for all $\mathbf{d}$, and its modules (this uses "framed" quivers). This bypasses the intricate geometry of $R_\mathbf{d}/G_\mathbf{d}$, since $R_\mathbf{d}$ is contractible!

CoHA is designed to fulfil a prediction in string theory postulating that to some supersymmetric quantum field theories, one can associate Lie superalgebras, called "algebras of BPS states". Mathematically, this is related to the theory of Donaldson–Thomas invariants (studying families of curves in algebraic three-folds), though for that one mainly is interested in *quivers with potentials*.

Throughout this talk, we are going let us assume that Q is *symmetric* (number of arrows from i to j is the same as the number of arrows from j to i). In this case $\mathcal{H}_Q$ is a (super)commutative algebra.

What this talk is about

Kontsevich and Soibelman used $\mathcal{H}_Q$ to define the “refined Donaldson–Thomas invariants of Q”. They conjectured, and Efimov (2011) proved that, as a commutative algebra, $\mathcal{H}_Q$ is freely generated by a super vector space of the form $W_Q \otimes \mathbb{Q}[t]$, which implies that the “refined Donaldson–Thomas invariants of Q” are equal to dimensions of multigraded components of W_Q, and, in particular, are non-negative integers.

In this talk, I shall discuss two new algebraic constructions which computes the refined DT invariants as dimensions of graded components of *something*, and I (not so) secretly hope that perhaps these “somethings” are known from a physics viewpoint to people in the audience.

A non-free commutative algebra

Denote by $(m_{i,j})$ the adjacency matrix of the quiver Q. The algebra $\mathcal{A}_Q$ is defined as follows. Its space of generators V_Q has a basis $a_{i,k}$ with $i \in Q_0$, $k \geq 0$. There are two groups of relations:

$$a_{i,k_1} a_{j,k_2} = (-1)^{m_{i,i} m_{j,j}} a_{j,k_2} a_{i,k_1} \quad \text{for all } i, j, k_1, k_2,$$

$$\sum_{k_1+k_2=k} \binom{k_2}{p} a_{i,k_1} a_{j,k_2} = 0 \quad \text{for all } k \geq 0 \text{ and } 0 \leq p < m_{i,j}.$$

The first of these is (super)commutativity: "parity is the number of loops at a vertex". The second is found among the coefficients of the power series

$$a_i(z) \frac{1}{p!} \frac{d^p}{dz^p} a_j(z) = 0,$$

where $a_i(z) = \sum_{k \geq 0} a_{i,k} z^k$. It ensures that $a_i(z) a_j(w)$ vanishes at $z = w$ with multiplicity $m_{i,j}$.

Examples

Let Q have one vertex and no edges. Then the algebra $\mathcal{A}_Q$ is generated by $a_0, a_1, \ldots$; the only relations are those of supercommutativity, and our algebra is the polynomial algebra $\mathbb{Q}[a_0, a_1, \ldots]$.

Let Q have one vertex and one loop. Then the algebra $\mathcal{A}_Q$ is generated by $a_0, a_1, \ldots$. Relations of the first group say that the generators anti-commute, and the only relation $a(z)a(z) = 0$ of the second group is redundant. Thus, our algebra is the infinite Grassmann algebra $\bigwedge(a_0, a_1, \ldots)$.

Examples

Let Q be the quiver with one vertex and two loops. Then the algebra $\mathcal{A}_Q$ is generated by commuting generators $a_0, a_1, \ldots$ modulo the relations

$$(a_0 + a_1 z + a_2 z^2 + \ldots)^2 = 0,$$
$$(a_0 + a_1 z + a_2 z^2 + \ldots)\frac{d}{dz}(a_0 + a_1 z + a_2 z^2 + \ldots) = 0.$$

The second relation follows from the first one by differentiation, so our algebra is

$$\frac{\mathbb{Q}[a_0, a_1, \ldots]}{\left(a_0^2, 2a_0a_1, 2a_0a_2 + a_1^2, \ldots, \sum_{i+j=k} a_i a_j\right)}.$$

This algebra was studied by B. Feigin and A. Stoyanovsky in 1990s in the context of level 1 modules over the Kac–Moody algebra $\widehat{\mathfrak{sl}}_2$.

Examples

Let Q be the quiver $\bullet \overset{\leftarrow}{\underset{\rightarrow}{}} \bullet$. Then the algebra $\mathcal{A}_Q$ is generated by $a_{0,0}, a_{0,1}, \dots$ and $a_{1,0}, a_{1,1}, \dots$. Relations of the first group say that the generators commute, and relations of the second group say that

$$(a_{0,0} + a_{0,1}z + a_{0,2}z^2 + \dots)(a_{1,0} + a_{1,1}z + a_{1,2}z^2 + \dots) = 0,$$

so our algebra is

$$\frac{\mathbb{Q}[a_{0,0}, a_{0,1}, \dots, a_{1,0}, a_{1,1}, \dots]}{\left(a_{0,0}a_{1,0}, a_{0,0}a_{1,1} + a_{0,1}a_{1,0}, \dots, \sum_{i+j=k} a_{0,i}a_{1,j}\right)}$$

This algebra is known as the jet algebra of the arc scheme of $\mathbb{Q}[x, y]/(xy)$. In fact, for every simple graph Γ, the algebra associated to its doubling has the same interpretation.

Poincaré series

To explain how these algebras can be used, we set

$$\deg(a_{i,k}) = (e_i, 2k + m_{i,i}) \in \mathbb{N}^{Q_0} \times \mathbb{N},$$

and extend this "degree" to $\mathcal{A}_Q$, creating the multigraded components $(\mathcal{A}_Q)^n_{\mathbf{d}}$; here $\mathbf{d}$ counts vertices of the quivers where the generators sit, and n is some kind of homological degree (refinement of parity). This allows us to define the Poincaré series

$$P(\mathcal{A}_Q, x, q) = \sum_{(\mathbf{d},n)\in\mathbb{N}^{Q_0}\times\mathbb{N}} (-1)^n \dim(\mathcal{A}_Q)^n_{\mathbf{d}} q^n x^{\mathbf{d}}.$$

It is a formal power series in variables q and x_i, $i \in Q_0$.

It is not easy to compute this series directly, since describing a basis in the algebra $\mathcal{A}_Q$ is, in general, hard. Using the specific form of defining relations, we shall rather examine the graded dual of $\mathcal{A}_Q$.

Functional realisation

For a dimension vector $\mathbf{d}$, and a linear function $\xi \in ((\mathcal{A}_Q)^n_{\mathbf{d}})^*$, let us compute

$$\xi\left(a_1(z_{1,1})\cdots a_1(z_{1,\mathbf{d}_1})a_2(z_{2,1})\cdots a_2(z_{2,\mathbf{d}_2})\cdots a_k(z_{k,1})\cdots a_k(z_{k,\mathbf{d}_k})\right).$$

It is a polynomial, symmetric (or anti-symmetric) in each group of variables $z_{i,1},\ldots,z_{i,\mathbf{d}_i}$, and, according to the defining relations, it is divisible by $(z_{i,s}-z_{j,t})^{m_{i,j}}$ for all i,j,s,t. Moreover, one can show that each such polynomial gives rise to a unique linear function.

Using this description of the graded dual, we immediately find that

$$P(\mathcal{A}_Q,x,q)=\sum_{\mathbf{d}\in\mathbb{N}^{Q_0}}\frac{(-q)^{\mathbf{d}^TM\mathbf{d}}}{\prod_{i\in Q_0}(q^2)_{\mathbf{d}_i}}x^{\mathbf{d}}.$$

(Here $(a)_n=(1-a)(1-a^2)\cdots(1-a^n)$.) Some versions of this formula are often referred to as a "Nahm sum" for the given matrix M.

Relationship to CoHA

In the context of cohomological Hall algebras, the conventions for the Poincaré series are a bit different, and

$$P(\mathcal{H}_Q, x, q) = \sum_{\mathbf{d} \in \mathbb{N}^{Q_0}} \frac{(-q^{\frac{1}{2}})^{\mathbf{d}^T(M-I)\mathbf{d}}}{\prod_{i \in Q_0}(q^{-1})_{\mathbf{d}_i}} x^{\mathbf{d}},$$

which is a formal power series in x_i, whose coefficients are formal Laurent series (in $q^{-\frac{1}{2}}$). Using the λ-ring structure on power series, the *refined Donaldson–Thomas invariants* $\mathrm{DT}_{Q,\mathbf{d}}(q)$ are defined by the formula

$$P(\mathcal{H}_Q, x, q) = \mathrm{Exp}\left(\frac{1}{1-q^{-1}} \sum_{\mathbf{d} \in \mathbb{N}^{Q_0}} (-1)^{\mathbf{d}^T(M-I)\mathbf{d}} \mathrm{DT}_{Q,\mathbf{d}}(q) x^{\mathbf{d}}\right).$$

Here Exp is the "plethystic exponential".

Observation. For each $\mathbf{d}$, the coefficient of $x^{\mathbf{d}}$ represents a rational function in $q^{\frac{1}{2}}$. In the ring $\mathbb{Q}(q^{\frac{1}{2}})[[x_i : i \in Q_0]]$, we have

$$P(\mathcal{H}_Q, x, q) = P(\mathcal{A}_Q, q^{\frac{1}{2}} x, q^{\frac{1}{2}}).$$

Koszul duality and DT invariants

A priori, $DT_{Q,\mathbf{d}}(q) \in \mathbb{Q}(q^{\frac{1}{2}})$. To say more, we shall use the Koszul duality theory. That theory assigns to any algebra $\mathcal{A}$ with quadratic relations another algebra $\mathcal{A}^!$ with quadratic relations. If $\mathcal{A}$ is supercommutative, the algebra $\mathcal{A}^!$ is isomorphic to the universal enveloping algebra of a certain Lie superalgebra $\mathfrak{g}(\mathcal{A})$.

Theorem. The Koszul dual Lie superalgebra $\mathfrak{g}_Q := \mathfrak{g}(\mathcal{A}_Q)$ has a Diff_1-module structure, and the action of $\mathbb{Q}[z] \subset \mathrm{Diff}_1$ on $\mathfrak{g}_Q$ is free, with the space of generators $\mathrm{Ker}(\partial)$, which happens to be a Lie subalgebra. Moreover, for each $\mathbf{d} \in \mathbb{N}^{Q_0}$

$$\mathrm{DT}_{Q,\mathbf{d}}(q) = \sum_n \dim(\mathrm{Ker}(\partial)^n_{\mathbf{d}}) q^{\frac{1}{2}n-1}.$$

Additionally, these sums are finite: $\mathrm{DT}_{Q,\mathbf{d}}(q) \in \mathbb{N}[q^{\pm\frac{1}{2}}]$

Sketch of the proof I

First, one may describe the Koszul dual Lie algebra explicitly. It has generators $b_{i,k}$ with $i \in Q_0$, $k \geq 0$. The multidegrees of these are $\deg(b_{i,k}) = (e_i, -2k - m_{i,i} - 1) \in \mathbb{N}^{Q_0} \times \mathbb{N}$. The relations are

$$\sum_{p=0}^{n_{i,j}} (-1)^p \binom{n_{i,j}}{p} [b_{i,k-p}, b_{j,l+p}] = 0,$$

where $n_{i,j} = \max(m_{i,j} - \delta_{i,j}, 0)$. In terms of the generating series, $(z-w)^{n_{i,j}}[b_i(z), b_j(w)] = 0$. (Locality relations: will get back to them later.)

This particular form of relations may be used as follows. Let us define endomorphisms p and q of the space of generators as $p(b_i(z)) = zb_i(z)$, $q(b_i(z)) = \partial_z b_i(z)$, and extend them as derivations to products of generators. It turns out that those derivations preserve all relations and thus act on $\mathfrak{g}_Q$.

Sketch of the proof II

For the derivations p and q of $\mathfrak{g}_Q$ that we defined, we have $pq - qp = n \cdot 1$ on Lie monomials involving n generators, and so if we redefine the operator ∂ on $\mathfrak{g}_Q$ setting $\partial = \frac{1}{n}q$ on Lie monomials involving n generators, we get an action of the Weyl algebra. Every module over the Weyl algebra that is graded and bounded from below is a free $\mathbb{Q}[z]$-module generated by $\mathrm{Ker}(\partial)$.
This already implies that

$$(1-q)P((\mathfrak{g}_Q^*)_{\mathbf{d}}, q) = \sum_n \dim(\mathrm{Ker}(\partial)_{\mathbf{d}}^n)q^{\frac{1}{2}n-1} \in \mathbb{N}[[q^{\pm\frac{1}{2}}]].$$

Since p and q are derivations, the kernel is a Lie subalgebra.
To show that $\sum_n \dim(\mathrm{Ker}(\partial)_{\mathbf{d}}^n)q^{\frac{1}{2}n-1} \in \mathbb{N}[q^{\pm\frac{1}{2}}]$ is finite, one may use a bound on the size of $\mathfrak{g}_Q$ coming from the theory of Gröbner–Shirshov bases for Lie superalgebras. Specifically, for each $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$, we have

$$(1-q^{|\mathbf{d}|})P((\mathfrak{g}_Q^*)_{\mathbf{d}}, q) \in \mathbb{N}[q^{\frac{1}{2}}].$$

Combining the two formulas, the necessary result follows.

Interlude: vertex algebras

Our result is a positivity result but for $\mathfrak{g}_Q$, not for refined DT invariants. Before proceeding, shall discuss my work with Sergey Mozgovoy relating the CoHA to vertex algebras. It started from a simple observation that as multigraded vector space, $\mathcal{H}_Q$ looks like the dual vector space of lattice vertex algebra associated with the "Euler form" of the quiver (given by the matrix $I - M$).

A key observation is that three vertex algebras coincide: the lattice vertex algebra associated with the Euler form, the free vertex algebra of a given non-negative locality function, and the universal envelope of a vertex Lie algebra. The first one gives the "size" of the CoHA, the third a cocommutative coproduct, and the second connects them together.

Sketch of the proof III

Let us denote by C_Q the free vertex Lie algebra corresponding to the locality function $N_Q(i,j) = m_{i,j} - \delta_{i,j}$. One can show that it is isomorphic to the free vertex Lie algebra corresponding to the non-negative locality function $N_Q^+(i,j) = \max(N_Q(i,j), 0)$.
Using the general theory of vertex Lie algebras, we may associate to the vertex Lie algebra C_Q an honest Lie algebra $\mathfrak{L}_Q$, the *coefficient algebra of* C_Q; moreover, we have a graded vector space decomposition

$$\mathfrak{L}_Q = \mathfrak{L}_Q^- \oplus \mathfrak{L}_Q^+,$$

where $\mathfrak{L}_Q^-$ and $\mathfrak{L}_Q^+$ are Lie subalgebras of $\mathfrak{L}_Q$.
It is established by Roitman (1999) that both the Lie algebra $\mathfrak{L}_Q$ and its subalgebra $\mathfrak{L}_Q^+$ admit explicit presentations by generators and relations which we shall use.

Sketch of the proof IV

The Lie algebra $\mathfrak{L}_Q$ is generated by elements $i(k)$ of degree $(e_i, 2k + m_{i,i} + 1)$, $i \in Q_0$, $k \in \mathbb{Z}$, subject to the relations

$$\sum_{p=0}^{N_Q^+(i,j)} (-1)^p \binom{N_Q^+(i,j)}{p} [i(k-p), j(l+p)] = 0 \qquad (1)$$

for all $i, j \in Q_0$, and the Lie algebra $\mathfrak{L}_Q^+$ is generated by elements $i(k)$ of degree $(\alpha_i, 2k + m_{i,i} + 1)$, $i \in Q_0$, $k \geq 0$, subject to those of the relations (1) that only contain the generators $i(k)$ with $k \geq 0$. The subalgebra $\mathfrak{L}_Q^-$ is defined more indirectly. We note that (up to multiplying gradings by -1) there is a Lie algebra isomorphism

$$\mathfrak{L}_Q^+ \cong \mathfrak{g}_Q$$

sending $i(k)$ to $b_{i,k}$.

Sketch of the proof V

Using the isomorphism $\mathfrak{L}_Q^+ \cong \mathfrak{g}_Q$ and the interpretation of the universal enveloping vertex algebra $\mathcal{U}(C_Q)$ as the dual of the CoHA $\mathcal{H}_Q$ which was discovered by myself and Mozgovoy (and confirms the intuition coming from the much more general work of Joyce), one can prove that

$$P(\mathcal{H}_Q, x, q)P(U(\mathfrak{g}_Q)^\vee, x, q) = 1.$$

That latter relation implies that computing the plethystic logarithm of A_Q is easily related to the plethystic logarithm of $P(U(\mathfrak{g}_Q)^\vee, x, q)$, equal to $P(\mathfrak{g}_Q^\vee, x, q)$.

We remark that this relation suggests that the algebras $\mathcal{A}_Q$ and $\mathfrak{g}_Q$ are Koszul; in fact, it expresses the so called "numerical Koszulness".We have been able to prove the Koszulness in some particular cases, and conjecture it to be true in general.

Concluding remarks

Let us note that in work with Mozgovoy, I identified the dual space of $\mathcal{H}_Q$ with the the universal envelope $\mathfrak{L}_Q^-$ of the "more complicated half" of the coefficient algebra, so there is some extra surprise relating two halves of the coefficient algebra, which generally are defined in a very asymmetric way.

Conjecturally, the relationship between the two approaches are also related by some version of the Koszul duality; specifically, it should be the Koszul duality between commutative vertex algebras and vertex Lie algebras.

Using the vertex algebra construction, Mozgovoy and myself also found interpretations of CoHA-modules on cohomologies of non-commutative Hilbert schemes (constructed by Franzen geometrically). A baby version of this for two-loop quiver was found in my work from some 15 years ago.