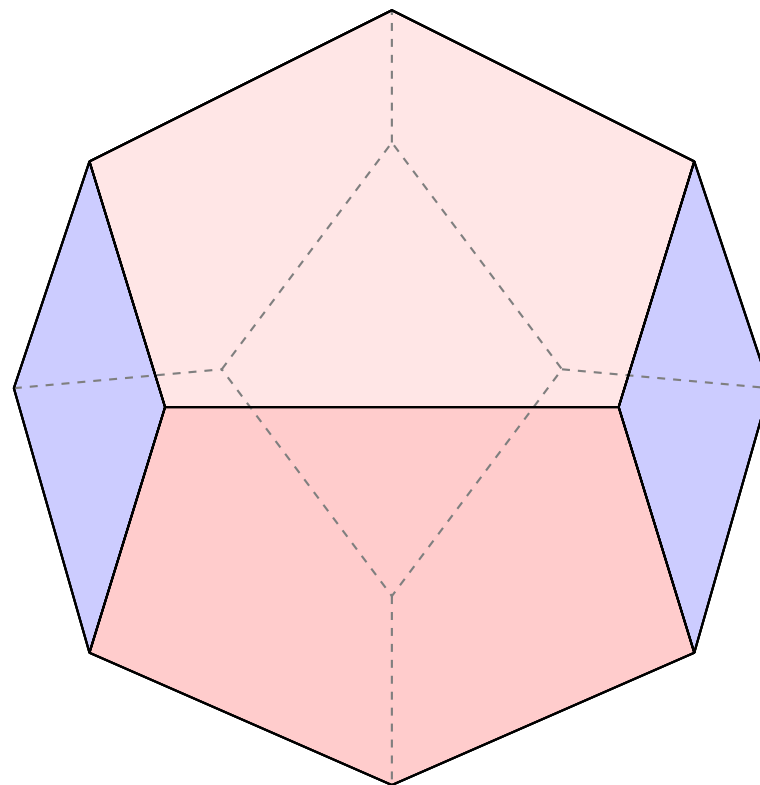


Scattering Amplitudes and their Geometric Properties

James Drummond
University of Southampton



LPTHE, February 2023

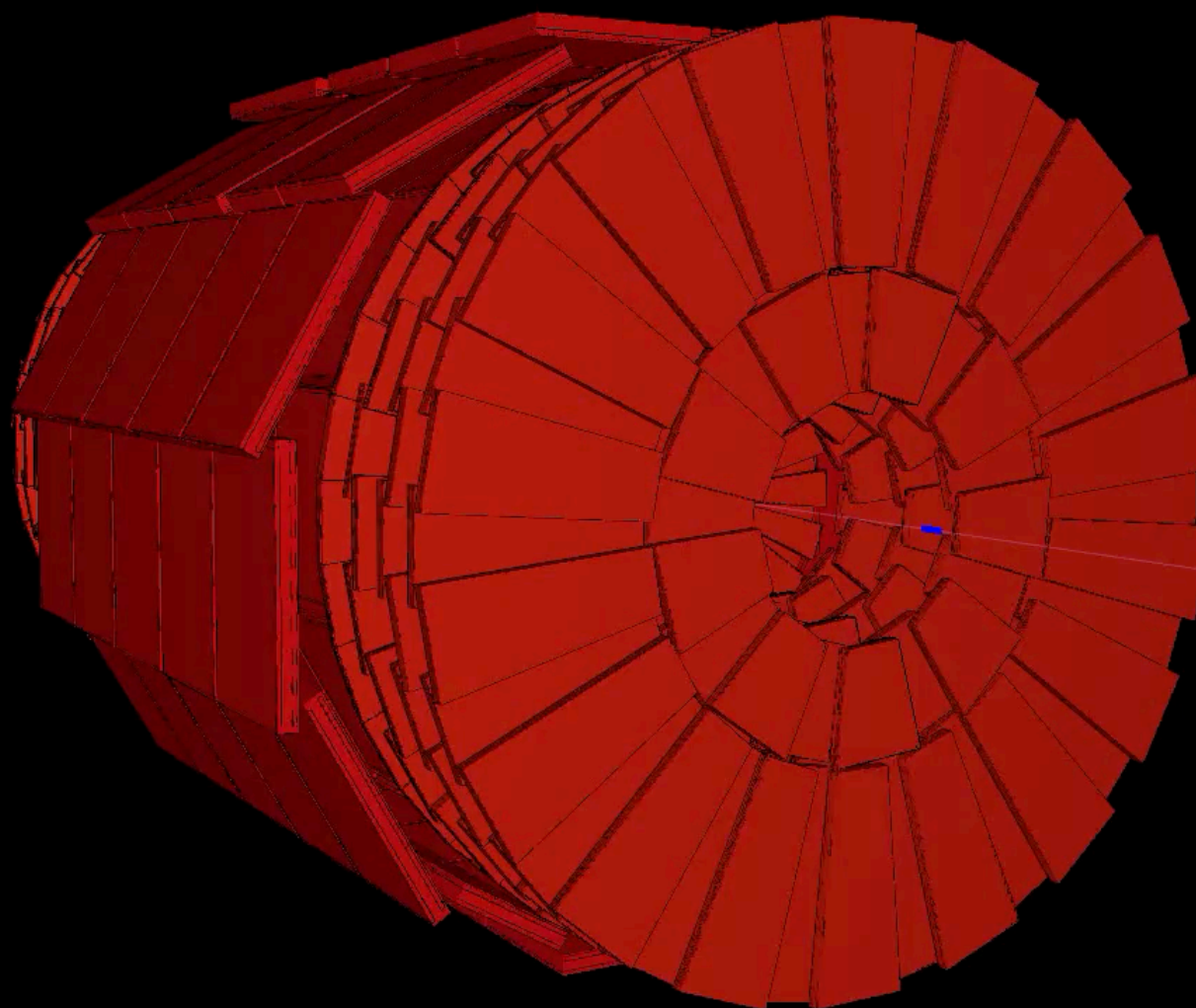


CMS

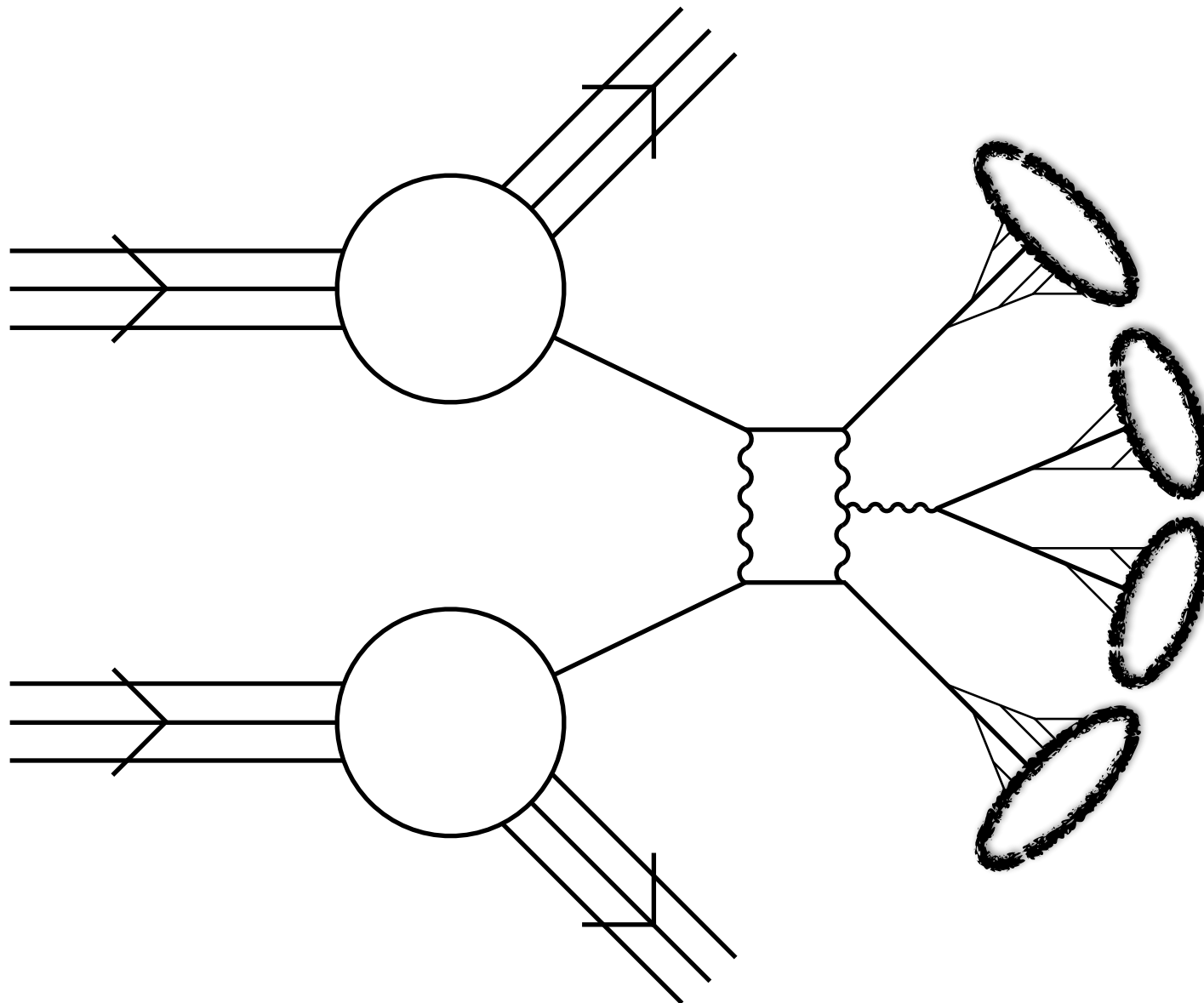
LHCb

ALICE

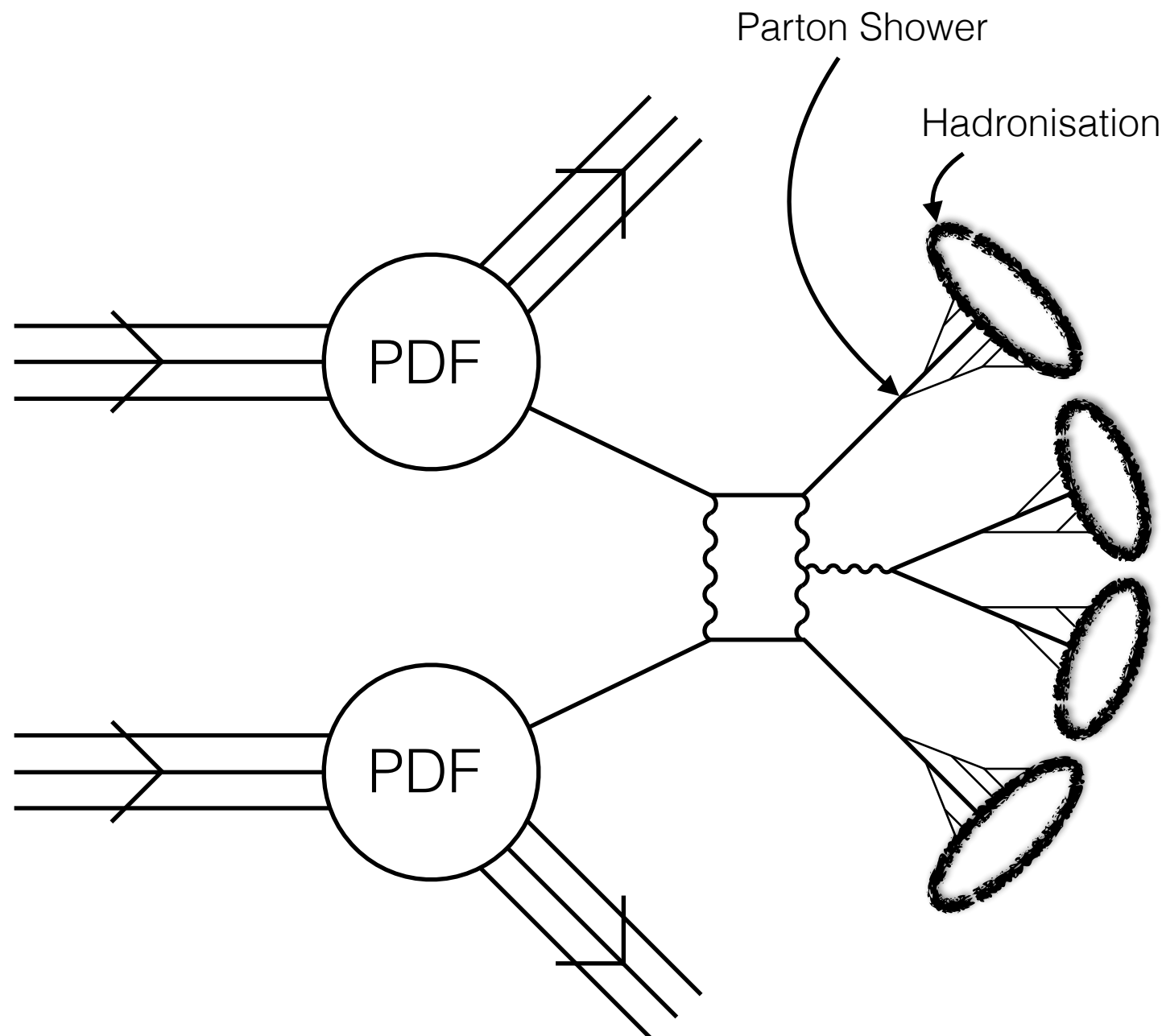
ATLAS



High energy hadron collisions



High energy hadron collisions



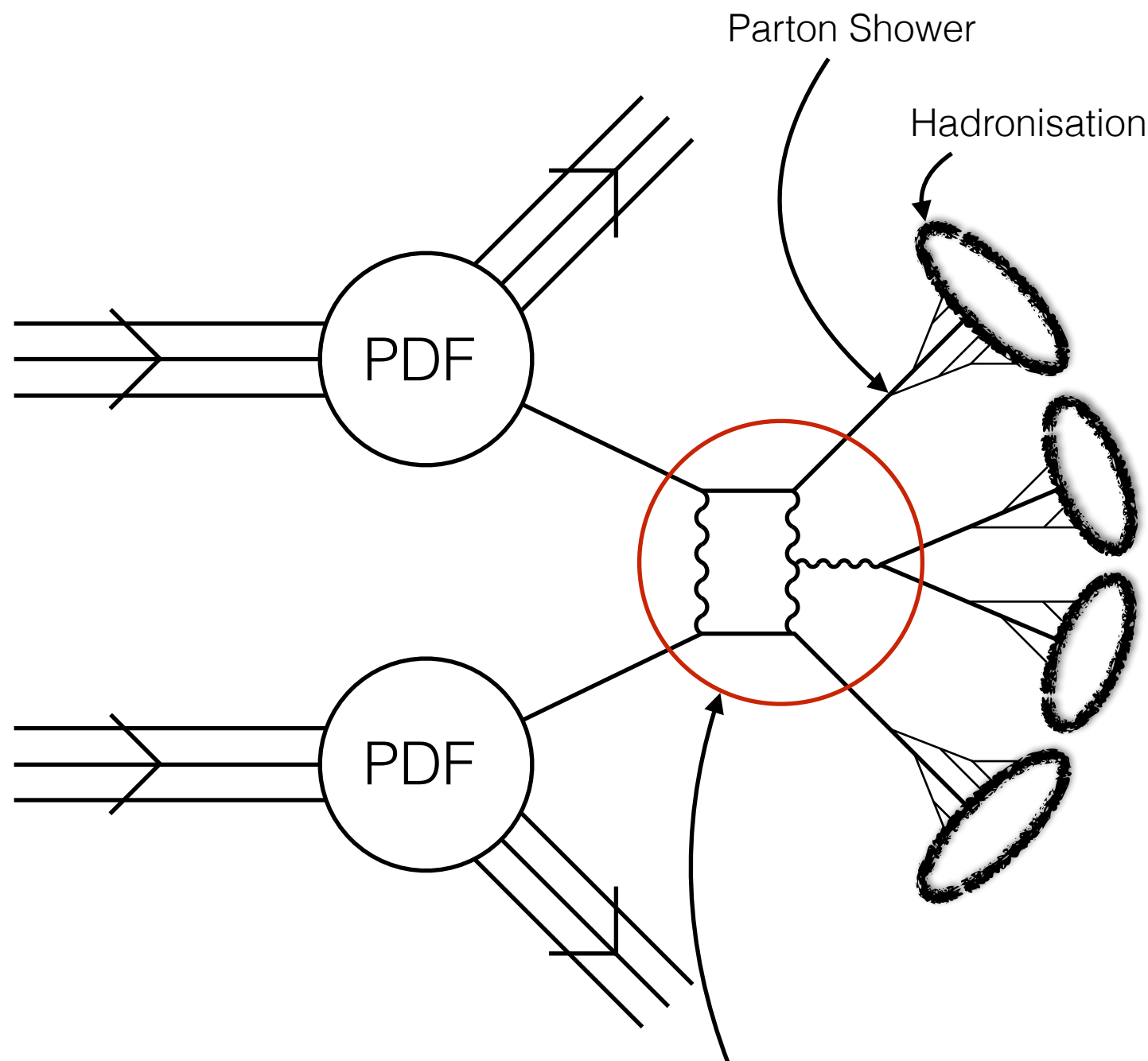
Strong Interactions governed by QCD.

Observed states are colour singlets.

Quarks and gluons bound into hadrons.

Non-perturbative effects.

High energy hadron collisions



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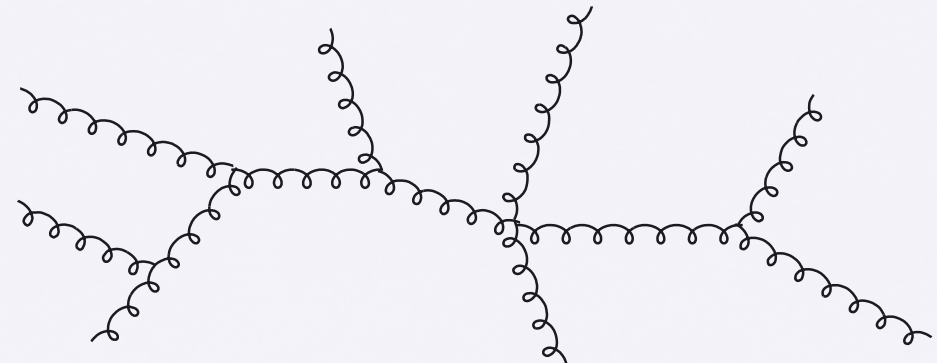
Non-perturbative effects.

The core of the event is described by a hard scattering part: **perturbative**. Higher orders - more accuracy (up to limits of non-perturbative physics).

Amplitudes

QFT: compute amplitudes as sum of Feynman diagrams (loop integrals).

Add up the answer.



Amplitudes

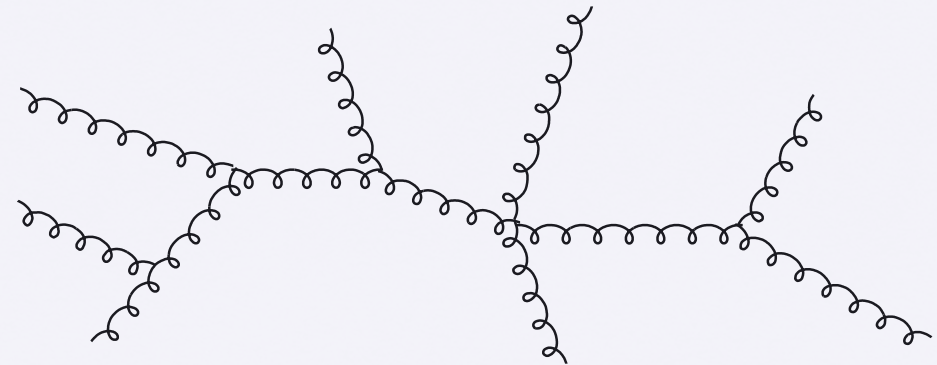
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OR:

[Parke, Taylor (86)]

$$\frac{\langle ij \rangle^4}{\langle 12 \rangle \dots \langle n1 \rangle}$$



Amplitudes

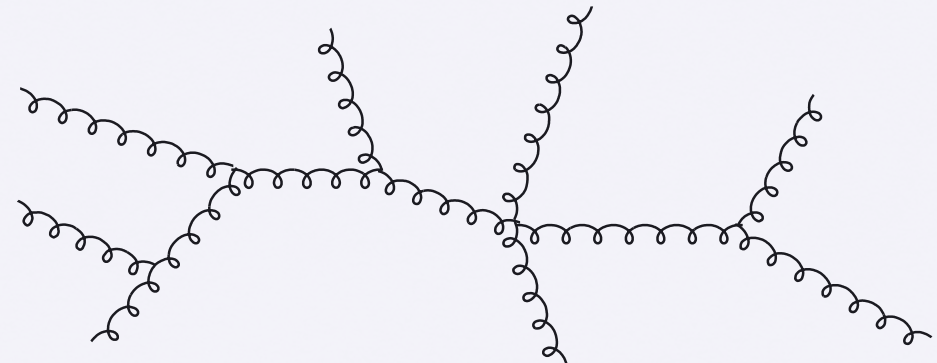
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Refinement: Use recursion relations, unitarity cuts to construct **Integrand**.

[Berends, Giele], [Britto, Cachazo, Feng, Witten],...

[Bern, Dixon, Dunbar, Kosower], [Ossola, Papadopoulos, Pittau]...

$$\text{Diagram} = \sum \left[c_{\square} \text{Diagram}_1 + c_{\triangle} \text{Diagram}_2 + c_{\circ} \text{Diagram}_3 \right]$$

Then integrate.

Amplitudes

Many techniques have been developed:

Tensor and i.b.p. reduction to basis of master integrals

Differential equations for master integrals [Kotikov], [Remiddi], [Gehrmann, Remiddi],...

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Factorised form: $dF(\epsilon, s) = \epsilon A(s)F(\epsilon, s)$ [Henn]

gives

$$F(\epsilon, s) = \sum_{k=-2l}^{\infty} \epsilon^k F_k(s)$$


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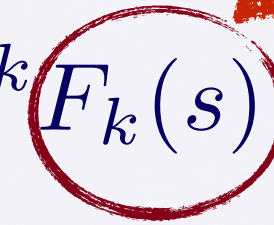
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More generally - elliptic polylogarithms and beyond...

[Walden, Weinzierl], [Brödel, Duhr, Dulat, Tancredi], [Duhr, Tancredi],...

Amplitudes

Current frontier for QCD:

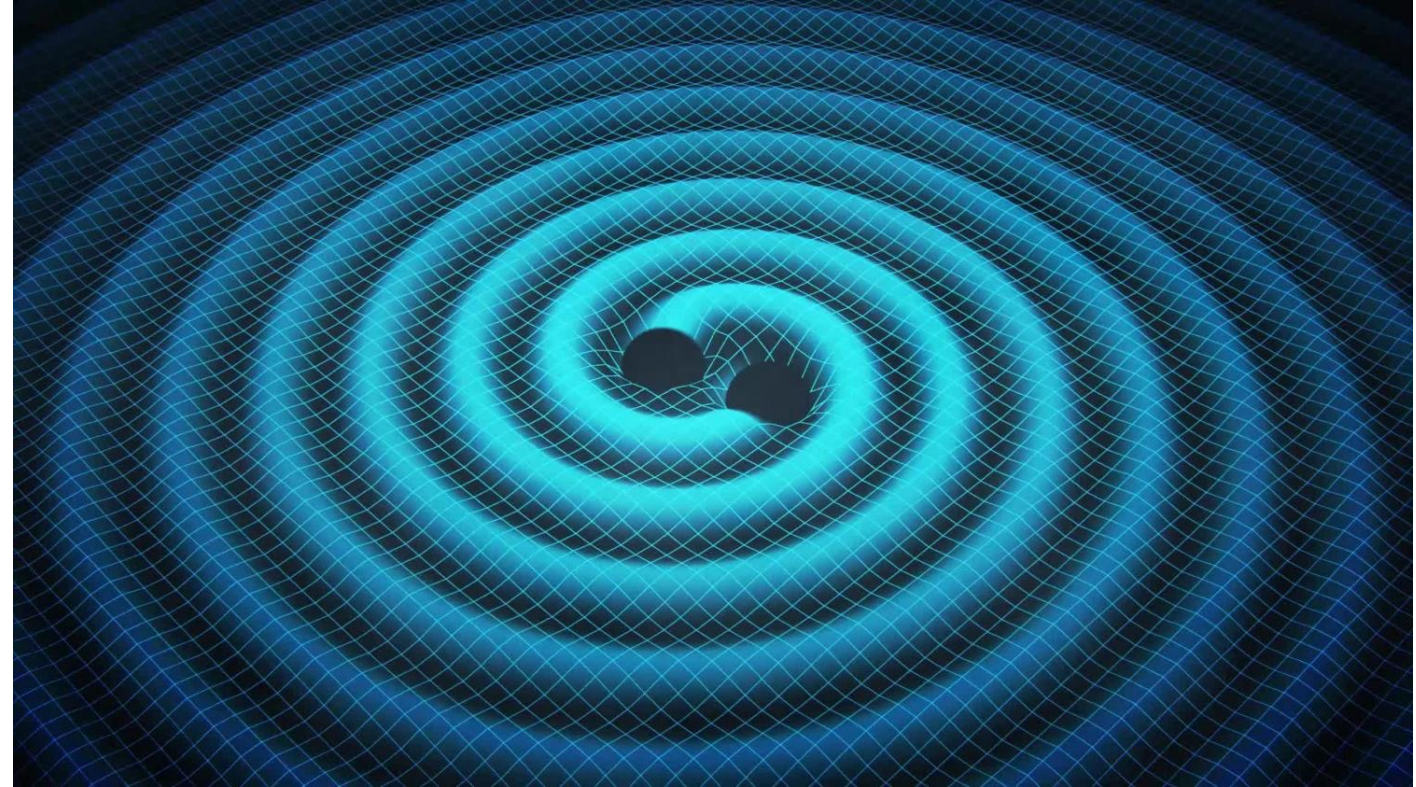
NNLO (2 loops) for 5-point (6-point) processes. [Abreu et al.],...

NLO for higher multiplicity.

Some processes - e.g. Drell-Yan cross-sections at NNNLO...

[Anastasiou, Duhr, Dulat, Herzog, Mistelberger], [Chen, Gehrmann, Glover, Huss, Mistelberger, Pelloni], [Baglio, Duhr, Mistelberger, Szafron],...

Application - Gravitational Inspiral



Amplitudes can be used to
extract corrections to potential

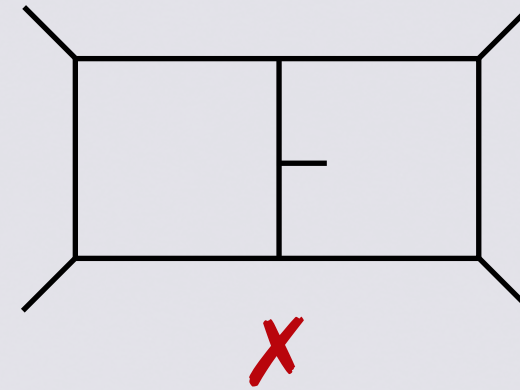
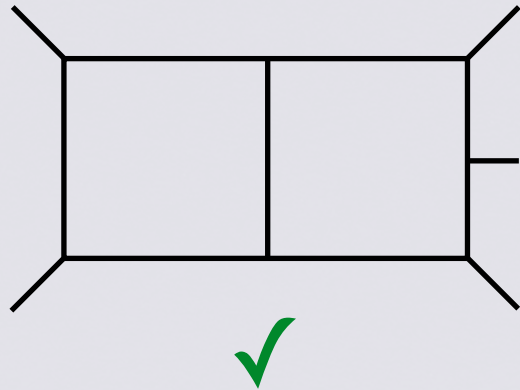
[Bern, Cheung, Roiban, Shen, Solon, Zeng], ...

$$\begin{aligned} &G(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots) \\ &G^2(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots) \\ &G^3(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots) \\ &G^4(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots) \\ &G^5(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots) \\ &G^6(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots) \\ &G^7(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots) \end{aligned}$$

Corrections to Newton potential (Fig: Snowmass report: 2204.05194)

Important Simplifications

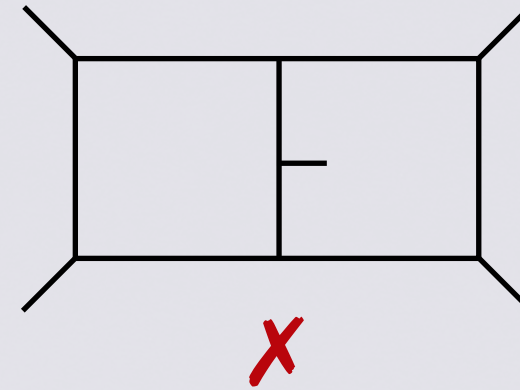
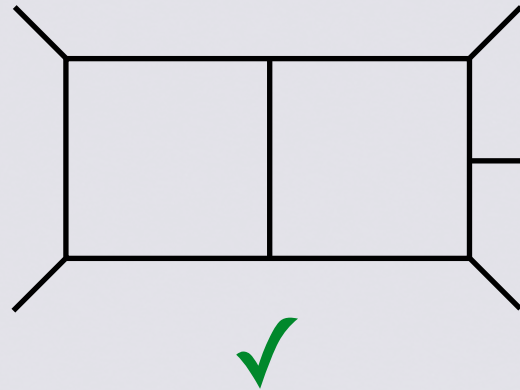
Large N limit ('t Hooft planar limit)



This reduces and simplifies the kind of Feynman integrals which appear.

Important Simplifications

Large N limit ('t Hooft planar limit)



This reduces and simplifies the kind of Feynman integrals which appear.

Simplify the theory: QCD \longrightarrow N=4 Super Yang-Mills theory

Often simplifies the analytic complexity of the final result.

Removes the complication of UV divergences.

Opens up connections to CFT techniques and relation to AdS/CFT

Both: Integrability and new symmetries - e.g. dual conformal symmetry

Amplitudes

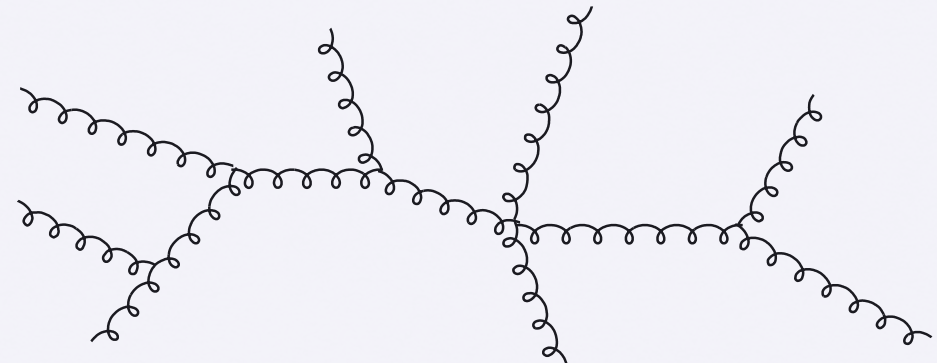
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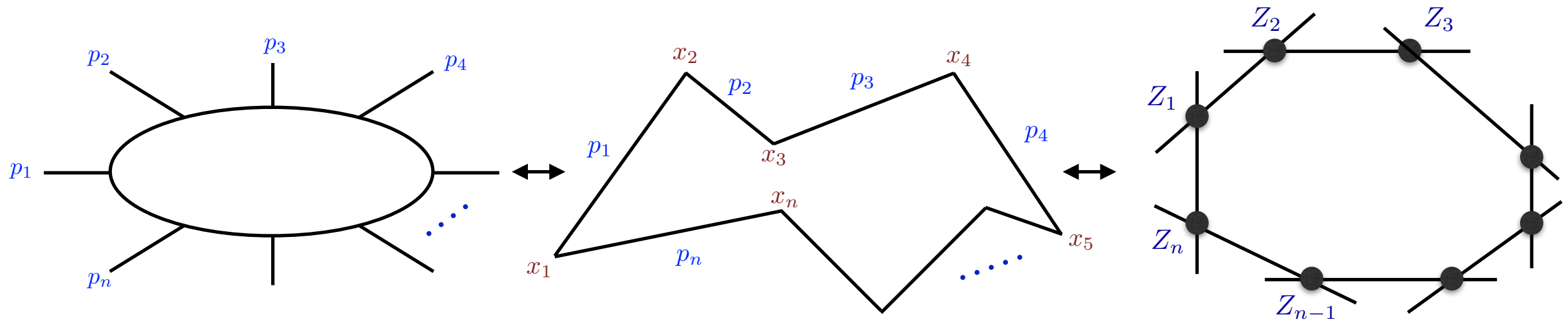
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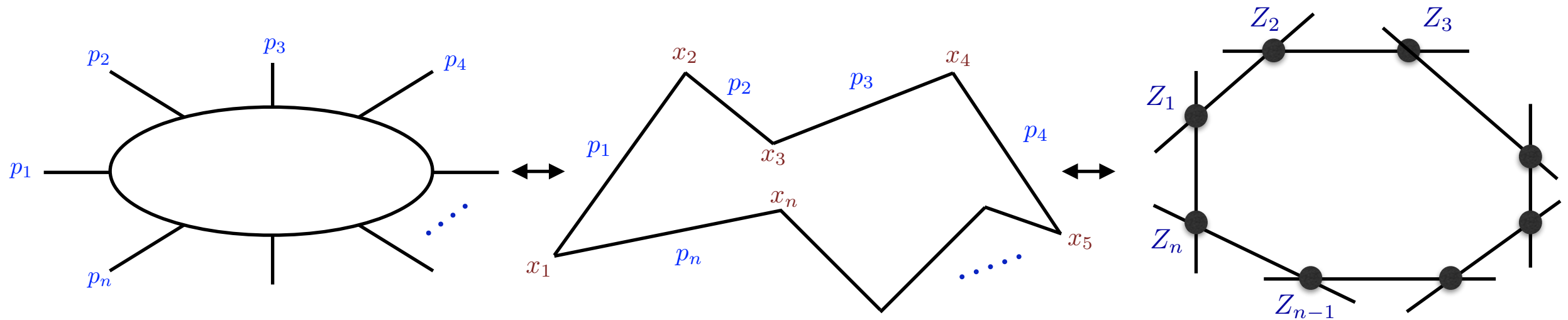
Then integrate.

OR: Understand geometry of amplitudes!

Massless planar scattering kinematics



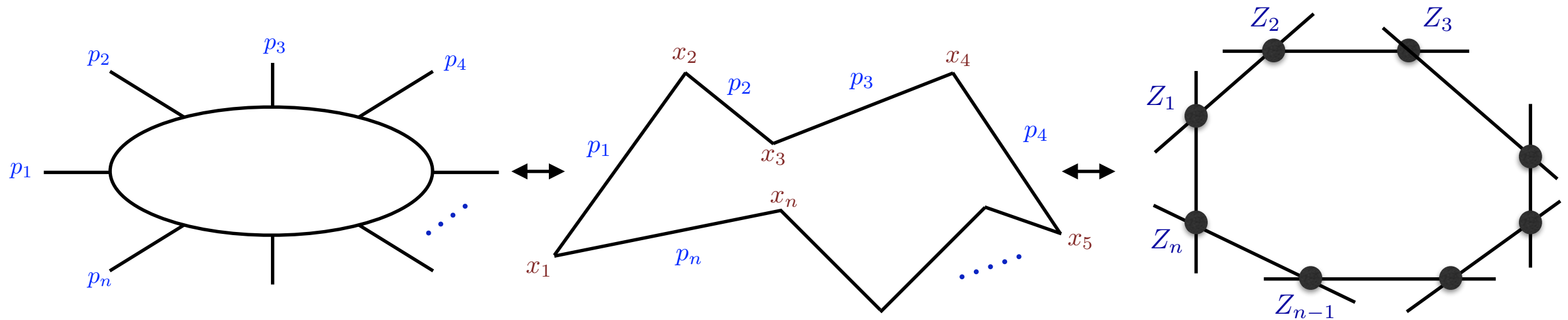
Massless planar scattering kinematics



$$\sum_i p_i = 0$$

$$p_i^2 = 0$$

Massless planar scattering kinematics



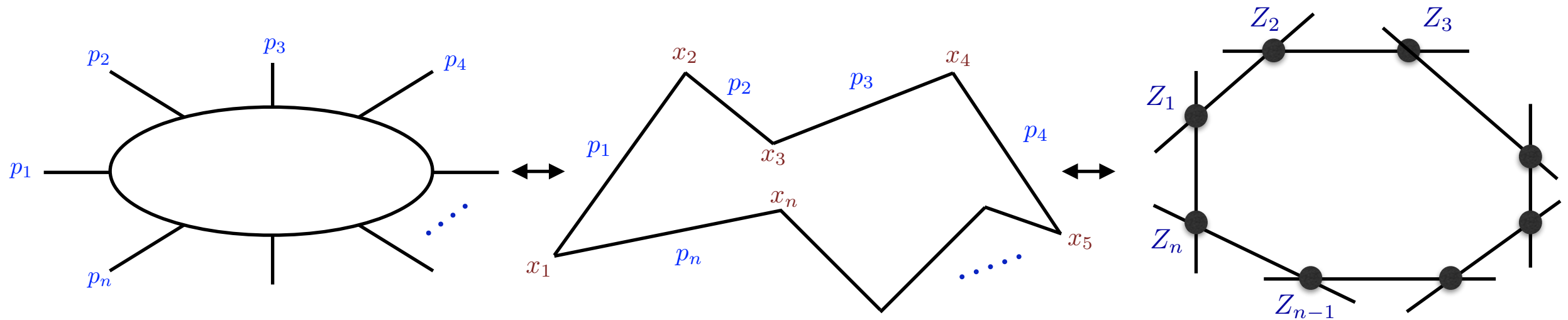
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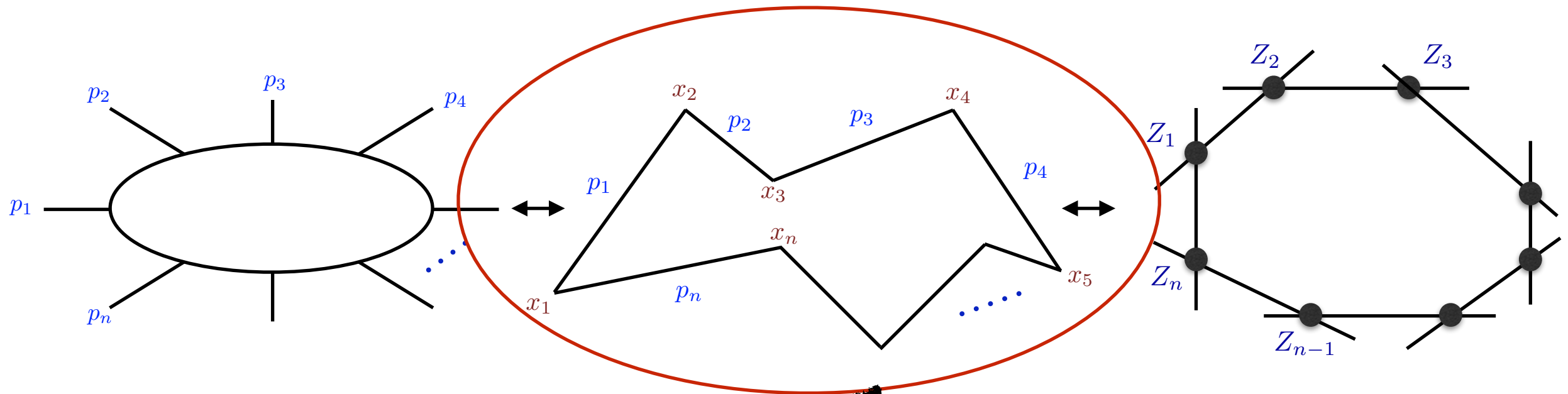
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$$Z_i^A = (\lambda_i^\alpha, \mu_i^{\dot{\alpha}})$$

[Hodges]

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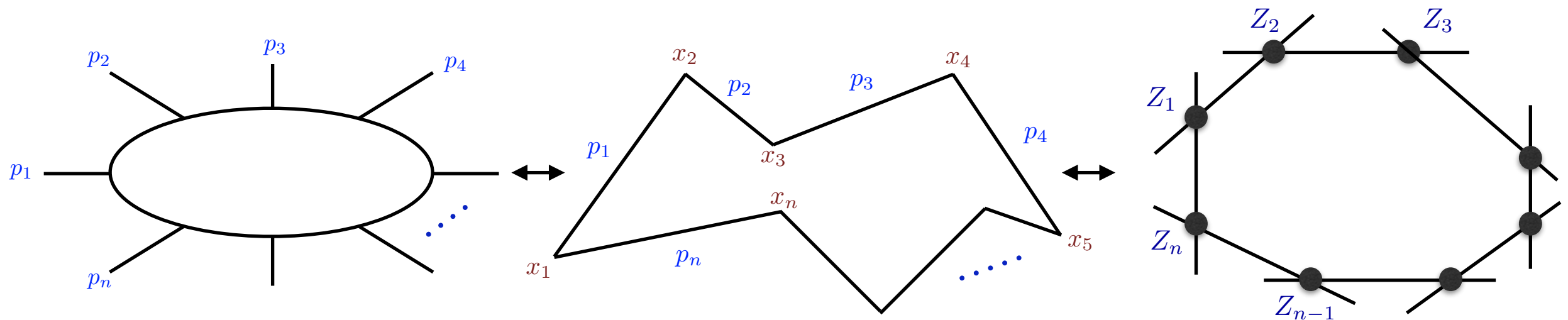
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[Hodges]

The middle picture corresponds to a light-like Wilson loop in planar N=4 SYM.

[Alday, Maldacena], [JMD, Korchemsky, Sokatchev], [Brandhuber, Heslop, Travaglini], [JMD, Henn, Korchemsky, Sokatchev], [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich], [Berkovits, Maldacena], [Beisert, Ricci, Tseytlin, Wolf], [Del Duca, Duhr, Smirnov],...

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[Hodges]

Amplitude depends on $\langle Z_i Z_j Z_k Z_l \rangle = \langle ijkl \rangle$ and $\langle \lambda_i \lambda_j \rangle = \langle ij \rangle$

Planar N=4 SYM: dual conformal symmetry: $\langle Z_i Z_j Z_k Z_l \rangle = \langle ijkl \rangle$ only

Planar N=4 SYM: Grassmannian $\text{Gr}(4,n)$

Arrange the twistors into a $(4 \times n)$ matrix: $(Z_i^A) \quad Z_i \in \mathbb{P}^3$

Dual conformal symmetry: mod out by sl_4 (4-planes in n dimensions)

Grassmannian: dimension = $(3n - 15)$

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Grassmannian: dimension = $(3n - 15)$ (4-pt and 5-pt fixed!)

Amplitudes are functions on this space as well as of the coupling

$$\mathcal{A}(\lambda, Z) = \sum_l \lambda^l \mathcal{A}^{(l)}(Z)$$

Low orders, multiplicities: $\mathcal{A}^{(l)}$ is a sum of iterated integrals of weight $2l$

Hexagons: perturbative expressions

One loop (weight 2):

$$\mathcal{E}_6^{(1)}(u_1, u_2, u_3) = \sum_{i=1}^3 \text{Li}_2(1 - 1/u_i)$$

Two loops (weight 4):

[Goncharov, Spradlin, Vergu, Volovich]

$$\mathcal{E}_6^{(2)}(u_1, u_2, u_3) = \sum_i \left[L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right] - \frac{1}{24} J^4 + \frac{1}{2} \zeta_2 J^2 + \frac{1}{2} \zeta_2^2$$

Iterated Integrals

[Chen], [Goncharov], [Brown], ...

Classical polylogarithms:

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t), \quad \mathrm{Li}_1(x) = -\log(1-x)$$

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$$df^{(k)} = \sum_{\phi} f_{\phi}^{(k-1)} d \log \phi, \quad f^{(1)} = \sum_{\phi} r_{\phi} \log \phi$$

The ‘letters’ ϕ run over a finite set of rational functions (later algebraic).

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Symbol:

$$S[f^{(k)}] = \sum_{\phi} S[f_{\phi}^{(k-1)}] \otimes \phi = \sum_{\vec{\phi}} c_{\vec{\phi}} [\phi_1 \otimes \dots \otimes \phi_k]$$

Integrability:

$$\sum_{\vec{\phi}} c_{\vec{\phi}} [\phi_1 \otimes \dots \otimes \phi_{i-1} \otimes \phi_{i+2} \otimes \dots \otimes \phi_k] d \log \phi_i \wedge d \log \phi_{i+1} = 0$$

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Differential properties:

$$d[\phi_1 \otimes \dots \otimes \phi_k] = [\phi_1 \otimes \dots \otimes \phi_{k-1}] d \log \phi_k$$

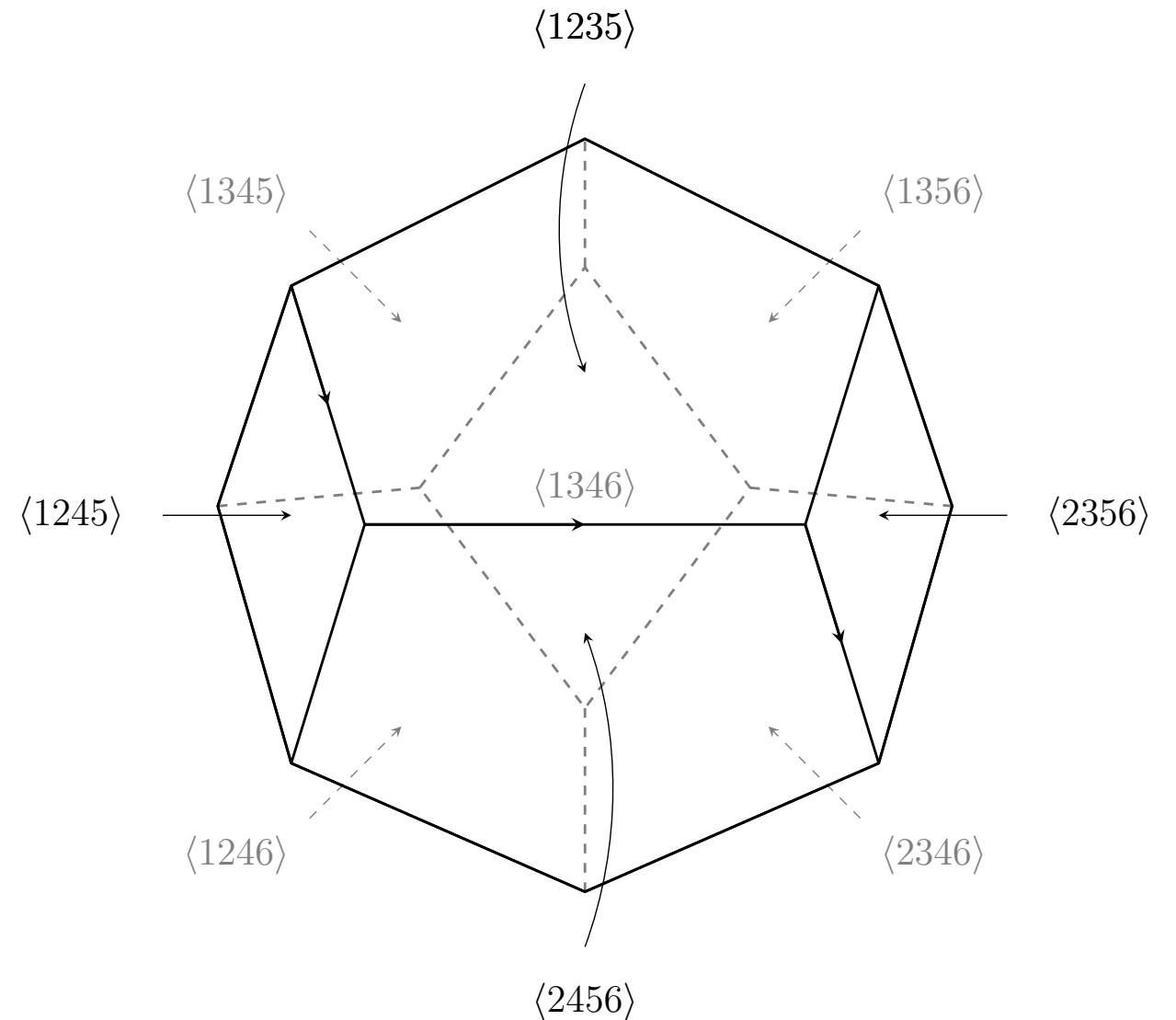
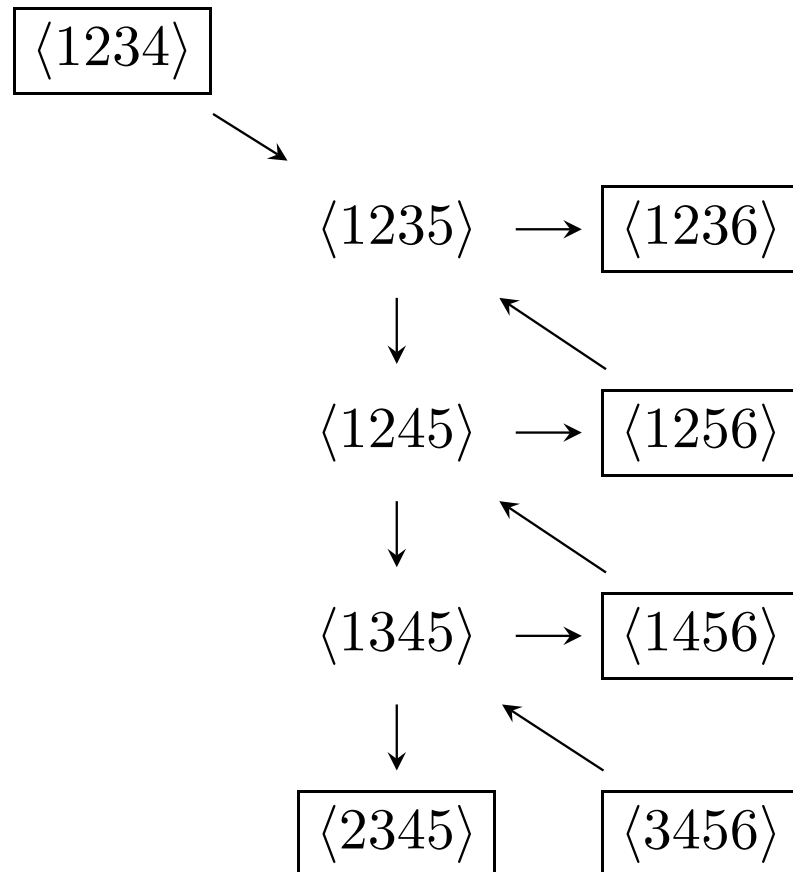
Discontinuities:

$$\mathrm{disc}_{\phi_1=0} [\phi_1 \otimes \dots \otimes \phi_k] = (2\pi i) [\phi_2 \otimes \dots \otimes \phi_k]$$

Cluster Algebras

[Golden, Goncharov, Spradlin, Vergu, Volovich] using results of [Caron-Huot]

Six-point case:



Mutation generates all 15 four-brackets (arranged in Stasheff polytope).

Nine homogeneous combinations (letters):

$$u_1 = \frac{\langle 1236 \rangle \langle 3456 \rangle}{\langle 1346 \rangle \langle 2356 \rangle}, \quad 1 - u_1 = \frac{\langle 1356 \rangle \langle 2346 \rangle}{\langle 1346 \rangle \langle 2356 \rangle}, \quad y_1 = \frac{\langle 1345 \rangle \langle 2456 \rangle \langle 1236 \rangle}{\langle 1235 \rangle \langle 3456 \rangle \langle 1246 \rangle} \quad \& \text{cyc.}$$

Heptagons $\text{Gr}(4,7)$

[JMD, Papathanasiou, Spradlin], [Dixon, JMD, Harrington, Mcleod, Papathanasiou, Spradlin]

Mutation generates all 35 four-brackets and 14 quadratics.

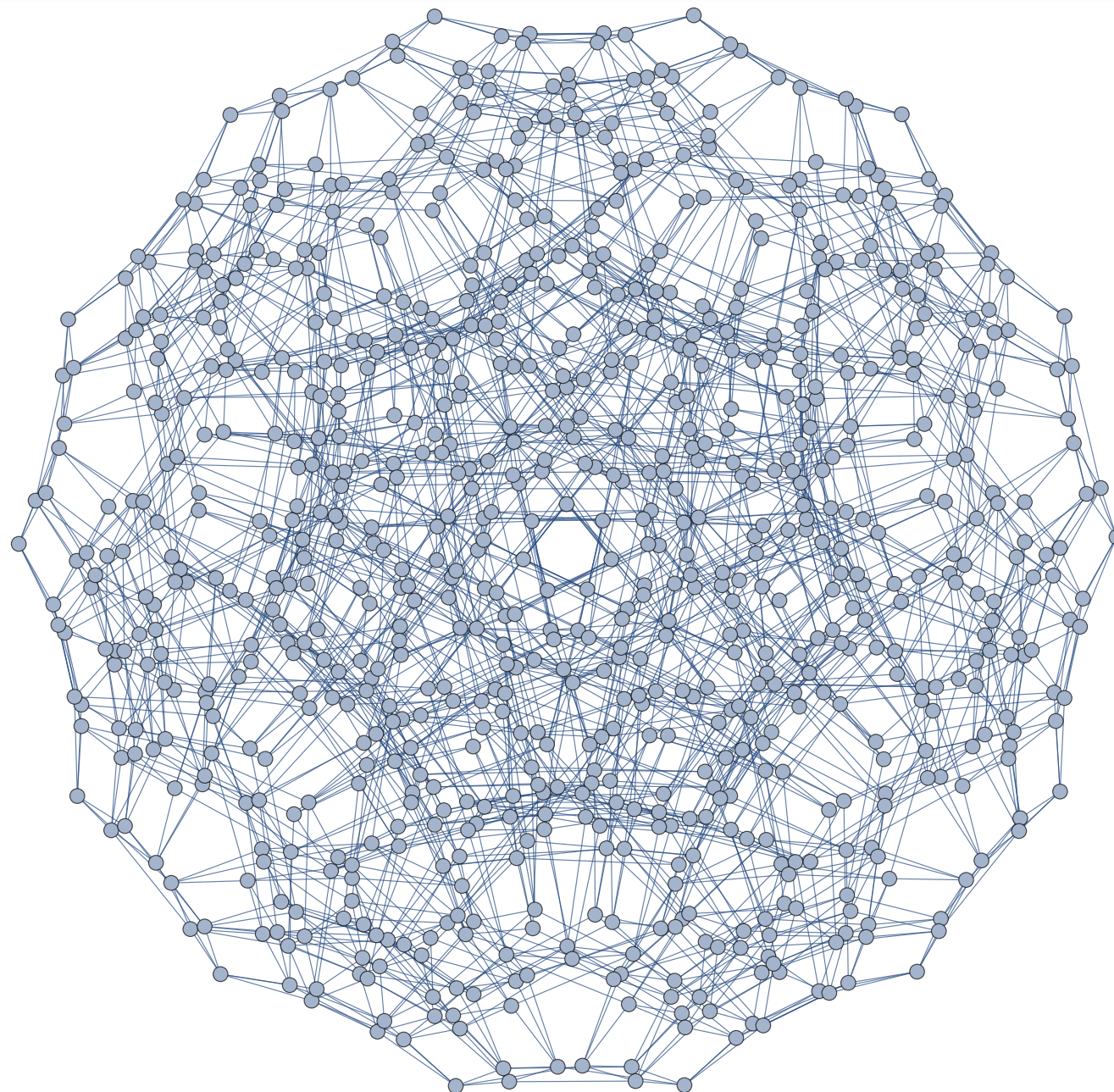
Arranged in a six-dimensional polytope with 833 clusters (hard to draw!).

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For heptagons we generate the following 42 homogeneous letters

$$a_{11} = \frac{\langle 1234 \rangle \langle 1567 \rangle \langle 2367 \rangle}{\langle 1237 \rangle \langle 1267 \rangle \langle 3456 \rangle},$$

$$a_{41} = \frac{\langle 2457 \rangle \langle 3456 \rangle}{\langle 2345 \rangle \langle 4567 \rangle},$$

$$a_{21} = \frac{\langle 1234 \rangle \langle 2567 \rangle}{\langle 1267 \rangle \langle 2345 \rangle},$$

$$a_{51} = \frac{\langle 1(23)(45)(67) \rangle}{\langle 1234 \rangle \langle 1567 \rangle},$$

$$a_{31} = \frac{\langle 1567 \rangle \langle 2347 \rangle}{\langle 1237 \rangle \langle 4567 \rangle},$$

$$a_{61} = \frac{\langle 1(34)(56)(72) \rangle}{\langle 1234 \rangle \langle 1567 \rangle}$$

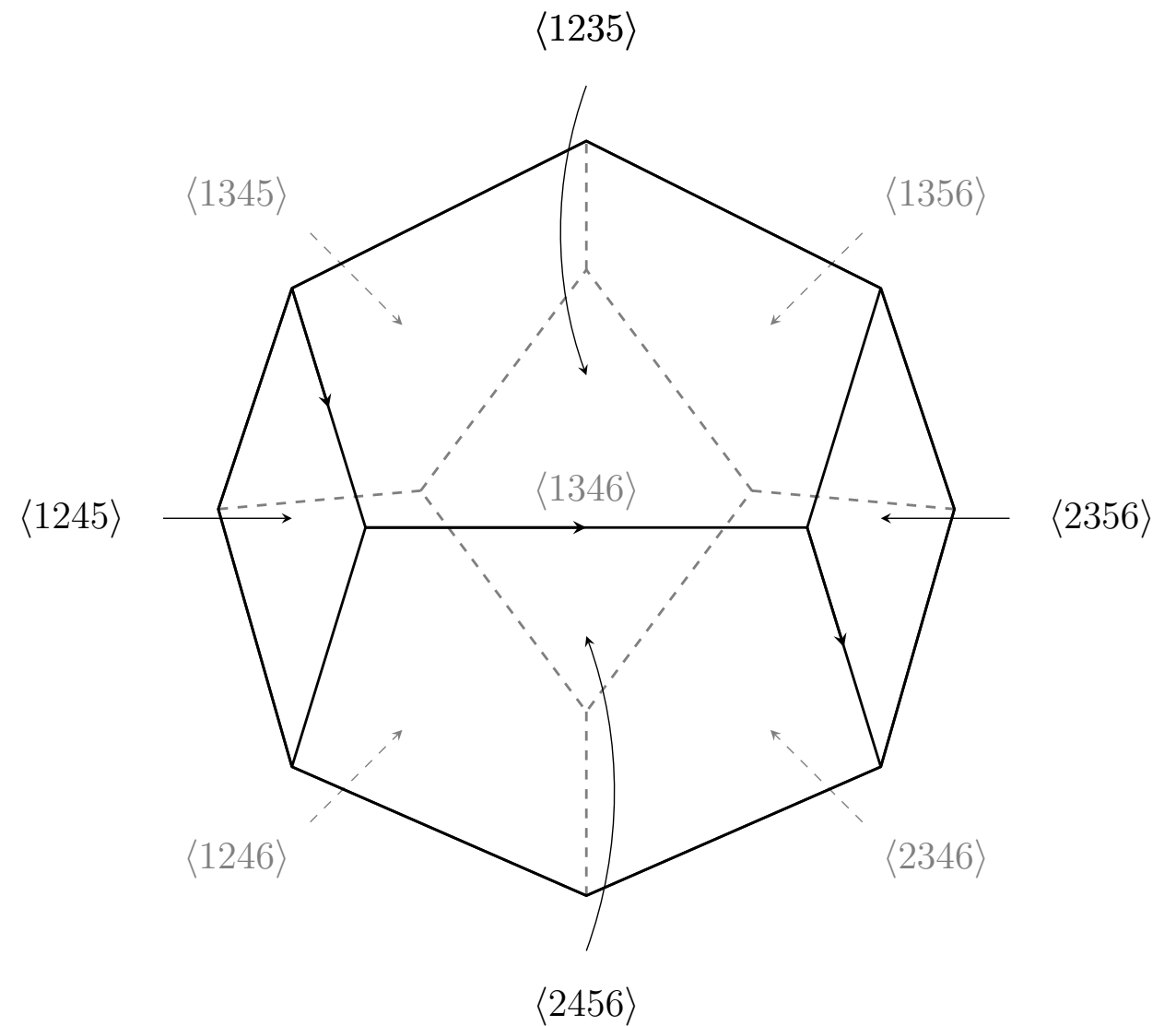
and those obtained by cyclic rotation of the labels.

$$\langle a(bc)(de)(fg) \rangle \equiv \langle abde \rangle \langle acfg \rangle - \langle abfg \rangle \langle acde \rangle$$

Cluster Adjacency

Steinmann relations - cannot have:

$$\langle 1245 \rangle \otimes \langle 2356 \rangle \otimes \dots$$



Cluster Adjacency

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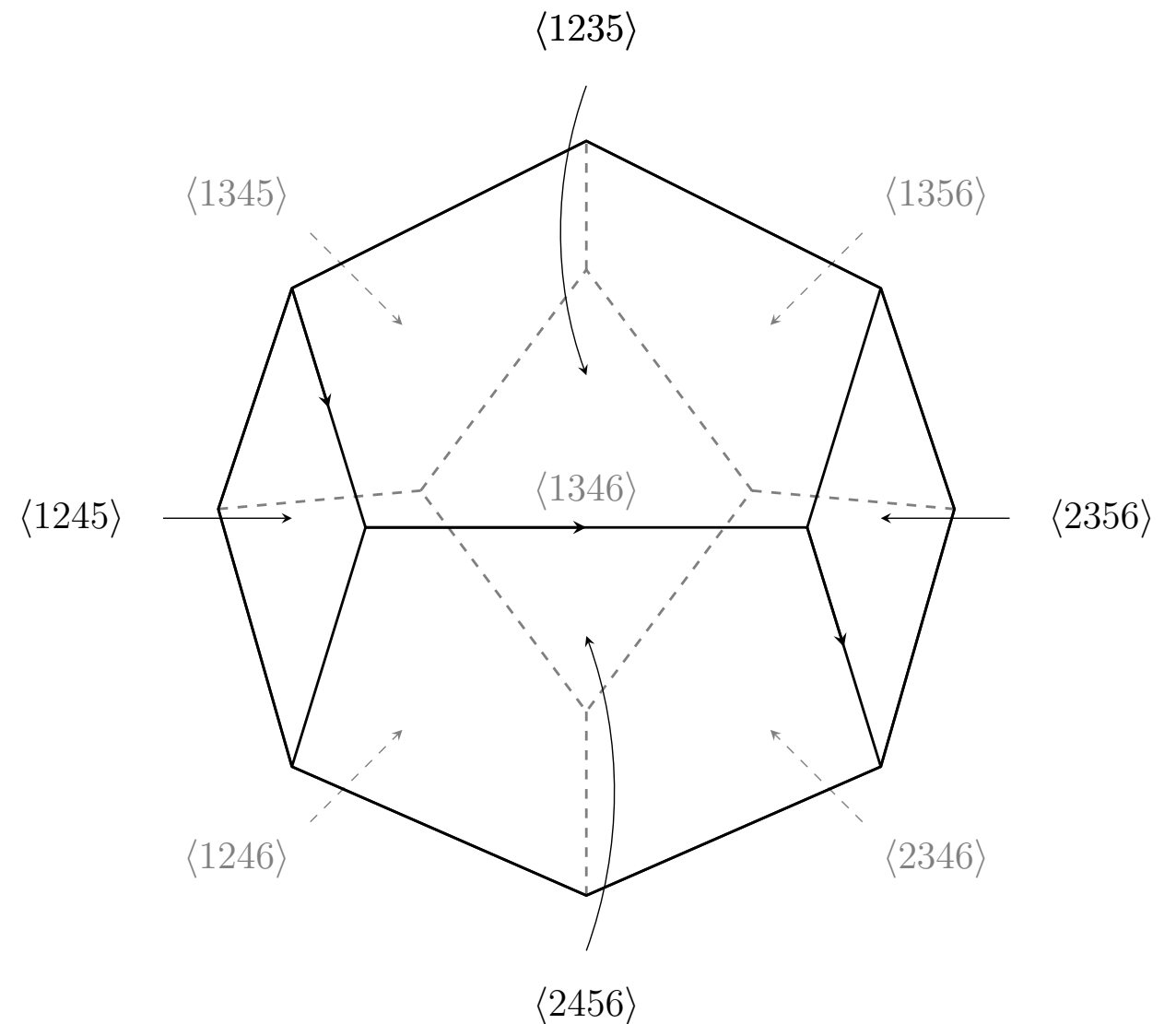
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By inspection we find also cannot have:

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$$\langle 1345 \rangle \otimes \langle 2456 \rangle \otimes \dots$$

True everywhere in symbol.



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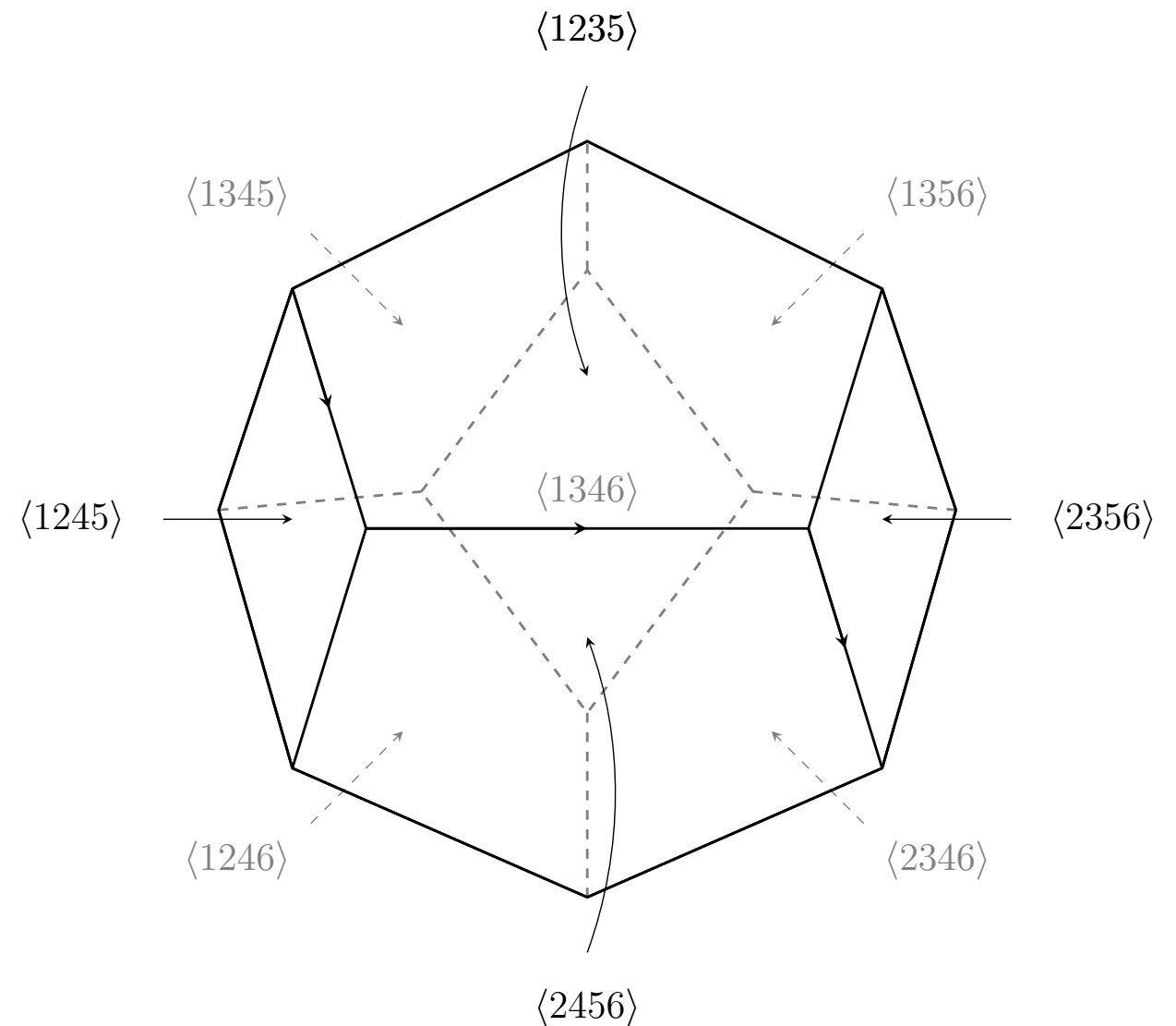
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Cluster Adjacency Principle:

Consecutive discontinuities around unfrozen nodes labelling separated faces forbidden.



[JMD, Foster, Gürdogan]

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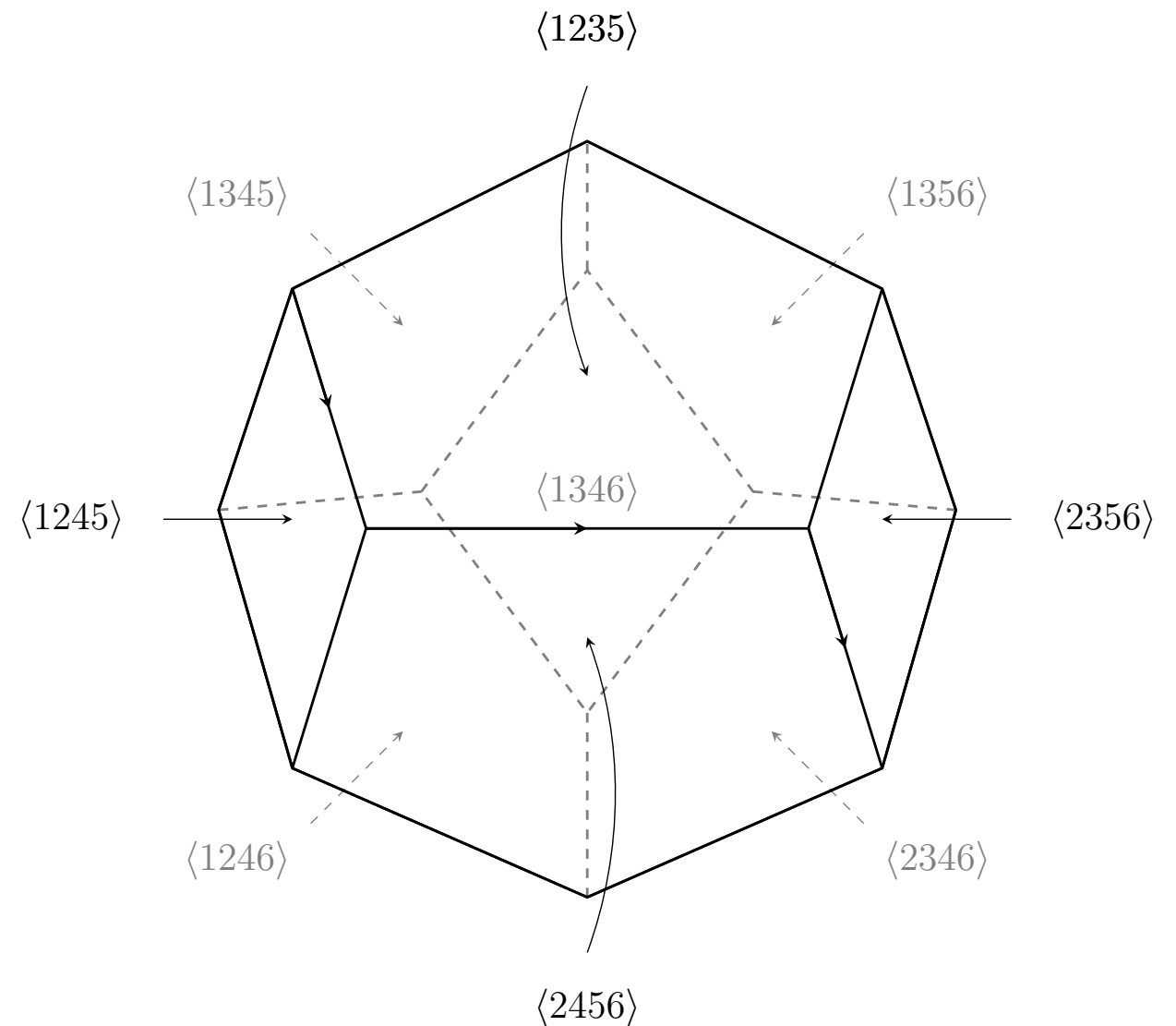
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[JMD, Foster, Gurdogan]

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Algebraic and geometric picture behind the analytic structure of amplitudes.



Bootstrap programme

Build integrable words in the symbol alphabet

Initial entries constrained by physical branch cuts, final entries by supersymmetry

Steinmann/cluster adjacency

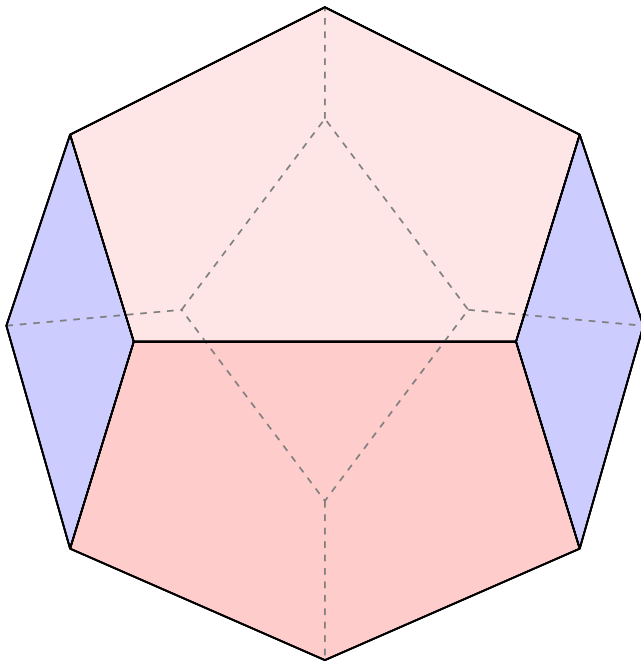
Well behaved in collinear limits

Fixes e.g. heptagon (and hence hexagon symbols) up to four loops

[Dixon, JMD, Henn],
[Dixon, JMD, von Hippel, Pennington],
[Dixon, JMD, Duhr, Pennington],
[Dixon, von Hippel]
[JMD, Papathanasiou, Spradlin],
[Dixon, von Hippel, McLeod],
[Caron-Huot, Dixon, von Hippel, McLeod],
[Dixon, JMD, Harrington, McLeod, Papathanasiou, Spradlin],
[Caron-Huot, Dixon, Dulat, von Hippel, McLeod],
[JMD, Foster, Gürdogan, Papathanasiou],
[Dixon, Gürdogan, McLeod, Wilhelm],...

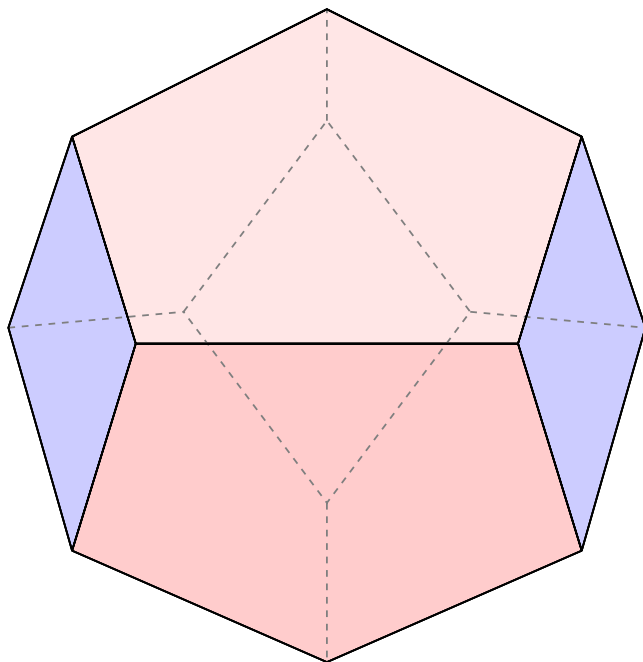
The approach generalises to form factors.

A Puzzle

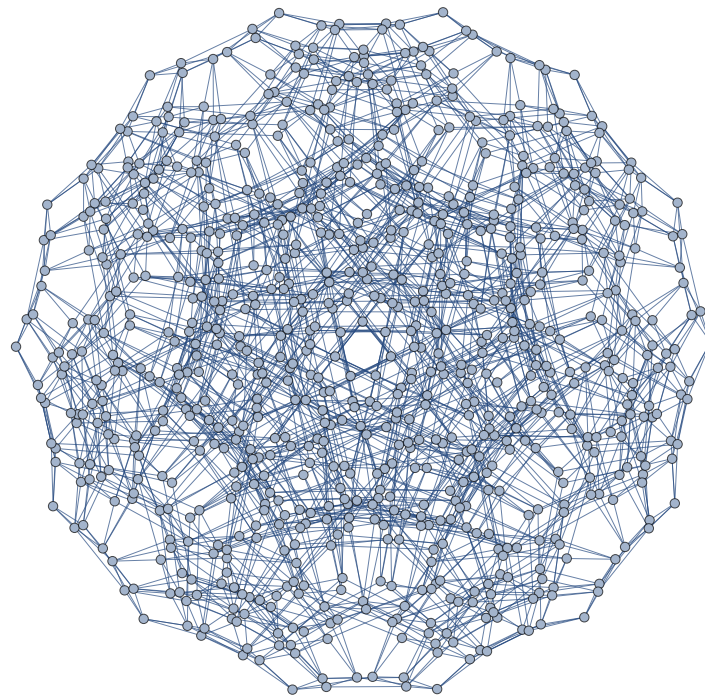


9 letters

A Puzzle

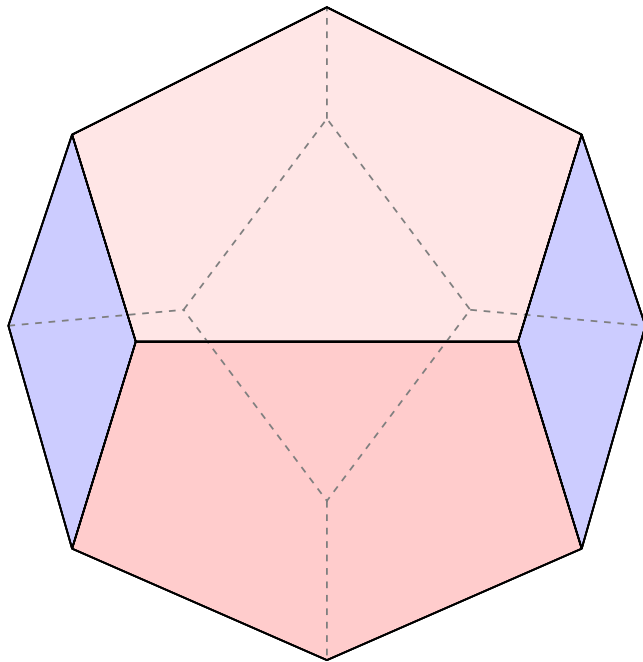


9 letters

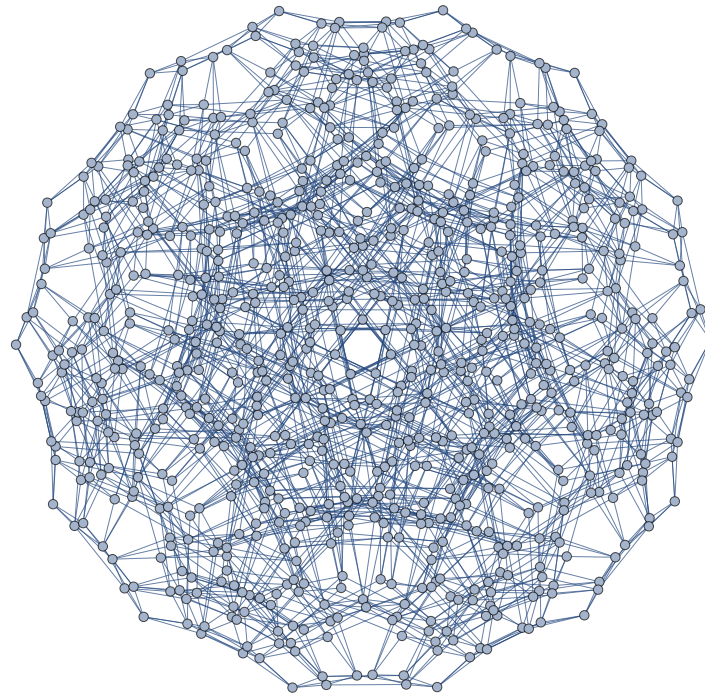


42 letters

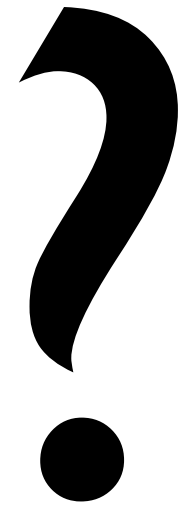
A Puzzle



9 letters



42 letters



Infinitely many?

Grassmannian $\text{Gr}(4,n)$

Plücker coordinates $\langle Z_i Z_j Z_k Z_l \rangle = \langle ijkl \rangle$

obey relations: $\langle ijk[l] \rangle \langle mnpq] \rangle = 0$

$$\langle ij[kl] \rangle \langle mnp]q \rangle = 0$$

We can *define* the Grassmannian through these relations

$$\langle ijkl \rangle \longrightarrow p_{ijkl}$$

Tropical Grassmannian

[Speyer, Sturmfels]

Plücker relations:

$$p_{ijk}[l p_{mnpq}] = 0$$

$$p_{ij}[kl p_{mnp}]_q = 0$$

(Plücker ideal)

Tropical Grassmannian

[Speyer, Sturmfels]

Plücker relations:

$$p_{ijk}[l p_{mnpq}] = 0$$

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Tropicalise:

multiplication	\longrightarrow	addition
addition	\longrightarrow	minimum

example: $xy + wz^2 + 2uv \longrightarrow \min(x + y, w + 2z, u + v)$

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Tropical polynomials divide the space into regions of piecewise linearity

Regions separated by *tropical hypersurfaces*

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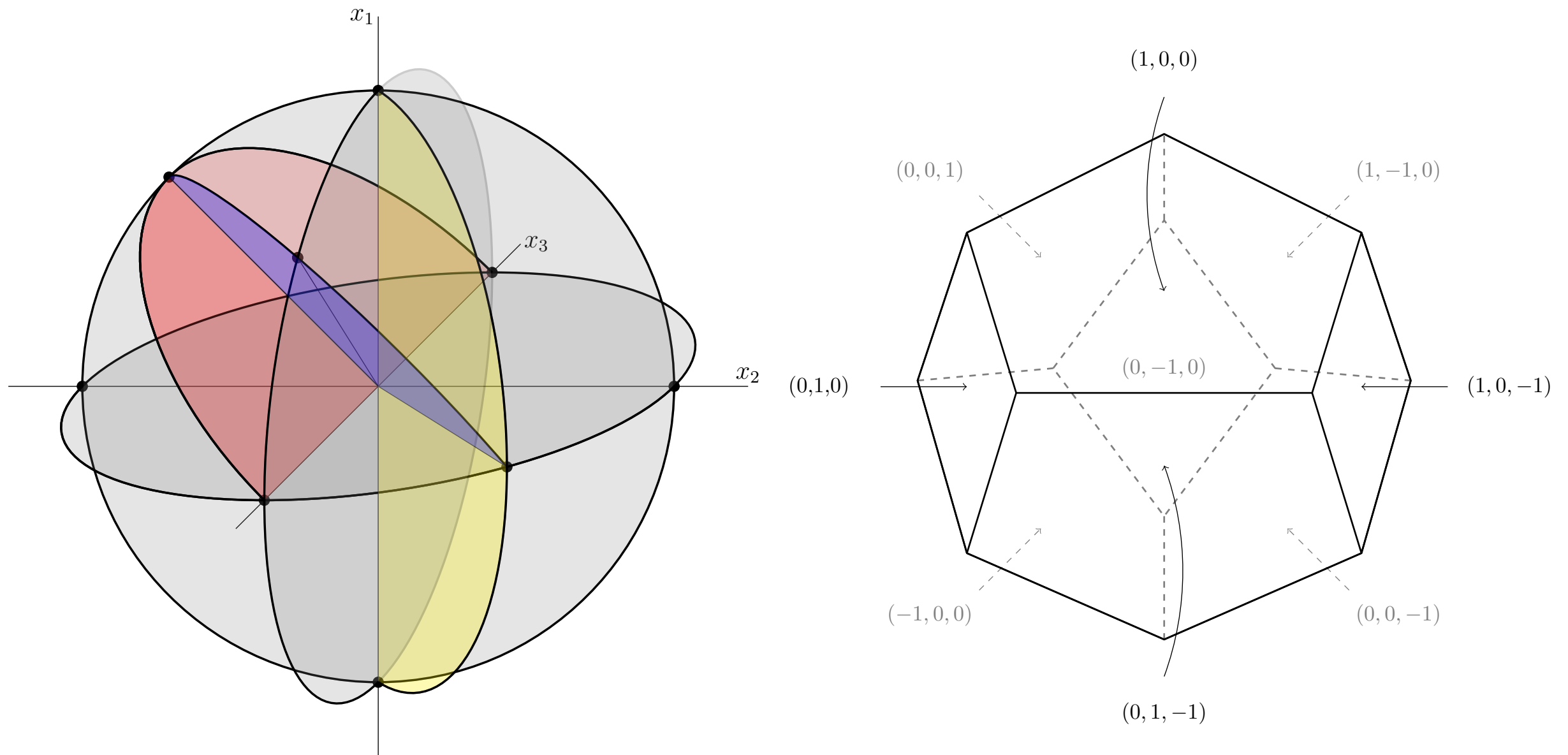
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Tropical polynomials divide the space into regions of piecewise linearity

Regions separated by *tropical hypersurfaces*

Tropical space captures many features of the original

Positive Tropical Grassmannian $\mathrm{Tr}^+(4,6)$



Tropical Grassmannian captures many features of the cluster algebra.

Positive Tropical Grassmannians $\text{Tr}^+(4,8)$

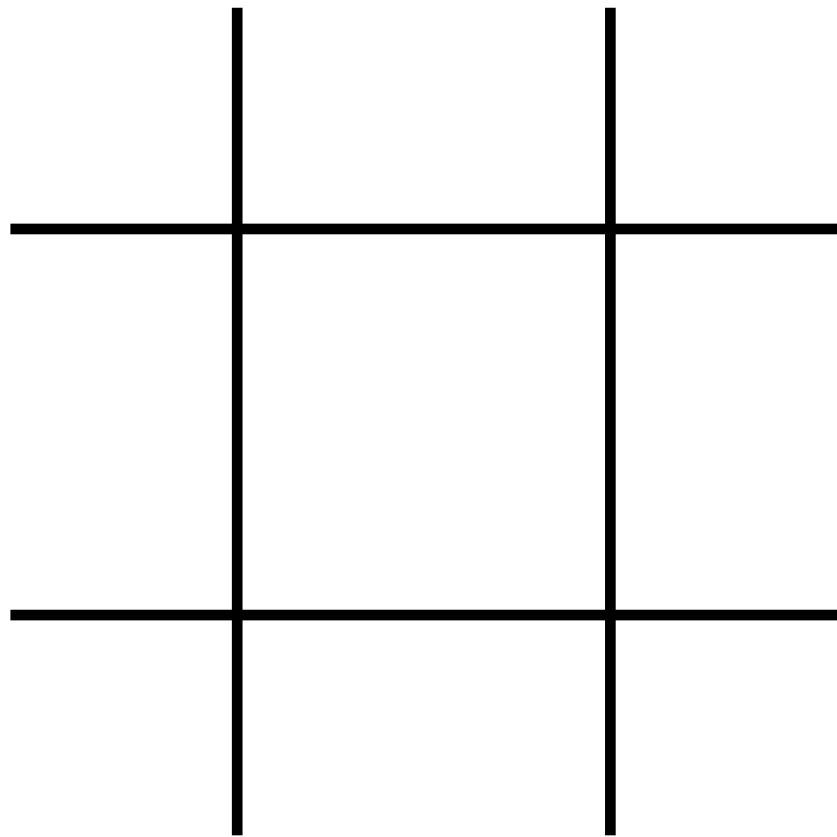
For $n=8$ we have infinite cluster algebra

- 1) Need a way to select a finite number of A-coords
- 2) We know: some letters are not rational functions so cannot be A-coords

Positive Tropical Grassmannians $\text{Tr}^+(4,8)$

For $n=8$ we have infinite (affine) cluster algebra

- 1) Need a way to select a finite number of A-coords
- 2) We know: some letters are not rational functions so cannot be A-coords



$$u = x\bar{x}, \quad v = (1-x)(1-\bar{x})$$

Four-mass box already present at one loop for 8-point amplitudes.

Positive Tropical Grassmannians $\mathrm{Tr}^+(4,8)$

For $n=8$ we have infinite (affine) cluster algebra

- 1) Need a way to select a finite number of A-coords
- 2) We know: some letters are not rational functions so cannot be A-coords

1st problem solved by picking a tropical fan

\mathcal{S}	g-vector rays	extra rays
$\{\langle ii+1jj+1 \rangle, \langle i-1ii+1j \rangle\}$	272	2
$\{\langle ijkl \rangle\}$	356	4
$\{\langle ijkl \rangle, \overline{\langle ijkl \rangle}\}$	544	4

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Match 18 letters of 2-loop NMHV octagon!

using results of [He, Li, Zhang]

square root letters
2nd problem also solved!

Back to generic massless planar theory

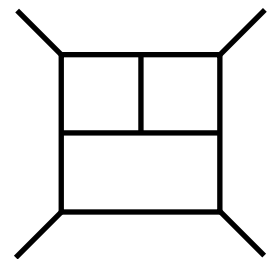
Much of the structure survives without dual conformal symmetry

Many results still expressed as iterated integrals over an alphabet of letters

Four-points - 2 letters:

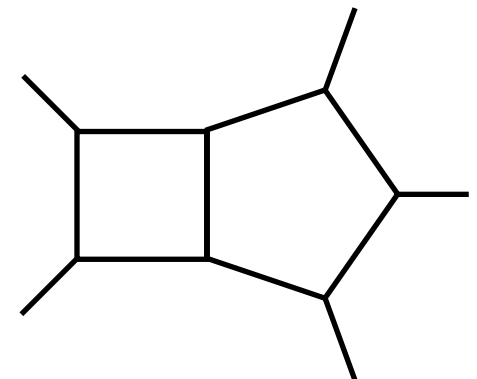
$$\{x, 1+x\} \quad x = \frac{t}{s}$$

Harmonic polylogarithms [Gehrmann, Remiddi]



Five-points - 26 letters:

Pentagon functions [Gehrmann, Henn, Lo Presti]



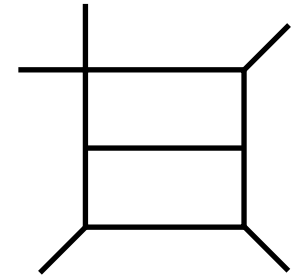
$$\{s_{12}, s_{12} + s_{23}, s_{12} - s_{45}, s_{12} + s_{23} - s_{45}, \Delta = |s_{ij}|, \frac{a+\sqrt{\Delta}}{a-\sqrt{\Delta}}\} \quad \& \text{cyc}$$

Rational functions of momentum twistor variables.

Six-points - ?

Relations to geometry?

Some things are known



Four-point, one-mass topology related to a folding of dual conformal hexagon case: $A_3 \rightarrow C_2$
[Henn, Papathanasiou] (gives six-letter alphabet)

This case is also related to a form factor:

[Dixon, McLeod, Wilhelm], [Dixon, Gürdogan, McLeod, Wilhelm]

Can also drop the requirement of planarity.

Full pentagon alphabet obtained by permutations of planar one.

[Chicherin, Henn, Mitev]

Summary

Grassmannian describes the kinematics of planar massless scattering

Cluster algebras provide sets of singularities and relations between them

Positive Tropical Grassmannian directly connected to cluster algebras

We have choices of fans associated to positive tropical Grassmannian

For each fan we have a natural class of tropically adjacent polylogarithms

Tropical fan and cluster algebra combine to provide algebraic letters

Hints that relations of amplitudes to geometry of kinematics go much deeper!

Outlook

Adjacency rules in the presence of algebraic singularities?

Beyond octagon we have wild cases (not affine)

Elliptic functions?

Can tropical geometry play a role for general massless scattering?

What plays the role of the cluster algebra?

Bases for cluster (tropically) adjacent polylogarithms