Classification of Rational Conformal Field Theories With A Single Critical Exponent

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Based on:

"Classification of Unitary RCFTs with Two Primaries and $c < 25$",
Sunil Mukhi and Brandon Rayhaun, arXiv:2208.05486.

"New Meromorphic CFTs from Cosets",
Arpit Das, Chethan N. Gowdigere and Sunil Mukhi, arXiv:
2207.04061.

"Meromorphic Cosets and the Classification of Three-Character CFT",
Arpit Das, Chethan N. Gowdigere and Sunil Mukhi, arXiv:
2212.03136.

Background:

"Towards a Classification of Two-Character Rational Conformal Field Theories",
A. Ramesh Chandra and Sunil Mukhi,
JHEP 1904 (2019) 153, arXiv:1810.09472.

"Curiosities above $c = 24$",
A. Ramesh Chandra and Sunil Mukhi,
SciPost Phys. 6 (2019) 5, 053, arXiv:1812.05109.

"On 2d Conformal Field Theories with Two Characters",
Harsha Hampapura and Sunil Mukhi,
JHEP 1601 (2016) 005, arXiv: 1510.04478.

"Cosets of Meromorphic CFTs and Modular Differential Equations",
Matthias Gaberdiel, Harsha Hampapura and Sunil Mukhi,
JHEP 1604 (2016) 156, arXiv: 1602.01022.

And previous work:

"Reconstruction of conformal field theories from modular geometry on the torus",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Nucl. Phys. B318 (1989) 483.

"On the classification of rational conformal field theories",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Phys. Lett. B213 (1988) 303.

"Differential equations for correlators and characters in arbitrary rational conformal field theories",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Nucl.Phys. B312 (1989).

- Two-dimensional Conformal Field Theory has been intensely studied since the mid 1980's [Belavin-Polyakov-Zamolodchikov 1984, Knizhnik-Zamolodchikov 1985].
- Physics motivations:
  - Critical systems in statistical physics
  - World-sheet of relativistic strings
  - Quantum/stringy version of AdS_3/CFT_2
  - Anyons and the fractional quantum Hall effect
  - Topological quantum computing
- Mathematical motivations:
  - Vertex operator algebras (VOA)
  - Modular tensor categories (MTC)
  - Vector-valued modular forms (VVMF)
  - Moonshine modules for sporadic groups

- 2d CFTs always have an infinite-dimensional symmetry algebra, the Virasoro algebra, generated by L_n.

- They often have additional infinite-dimensional symmetries beyond Virasoro, such as Kac-Moody algebras associated to a Lie algebra, or W-algebras etc.

- The full symmetry algebra is called the chiral algebra.

- Let us label the chiral algebra generators of a given theory by $\{A_n^\alpha\} = \{L_n, J_n^a, \cdots\}$.

- Then the Hilbert space decomposes into towers (modules) over highest-weight states $|\phi_i\rangle$ called primaries satisfying:

$$A_n^\alpha|\phi_i\rangle = \bar{A}_n^\alpha|\phi_i\rangle = 0, \quad n > 0, \text{ all } \alpha$$

- The negative modes $A_{-n}^\alpha, n > 0$ generate descendants.

- If the number n of primaries $|\phi_i\rangle$ is finite then we have a Rational Conformal Field Theory (RCFT).
- This is equivalent to the statement that c, h_i are rational numbers [Anderson-Moore 1988].
- It is thought that Rational 2d CFT are classified, at least in principle.
- In fact the integrable representations of specific chiral algebras are classified [Belavin-Polyakov-Zamolodchikov 1984, Knizhnik-Zamolodchikov 1985].
- Additionally, if $\mathcal{V}_1, \mathcal{V}_2$ are two chiral algebras with $\mathcal{V}_2 \subset \mathcal{V}_1$, one can take the coset $\mathcal{V}_1/\mathcal{V}_2$ of the corresponding CFTs [Goddard-Kent-Olive 1985] to generate new ones.

- However this does not help us answer simple questions, for example:
  - What are all RCFTs that have just one primary $\mathbf{1}$?

    $= {}^{(1)}$CFT = no critical exponents = meromorphic vertex operator algebras
  - What are all RCFTs that have exactly p primaries $\mathbf{1}, \mathbf{\Phi_1}, \mathbf{\Phi_2}, \cdots, \mathbf{\Phi_{p-1}}$?

    $= {}^{(p)}$CFT = $p-1$ critical exponents = vertex operator algebras with p simple modules

- There is no complete solution to these questions.
- However there was progress on them during 1984 – 1992:
  - Meromorphic CFT [Goddard-Olive 1984, Goddard 1988, Schellekens 1992]
  - Classification of CFT via Modular Linear Differential Equations [Mathur-Mukhi-Sen 1988-89, Naculich 1989].
- These two developments appeared independent, but eventually converged [Gaberdiel-Hampapura-Mukhi 2016].
- In the last decade there has been fresh progress on both questions, and here I will present some recent results.

- I will present the complete classification of unitary CFT with $c < 25$ and two primaries $\mathbf{1},\ \mathbf{\Phi}$ [Mukhi-Rayhaun 2022, Comm. Math. Phys, in press].
- In mathematical terminology, this is the classification of strongly regular VOAs with central charge $c < 25$ and two simple modules.
- The result is a set of 123 theories.
- I will also briefly present:
    - The complete classification of three-character CFT with vanishing Wronskian index [Das-Gowdigere-Mukhi 2022b].
    - A new method to construct meromorphic CFTs with $c \geq 32$ [Das-Gowdigere-Mukhi 2022a].

Outline

- Meromorphic CFT have just the identity primary $\mathbf{1}$.
- Their partition function has the form:

$$Z(\tau,\bar\tau) = |\chi(\tau)|^2$$

where $\chi(\tau)$, the character, counts the degeneracies under the holomorphic the chiral algebra.

- The partition function must be modular invariant:

$$Z\left(\tfrac{a\tau+b}{c\tau+d}, \tfrac{a\bar\tau+b}{c\bar\tau+d}\right) = Z(\tau,\bar\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL(2,Z)}$$

- It follows that $\chi(\tau)$ must be modular invariant upto a phase, and hence is a function of the Klein j-invariant:

$$j(q) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

- Meromorphic CFT can only exist for c a multiple of 8.
- Some examples:

$$
\begin{array}{lll}
c=8: & \chi(\tau)=j(\tau)^{\frac{1}{3}} & \mathsf{E}_{8,1} \quad \text{(unique)} \\
c=16: & \chi(\tau)=j(\tau)^{\frac{2}{3}} & \mathsf{E}_{8,1}\times\mathsf{E}_{8,1},\ \mathsf{D}^{+}_{16,1} \\
c=24: & \chi(\tau)=j(\tau)+\mathcal{N} & \text{Niemeier lattices} \\
c=32: & \chi(\tau)=j(\tau)^{\frac{1}{3}}\big(j(\tau)+\mathcal{N}\big) & \text{Even unimodular 32d lattices}
\end{array}
$$

  where $\mathsf{X}_{r,k}$ = Kac-Moody algebra X of rank r and level k.

- These examples correspond to "lattice theories": c free bosons compactified on a torus $\mathbb{R}^c/\Gamma$, where Γ is an even, unimodular lattice.
- Starting from $c \geq 24$, there are more general (non-lattice) possibilities.

- The most general allowed character at $c = 24$ is:

$$\chi(\tau) = j(\tau) + \mathcal{N}$$

where $\mathcal{N}$ is any integer ≥ -744, but there are just 71 CFT's [Schellekens 1992].

- These include 24 lattice theories and a finite number of generalisations involving orbifolding etc.

- Examples:
  - Schellekens #59: $\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1}$(lattice theory)
  - Schellekens #34: $\mathsf{A}_{3,1}\mathsf{D}_{7,3}\mathsf{G}_{2,1}$(non-lattice theory)

- These are special modular invariant combinations ("extensions") of characters for the given non-simple KM algebras.

- 70 of the 71 Schellekens theories are meromorphic extensions of non-simple KM algebras.
- This means we treat some primaries of integer dimension as chiral generators of higher spin, which then organises the theory into a smaller number of primaries.
- The 71st Schellekens theory is also a meromorphic extension, not of KM algebras, but of $(\text{Ising model})^{48}$.
- This extension is called the Monster CFT.
- At $c = 32$ there are around $\sim 10^9$ even, unimodular lattices (and an unknown number of non-lattice theories), so complete classification seems very difficult.

Outline

- Let us now try to classify CFT by the number of primaries.
- For this we define the character:

$$\chi_i(q) = \mathrm{tr}_i\, q^{L_0 - \frac{c}{24}}, \qquad q \equiv e^{2\pi i\tau}$$

where the trace runs over only holomorphic descendants.

- Multiple primaries can have the same character, e.g. if Φ is complex then $\chi_\Phi = \chi_{\bar{\Phi}}$. Thus, number n of independent characters $\leq$ number p of independent primaries.
- The partition function is then:

$$Z(q,\bar{q}) = \sum_{i=0}^{n-1} M_i\, \chi_i(q)\, \bar{\chi}_{\bar{i}}(\bar{q})$$

where M_i is the multiplicity of the character.

- Modular invariance of $Z \iff$ characters go into linear combinations of themselves under SL(2,Z):

$$\chi_i\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{j=0}^{n-1} \varrho_{ij}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\chi_j(\tau), \quad \varrho^\dagger\varrho = 1$$

- Thus they are vector-valued modular functions or VVMF (of weight 0).

- ϱ is an n-dimensional representation of SL(2,Z), the modular representation of the characters.

- A holomorphic modular bootstrap method was originally proposed in [Mathur-Mukhi-Sen (1988)] to find candidate VVMFs.

- These should have an expansion in $q \equiv e^{2\pi i\tau}$:

$$\chi_i(q) = q^{\alpha_i}\left(a_0^{(i)} + a_1^{(i)}\, q + a_2^{(i)}\, q^2 + \cdots\right),\ i = 0, 1, \cdots, n-1$$

with non-negative integer coefficients $a_m^{(i)}$, the degeneracies of descendants.

- Generic VVMFs do not have positive or integral coefficients. So one needs to isolate the admissible ones, for which:

$$a_m^{(i)} \in \mathbb{Z}_{\geq 0} \text{ (potentially giving degeneracies)}$$
$$a_0^{(0)} = 1 \text{ (non-degenerate vacuum state)}$$

- Then one has to verify which of these admissible characters correspond to actual CFT.
- Thus the classification of RCFT involves two steps:

  I. Classify admissible characters.

  II. Within this set, search for actual CFT.

- There has been some kind of folklore that most often, $\text{I} \equiv \text{II}$, i.e. each set of admissible characters describes a unique CFT.
- As we will see, this is wrong in two ways:
  - Most admissible characters do not describe any CFT,
  - Some admissible characters describe multiple CFT.

- The starting point is that every VVMF $\chi_i(q)$ can be written as the n independent solutions of a Modular Invariant Linear Differential Equation (MLDE) in τ.

- So we write the most general MLDE and scan it for admissible solutions. For $n = 2$:

$$\Big(D_\tau^2 + \phi_2(\tau) D_\tau + \phi_4(\tau)\Big)\chi(\tau) = 0$$

where:

$$D_\tau \equiv \frac{1}{2\pi i}\frac{\partial}{\partial\tau} - \frac{k}{12}E_2(\tau): \quad \mathcal{M}_k \to \mathcal{M}_{k+2}$$

is the Ramanujan-Serre derivative, k is the weight of the modular form on which it acts, and $E_2(\tau)$ is an Eisenstein series.

- The equation will be modular invariant iff $\phi_2(\tau), \phi_4(\tau)$ are modular of weight $2, 4$ respectively.

- Suppose we are given a VVMF χ_0, χ_1. A general linear combination of them, χ, satisfies the following equation:

$$\begin{vmatrix} \chi_0 & \chi_1 & \chi \\ D_\tau \chi_0 & D_\tau \chi_1 & D_\tau \chi \\ D_\tau^2 \chi_0 & D_\tau^2 \chi_1 & D_\tau^2 \chi \end{vmatrix} = 0$$

- Expanding by the last column, we get:

$$\begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau \chi_0 & D_\tau \chi_1 \end{vmatrix} D_\tau^2 \chi - \begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau^2 \chi_0 & D_\tau^2 \chi_1 \end{vmatrix} D_\tau \chi + \begin{vmatrix} D_\tau \chi_0 & D_\tau \chi_1 \\ D_\tau^2 \chi_0 & D_\tau^2 \chi_1 \end{vmatrix} \chi = 0$$

- Hence:

$$\phi_2 = -\frac{\begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau^2 \chi_0 & D_\tau^2 \chi_1 \end{vmatrix}}{\begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau \chi_0 & D_\tau \chi_1 \end{vmatrix}}, \quad \phi_4 = \frac{\begin{vmatrix} D_\tau \chi_0 & D_\tau \chi_1 \\ D_\tau^2 \chi_0 & D_\tau^2 \chi_1 \end{vmatrix}}{\begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau \chi_0 & D_\tau \chi_1 \end{vmatrix}}$$

- Both ϕ_2, ϕ_4 can have poles wherever the denominator, which we call the Wronskian W, has zeroes.

- The number of such poles in the interior of moduli space is denoted $\frac{\ell}{6}$, where $\ell = 0, 2, 3, 4 \cdots$.
- The factor of $\frac{1}{6}$ arises because moduli space has special points $\rho \equiv e^{\frac{2\pi i}{3}}, i$.
- ℓ is called the Wronskian index.
- With two characters it can be shown that ℓ is even: $\ell = 0, 2, 4, \cdots$ [Naculich 1989].
- For any given ℓ there is a finite basis of functions of E_4, E_6 from which ϕ_2, ϕ_4 are built. Thus the MLDE has a finite number of parameters that grows with ℓ.

- Now we fix ℓ to various small values and scan the parameter space to look for solutions that are admissible characters.
- In order of simplicity we start with $\ell = 0$. Then $\phi_2 = 0$ and $\phi_4(\tau) = \mu E_4(\tau)$, where E_4 is an Eisenstein series and μ is a real parameter.
- This leads to the "MMS equation":

$$\Big(D_\tau^2 + \mu E_4(\tau)\Big)\chi = 0$$

- The parameter μ completely determines the solutions up to overall normalisations.

- The leading terms in the solutions are denoted $q^{\alpha_0}, q^{\alpha_1}$ where α_0, α_1 are the critical indices or exponents.
- We write:

$$\alpha_0 = -\tfrac{c}{24}, \quad \alpha_1 = -\tfrac{c}{24} + h$$

  If the solutions describe a CFT then (c, h) will have the usual meaning.
- Next we solve the MLDE recursively by the Frobenius method, order by order in q.
- Let's look at two examples.

- MMS equation with $\mu = -\frac{119}{3600}$ gives admissible characters:

$$\begin{aligned}\chi_0(q) = q^{-7/60}(1 &+ 14q + 42q^2 + 140q^3 + 350q^4 + 850q^5 \\ &+ 1827q^6 + 3858q^7 + 7637q^8 + 14756q^9 + \cdots) \\ \chi_1(q) = q^{17/60}(1 &+ \tfrac{34}{7}q + 17q^2 + 46q^3 + 117q^4 + 266q^5 \\ &+ 575q^6 + 1174q^7 + 2311q^8 + 4380q^9 + \cdots)\end{aligned}$$

  $c = \frac{14}{5}, h = \frac{2}{5}$. Normalising second character by 7, it becomes admissible. These characters can be identified with the CFT $\mathsf{G}_{2,1}$.

- MMS equation with $\mu = -\frac{143}{4800}$ gives non-admissible characters:

$$\begin{aligned}\chi_0(q) = q^{-13/120}(1 &+ \tfrac{455}{37}q + \tfrac{121784}{3589}q^2 + \tfrac{60836763}{563473}q^3 + \tfrac{4525367613}{17467663}q^4 \\ &+ \tfrac{2893074116179}{4838542651}q^5 + \tfrac{2046920234847579}{1630588873387}q^6 + \cdots) \\ \chi_1(q) = q^{11/40}(1 &+ \tfrac{363}{83}q + \tfrac{15849}{1079}q^2 + \tfrac{90512}{2407}q^3 + \tfrac{58528917}{633041}q^4 \\ &+ \tfrac{128150964}{633041}q^5 + \tfrac{102972265445}{242454703}q^6 + \cdots)\end{aligned}$$

  Formally $c = \frac{13}{5}, h = \frac{23}{60}$, but clearly this is not a CFT.

- We found a finite and very interesting set of admissible characters, all with $0 < c < 8$, and guessed their identification with various known RCFT:

m_1	c	h	Identification
1	$\frac{2}{5}$	$\frac{1}{5}$	$c=-\frac{22}{5}$ minimal model ($c \leftrightarrow c-24h$)
3	1	$\frac{1}{4}$	$k=1$ SU(2) WZW model
8	2	$\frac{1}{3}$	$k=1$ SU(3) WZW model
14	$\frac{14}{5}$	$\frac{2}{5}$	$k=1$ G_2WZW model
28	4	$\frac{1}{2}$	$k=1$ SO(8) WZW model
52	$\frac{26}{5}$	$\frac{3}{5}$	$k=1$ F_4 WZW model
78	6	$\frac{2}{3}$	$k=1$ E_6 WZW model
133	7	$\frac{3}{4}$	$k=1$ E_7 WZW model
190	$\frac{38}{5}$	$\frac{4}{5}$	?
248	8	$\frac{5}{6}$	$\supset k=1$ E_8 WZW model

- This brings together several distinct level-1 KM characters, and a few curious entries that have negative fusion rules. Today I will ignore those (they are now called Intermediate Vertex Operator Algebras or IVOA).

- From now on we will restrict to unitary CFT with exactly two primaries.
- Thus we must discard:
  - One primary: $\mathsf{E}_{8,1}$
  - More than two primaries: $\mathsf{A}_{2,1}, \mathsf{D}_{4,1}, \mathsf{E}_{6,1}$. For example $\mathsf{A}_{2,1}$ has three primaries $\mathbf{1}$, $\mathbf{3}$ and $\bar{\mathbf{3}}$ but the latter two have the same character.
- This leaves just four theories with $(p, \ell) = (2, 0)$, which we identified with the affine theories:

$$\mathsf{A}_{1,1}, \mathsf{G}_{2,1}, \mathsf{F}_{4,1}, \mathsf{E}_{7,1} \qquad \text{"MMS set"}$$

with:

$$c = 1, \tfrac{14}{5}, \tfrac{26}{5}, 7 \text{ respectively.}$$

- Recently this identification was shown to be unique [Mason-Nagatomo-Sakai 2018].

Outline

- Now that $\ell = 0$ is classified, we move to the next case, $\ell = 2$. Historically, this case provided the insights for the more general classification.

- The MLDE is:

$$\left(D_\tau^2 + \frac{E_6}{3E_4} D_\tau + \mu E_4(\tau)\right)\chi = 0$$

  and we see that ϕ_2 has a pole where E_4 vanishes.

- This MLDE was solved in [Naculich 1989, Hampapura-Mukhi 2015] and four admissible two-primary VVMF's were found, with central charges:

$$17, \frac{94}{5}, \frac{106}{5}, 23$$

- These solutions have central charges $24 - c$ and conformal dimensions $2 - h$ relative to the MMS set.

- These solutions were not identified as CFTs for nearly three decades! This was finally achieved in [Gaberdiel-Hampapura-Mukhi 2016].

- We used the coset construction of RCFT's [Goddard-Kent-Olive 1984,1985] where one embeds a chiral algebra $\mathcal{V}_2 \subset \mathcal{V}_1$ to define $\frac{\mathcal{V}_1}{\mathcal{V}_2}$.

- One has $c_{\rm coset} = c_{\mathcal{V}_1} - c_{\mathcal{V}_2}$.

- Most examples in the physics literature take $\mathcal{V}_1, \mathcal{V}_2$ to be WZW theories, then embed the KM algebra of the denominator in that of the numerator.

- However the cosets we need are actually simpler. They are cosets of a meromorphic theory $\mathcal{A}$ by an affine theory $\mathcal{V} \subset \mathcal{A}$ leading to $\mathcal{V}' = \frac{\mathcal{A}}{\mathcal{V}}$. [Moore-Seiberg 1988, Schellekens et al 1990, Frenkel-Zhu 1991, Fröhlich et al 2006].

- Such cosets can sometimes be defined by embedding KM algebras, but they exist in greater generality.

- Since $\mathcal{A}$ is meromorphic, the denominator $\mathcal{V}$ and the coset $\mathcal{V}'$ are both ${}^{(p)}$CFTs for the same p and their characters satisfy a holomorphic bilinear relation:

$$\sum_{i=0}^{p-1} \chi_i^{\mathcal{V}}(q)\, \chi_i^{\mathcal{V}'}(q) = \chi^{\mathcal{A}}(q)$$

- Let's clarify this by an example. Recall #34 in Schellekens' list with KM algebra $\mathsf{A}_{3,1}\mathsf{D}_{7,3}\mathsf{G}_{2,1}$.

- We can now take the quotient:

$$\frac{\mathcal{S}_{\#34}}{\mathsf{G}_{2,1}}$$

  by deleting $\mathsf{G}_{2,1}$ from the numerator. This leads to a ${}^{(2)}$CFT with algebra $\mathsf{A}_{3,1}\mathsf{D}_{7,3}$ and $c = 24 - \frac{14}{5} = \frac{106}{5}$.

- Such quotients can be shown to have Wronskian index $\ell = 2$.

- There are 15 such ${}^{(2)}$CFT, with $c = 17, \frac{94}{5}, \frac{106}{5}, 23$, corresponding to cosets of entries in the Schellekens list by the four MMS solutions.

- Generalising this method to other cosets of meromorphic theories by MMS theories generates many CFTs with various values of the Wronskian index ℓ.
- Remarkably this procedure is exhaustive: every theory with two primaries arises by taking cosets of meromorphic theories by the MMS set!
- This result [Mukhi-Rayhaun 2022] makes use of two previous results:

    1. All admissible modular representations ϱ of rank 2 arise from the MMS set by conjugation and/or multiplication by the phase $\omega = e^{2\pi i/3}$.
    2. With two primaries, the modular representation completely determines the modular tensor category.

- All modular representations are generated by two MMS CFTs, which we can take to be: ϱ_{A} for $\mathsf{A}_{1,1}$ and ϱ_{G} for $\mathsf{G}_{2,1}$.
- From these we can make a total of 12 modular representations:

$$\begin{array}{cccccc} \varrho_{\mathsf{A}} & ,\varrho_{\mathsf{G}}, & \omega\varrho_{\mathsf{A}}, & \omega\varrho_{\mathsf{G}}, & \omega^2\varrho_{\mathsf{A}}, & \omega^2\varrho_{\mathsf{G}} \\ \varrho_{\mathsf{A}}^{\dagger}, & \varrho_{\mathsf{G}}^{\dagger}, & \omega\varrho_{\mathsf{A}}^{\dagger}, & \omega\varrho_{\mathsf{G}}^{\dagger}, & \omega^2\varrho_{\mathsf{A}}^{\dagger}, & \omega^2\varrho_{\mathsf{G}}^{\dagger} \end{array}$$

  Here $\omega\varrho_A^{\dagger}, \omega\varrho_G^{\dagger}$ correspond to $\mathsf{E}_{7,1}$ and $\mathsf{F}_{4,1}$ respectively.
- If now we pair any representation with one containing its conjugate, e.g ϱ_A and $\omega^n\varrho_A^{\dagger}$ then the result is a pure phase ω^n and we get a bilinear relation to a meromorphic theory in dimension $8n \bmod 24$.

- But the converse is also true – any CFT in one of these representations must pair up with its conjugate to a meromorphic theory.
- The reason is that the Modular Tensor Category (MTC) associated to the CFT is uniquely specified by the modular representation – but only for rank $2, 3$.
- Thus all ${}^{(2)}$CFT can be found by taking cosets of meromorphic CFT by $\mathsf{A}_{1,1}, \mathsf{G}_{2,1}, \mathsf{F}_{4,1}, \mathsf{E}_{7,1}$ – the MMS set.

- However, the classification of "all meromorphic CFTs of central charge c" exists only up to $c = 24$, and beyond that it is impractical.
- So the best we can do is classify all $^{(2)}$CFT with $c < 24$. We take all meromorphic theories up to $c = 24$ and quotient in all possible ways by the MMS set.
- The possible central charges we will get in this way are:

$$c = c_M - 1,\ c_M - \tfrac{14}{5},\ c_M - \tfrac{26}{5},\ c_M - 7$$

with $c_M = 8, 16, 24$. The maximum value is 23.

- A small trick allows us to stretch this range a little bit.
- The minimum and maximum central charges of the MMS set are $1, 7$.
- So meromorphic theories at $c_M = 24, 32$ gives maximum/minimum central charges of $23, 25$ respectively.
- Hence there is no $^{(2)}$CFT with $23 < c < 25$. So we can push our upper limit to 25.

- From the Riemann-Roch theorem:

$$\ell = \frac{c}{2} - 6h + 1$$

  unitary theories with $0 < c < 25$ can only have Wronskian index $0, 2, 4, 6, 8, 10, 12$.

- We already classified $\ell = 0, 2$ to get $4 + 15$ theories.

- From the allowed modular representations one can show there are no admissible characters in this range for $\ell = 6, 10, 12$.

- That only leaves $\ell = 4, 8$. These arise from embeddings of the MMS set in meromorphic theories of $c = 16, 24$ respectively.

- Finally one computes all possible such embeddings.
- This is a complicated exercise involving Dynkin and embedding indices, so I will skip the details.
- The result is a set of $6 + 98$ theories at $\ell = 4, 8$ respectively.
- As an example, for the Schellekens theory with chiral algebra $\mathsf{A}_{3,1}\,\mathsf{D}_{7,3}\,\mathsf{G}_{2,1}$, we can embed $\mathsf{A}_{1,1}$ in two ways:

$$\mathsf{A}_{1,1} \hookrightarrow \mathsf{A}_{3,1}, \qquad \mathsf{A}_{1,1} \hookrightarrow \mathsf{G}_{2,1}$$

- We cannot embed $\mathsf{A}_{1,1}$ into $\mathsf{D}_{7,3}$ because the level of the embedding algebra has to be $\geq$ the level of the numerator.

- In total we found 123 CFT's with two primaries and $c < 25$, and 100 of these are new.

- Some features of the final table:

  - Wronskian indices $\ell = 0, 2, 4, 8$ arise.
  - Some theories have complete KM algebras and others have incomplete ones together with minimal models, the latter being one of $c = \frac{7}{10}, \frac{4}{5}, \frac{1}{2} \oplus \frac{7}{10}$ (not the case in Schellekens theories).
  - Some theories have both non-Abelian and Abelian factors (not the case in Schellekens list).
  - There are theories with the same c but different conformal dimension h, and also multiple theories with the same (c, h). For example we find:

$$2 \text{ theories with } (c,h) = \left(\tfrac{106}{5}, \tfrac{8}{5}\right)$$

$$27 \text{ theories with } (c,h) = \left(\tfrac{106}{5}, \tfrac{3}{5}\right)$$

- Some subtleties we encountered:
  (i) Possible inequivalence of embeddings into different copies of the same factor.
    - The problem arises when there are multiple copies of one factor. For example many Schellekens theories have factors of $\mathsf{A}_{1,1}^m$ for $m \geq 2$. When taking a coset by $\mathsf{A}_{1,1}$, does it matter which of the numerator factors we delete?
    - This was resolved in [Betsumiya-Lam-Shimakura 2022] and private communication with the authors. One example is:

$$\frac{\mathsf{D}_{6,5}\mathsf{A}_{1,1}\mathsf{A}'_{1,1}}{\mathsf{A}_{1,1}} \neq \frac{\mathsf{D}_{6,5}\mathsf{A}_{1,1}\mathsf{A}'_{1,1}}{\mathsf{A}'_{1,1}}$$

    - However this happens in just two cases. In the remaining ones, the multiple copies are permuted by outer automorphisms of the algebra and in this case the two embeddings are equivalent.

(ii) Linear equivalence vs equivalence of embeddings.

- We computed linearly inequivalent embeddings using suitable software. However in some specific cases, linear equivalence does not imply equivalence.
- The complete set of conditions when this can happen were described in [Minchenko 2006]. We were able to check that for all our cases, linear equivalence corresponds to equivalence of embeddings.

No.	Theory	$(c,\mathscr{C})$	h	ℓ	Subalgebra	d
$\star$ 1	$\mathsf{A}_{1,1}$	$(1,\text{Sem})$	$1/4$	0	$\mathsf{A}_{1,1}$	2
2	$\mathsf{G}_{2,1}$	$(14/5,\text{Fib})$	$2/5$	0	$\mathsf{G}_{2,1}$	7
3	$\mathsf{F}_{4,1}\cong\mathsf{E}_{8,1}/\mathsf{G}_{2,1}$	$(26/5,\overline{\text{Fib}})$	$3/5$	0	$\mathsf{F}_{4,1}$	26
$\star$ 4	$\mathsf{E}_{7,1}\cong\mathsf{E}_{8,1}/\mathsf{A}_{1,1}$	$(7,\overline{\text{Sem}})$	$3/4$	0	$\mathsf{E}_{7,1}$	56
$\star$ 5	$\mathsf{E}_{8,1}\mathsf{A}_{1,1}\cong\mathsf{E}_{8,1}^2/\mathsf{E}_{7,1}$	$(9,\text{Sem})$	$1/4$	4	$\mathsf{A}_{1,1}\mathsf{E}_{8,1}$	2
6	$\mathsf{E}_{8,1}\mathsf{G}_{2,1}\cong\mathsf{E}_{8,1}^2/\mathsf{F}_{4,1}$	$(54/5,\text{Fib})$	$2/5$	4	$\mathsf{G}_{2,1}\mathsf{E}_{8,1}$	7
7	$\mathsf{E}_{8,1}\mathsf{F}_{4,1}\cong\mathsf{E}_{8,1}^2/\mathsf{G}_{2,1}$	$(66/5,\overline{\text{Fib}})$	$3/5$	4	$\mathsf{F}_{4,1}\mathsf{E}_{8,1}$	26
8	$\mathsf{D}_{16,1}^+/\mathsf{G}_{2,1}$	$(66/5,\overline{\text{Fib}})$	$3/5$	4	$\mathsf{B}_{12,1}\mathsf{L}_{7/10}$	26
$\star$ 9	$\mathsf{E}_{8,1}\mathsf{E}_{7,1}\cong\mathsf{E}_{8,1}^2/\mathsf{A}_{1,1}$	$(15,\overline{\text{Sem}})$	$3/4$	4	$\mathsf{E}_{7,1}\mathsf{E}_{8,1}$	56
$\star$ 10	$\mathsf{D}_{16,1}^+/\mathsf{A}_{1,1}$	$(15,\overline{\text{Sem}})$	$3/4$	4	$\mathsf{D}_{14,1}\mathsf{A}_{1,1}$	56
$\star$ 11	$\mathbf{S}(\mathsf{D}_{10,1}\mathsf{E}_{7,1}^2)/(\mathsf{E}_{7,1}\hookrightarrow\mathsf{E}_{7,1})$	$(17,\text{Sem})$	$5/4$	2	$\mathsf{D}_{10,1}\mathsf{E}_{7,1}$	1632
$\star$ 12	$\mathbf{S}(\mathsf{A}_{17,1}\mathsf{E}_{7,1})/(\mathsf{E}_{7,1}\hookrightarrow\mathsf{E}_{7,1})$	$(17,\text{Sem})$	$5/4$	2	$\mathsf{A}_{17,1}$	1632
$\star$ 13	$\mathsf{E}_{8,1}^2\mathsf{A}_{1,1}\cong\mathsf{E}_{8,1}^3/\mathsf{E}_{7,1}$	$(17,\text{Sem})$	$1/4$	8	$\mathsf{E}_{8,1}^2\mathsf{A}_{1,1}$	2
$\star$ 14	$\mathbf{S}(\mathsf{D}_{16,1}\mathsf{E}_{8,1})/(\mathsf{E}_{7,1}\hookrightarrow\mathsf{E}_{8,1})$	$(17,\text{Sem})$	$1/4$	8	$\mathsf{D}_{16,1}\mathsf{A}_{1,1}$	2
15	$\mathbf{S}(\mathsf{C}_{8,1}\mathsf{F}_{4,1}^2)/(\mathsf{F}_{4,1}\hookrightarrow\mathsf{F}_{4,1})$	$(94/5,\text{Fib})$	$7/5$	2	$\mathsf{C}_{8,1}\mathsf{F}_{4,1}$	4794
16	$\mathbf{S}(\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{F}_{4,1})/(\mathsf{F}_{4,1}\hookrightarrow\mathsf{F}_{4,1})$	$(94/5,\text{Fib})$	$7/5$	2	$\mathsf{E}_{7,2}\mathsf{B}_{5,1}$	4794
17	$\mathbf{S}(\mathsf{E}_{6,1}^4)/\mathsf{F}_{4,1}$	$(94/5,\text{Fib})$	$2/5$	8	$\mathsf{E}_{6,1}^3\mathsf{L}_{4/5}$	1
18	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{F}_{4,1}\hookrightarrow\mathsf{E}_{6,1})$	$(94/5,\text{Fib})$	$2/5$	8	$\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{L}_{4/5}$	1
19	$\mathbf{S}(\mathsf{D}_{10,1}\mathsf{E}_{7,1}^2)/(\mathsf{F}_{4,1}\hookrightarrow\mathsf{E}_{7,1})$	$(94/5,\text{Fib})$	$2/5$	8	$\mathsf{D}_{10,1}\mathsf{E}_{7,1}\mathsf{A}_{1,3}$	3
20	$\mathbf{S}(\mathsf{A}_{17,1}\mathsf{E}_{7,1})/(\mathsf{F}_{4,1}\hookrightarrow\mathsf{E}_{7,1})$	$(94/5,\text{Fib})$	$2/5$	8	$\mathsf{A}_{17,1}\mathsf{A}_{1,3}$	3
21	$\mathsf{E}_{8,1}^2\mathsf{G}_{2,1}\cong\mathsf{E}_{8,1}^3/\mathsf{F}_{4,1}$	$(94/5,\text{Fib})$	$2/5$	8	$\mathsf{E}_{8,1}^2\mathsf{G}_{2,1}$	7
22	$\mathbf{S}(\mathsf{D}_{16,1}\mathsf{E}_{8,1})/(\mathsf{F}_{4,1}\hookrightarrow\mathsf{E}_{8,1})$	$(94/5,\text{Fib})$	$2/5$	8	$\mathsf{D}_{16,1}\mathsf{G}_{2,1}$	7
23	$\mathbf{S}(\mathsf{E}_{6,3}\mathsf{G}_{2,1}^3)/\mathsf{G}_{2,1}$	$(106/5,\overline{\text{Fib}})$	$8/5$	2	$\mathsf{E}_{6,3}\mathsf{G}_{2,1}^2$	15847
24	$\mathbf{S}(\mathsf{D}_{7,3}\mathsf{A}_{3,1}\mathsf{G}_{2,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{G}_{2,1})$	$(106/5,\overline{\text{Fib}})$	$8/5$	2	$\mathsf{D}_{7,3}\mathsf{A}_{3,1}$	15847
25	$\mathbf{S}(\mathsf{D}_{6,2}\mathsf{C}_{4,1}\mathsf{B}_{3,1}^2)/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{B}_{3,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{D}_{6,2}\mathsf{C}_{4,1}\mathsf{B}_{3,1}\mathsf{L}_{7/10}$	1
26	$\mathbf{S}(\mathsf{A}_{9,2}\mathsf{A}_{4,1}\mathsf{B}_{3,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{B}_{3,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{A}_{9,2}\mathsf{A}_{4,1}\mathsf{L}_{7/10}$	1
27	$\mathbf{S}(\mathsf{D}_{4,1}^6)/\mathsf{G}_{2,1}$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{D}_{4,1}^5\mathsf{L}_{1/2}\mathsf{L}_{7/10}$	2
28	$\mathbf{S}(\mathsf{A}_{5,1}^4\mathsf{D}_{4,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{D}_{4,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{A}_{5,1}^4\mathsf{L}_{1/2}\mathsf{L}_{7/10}$	2
29	$\mathbf{S}(\mathsf{D}_{8,2}\mathsf{B}_{4,1}^2)/\mathsf{G}_{2,1}$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{D}_{8,2}\mathsf{B}_{4,1}\mathsf{U}_1\mathsf{L}_{7/10}$	3
30	$\mathbf{S}(\mathsf{C}_{6,1}^2\mathsf{B}_{4,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{B}_{4,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{C}_{6,1}^2\mathsf{U}_1\mathsf{L}_{7/10}$	3
31	$\mathbf{S}(\mathsf{A}_{7,1}^2\mathsf{D}_{5,1}^2)/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{D}_{5,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{A}_{7,1}^2\mathsf{D}_{5,1}\mathsf{A}_{1,2}\mathsf{L}_{7/10}$	4
32	$\mathbf{S}(\mathsf{C}_{8,1}\mathsf{F}_{4,1}^2)/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{F}_{4,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{C}_{8,1}\mathsf{F}_{4,1}\mathsf{A}_{1,8}$	5
33	$\mathbf{S}(\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{F}_{4,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{B}_{5,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{E}_{7,2}\mathsf{A}_{1,1}^2\mathsf{F}_{4,1}\mathsf{L}_{7/10}$	5
34	$\mathbf{S}(\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{F}_{4,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{F}_{4,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{A}_{1,8}$	5
35	$\mathbf{S}(\mathsf{D}_{6,1}^4)/\mathsf{G}_{2,1}$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{D}_{6,1}^3\mathsf{B}_{2,1}\mathsf{L}_{7/10}$	6
36	$\mathbf{S}(\mathsf{A}_{9,1}^2\mathsf{D}_{6,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{D}_{6,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{A}_{9,1}^2\mathsf{B}_{2,1}\mathsf{L}_{7/10}$	6
37	$\mathbf{S}(\mathsf{C}_{10,1}\mathsf{B}_{6,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{B}_{6,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{C}_{10,1}\mathsf{A}_{3,1}\mathsf{L}_{7/10}$	7
38	$\mathbf{S}(\mathsf{E}_{6,1}^4)/\mathsf{G}_{2,1}$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{E}_{6,1}^3\mathsf{A}_{2,2}$	8
39	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{D}_{7,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{A}_{11,1}\mathsf{B}_{3,1}\mathsf{E}_{6,1}\mathsf{L}_{7/10}$	8
40	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{E}_{6,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{A}_{2,2}$	8
41	$\mathbf{S}(\mathsf{D}_{8,1}^3)/\mathsf{G}_{2,1}$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{D}_{8,1}^2\mathsf{B}_{4,1}\mathsf{L}_{7/10}$	10
42	$\mathbf{S}(\mathsf{E}_{8,2}\mathsf{B}_{8,1})/\mathsf{G}_{2,1}$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{E}_{8,2}\mathsf{D}_{5,1}\mathsf{L}_{7/10}$	11
43	$\mathbf{S}(\mathsf{A}_{15,1}\mathsf{D}_{9,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{D}_{9,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{A}_{15,1}\mathsf{B}_{5,1}\mathsf{L}_{7/10}$	12
44	$\mathbf{S}(\mathsf{D}_{10,1}\mathsf{E}_{7,1}^2)/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{D}_{10,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{B}_{6,1}\mathsf{E}_{7,1}^2\mathsf{L}_{7/10}$	14
45	$\mathbf{S}(\mathsf{D}_{10,1}\mathsf{E}_{7,1}^2)/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{E}_{7,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{D}_{10,1}\mathsf{E}_{7,1}\mathsf{C}_{3,1}$	14
46	$\mathbf{S}(\mathsf{A}_{17,1}\mathsf{E}_{7,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{E}_{7,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{A}_{17,1}\mathsf{C}_{3,1}$	14
47	$\mathbf{S}(\mathsf{D}_{12,1}^2)/\mathsf{G}_{2,1}$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{D}_{12,1}\mathsf{B}_{8,1}\mathsf{L}_{7/10}$	18
48	$\mathsf{E}_{8,1}^2\mathsf{F}_{4,1}\cong\mathsf{E}_{8,1}^3/\mathsf{G}_{2,1}$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{E}_{8,1}^2\mathsf{F}_{4,1}$	26
49	$\mathbf{S}(\mathsf{D}_{16,1}\mathsf{E}_{8,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{D}_{16,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{B}_{12,1}\mathsf{E}_{8,1}\mathsf{L}_{7/10}$	26
50	$\mathsf{D}_{16,1}^+\mathsf{F}_{4,1}\cong\mathbf{S}(\mathsf{D}_{16,1}\mathsf{E}_{8,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{E}_{8,1})$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{D}_{16,1}\mathsf{F}_{4,1}$	26
51	$\mathbf{S}(\mathsf{D}_{24,1})/\mathsf{G}_{2,1}$	$(106/5,\overline{\text{Fib}})$	$3/5$	8	$\mathsf{B}_{20,1}\mathsf{L}_{7/10}$	42
$\star$ 52	$\mathbf{S}(\mathsf{A}_{1,1}^{24})/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$7/4$	2	$\mathsf{A}_{1,1}^{23}$	32384
53	$\mathbf{S}(\mathsf{A}_{3,2}^4\mathsf{A}_{1,1}^4)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$7/4$	2	$\mathsf{A}_{3,2}^4\mathsf{A}_{1,1}^3$	32384
54	$\mathbf{S}(\mathsf{A}_{5,3}\mathsf{D}_{4,3}\mathsf{A}_{1,1}^3)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$7/4$	2	$\mathsf{A}_{5,3}\mathsf{D}_{4,3}\mathsf{A}_{1,1}^2$	32384
55	$\mathbf{S}(\mathsf{A}_{7,4}\mathsf{A}'_{1,1}\mathsf{A}_{1,1}^2)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$7/4$	2	$\mathsf{A}_{7,4}\mathsf{A}_{1,1}^2$	32384
56	$\mathbf{S}(\mathsf{A}_{7,4}\mathsf{A}'_{1,1}\mathsf{A}_{1,1}^2)/\mathsf{A}'_{1,1}$	$(23,\overline{\text{Sem}})$	$7/4$	2	$\mathsf{A}_{7,4}\mathsf{A}_{1,1}^2$	32384
57	$\mathbf{S}(\mathsf{D}_{5,4}\mathsf{C}_{3,2}\mathsf{A}_{1,1}^2)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$7/4$	2	$\mathsf{D}_{5,4}\mathsf{C}_{3,2}\mathsf{A}_{1,1}$	32384
58	$\mathbf{S}(\mathsf{D}_{6,5}\mathsf{A}_{1,1}\mathsf{A}'_{1,1})/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$7/4$	2	$\mathsf{D}_{6,5}\mathsf{A}_{1,1}$	32384
59	$\mathbf{S}(\mathsf{D}_{6,5}\mathsf{A}_{1,1}\mathsf{A}'_{1,1})/\mathsf{A}'_{1,1}$	$(23,\overline{\text{Sem}})$	$7/4$	2	$\mathsf{D}_{6,5}\mathsf{A}_{1,1}$	32384
60	$\mathbf{S}(\mathsf{C}_{5,3}\mathsf{G}_{2,2}\mathsf{A}_{1,1})/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$7/4$	2	$\mathsf{C}_{5,3}\mathsf{G}_{2,2}$	32384
$\star$ 61	$\mathbf{S}(\mathsf{A}_{2,1}^{12})/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{2,1}^{11}\mathsf{U}_1$	2
62	$\mathbf{S}(\mathsf{D}_{4,2}^2\mathsf{B}_{2,1}^4)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{4,2}^2\mathsf{B}_{2,1}^3\mathsf{A}_{1,1}\mathsf{L}_{1/2}$	2
63	$\mathbf{S}(\mathsf{A}_{5,2}^2\mathsf{B}_{2,1}\mathsf{A}_{2,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{2,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{5,2}^2\mathsf{C}_{2,1}\mathsf{A}_{2,1}\mathsf{U}_1$	2
64	$\mathbf{S}(\mathsf{A}_{5,2}^2\mathsf{B}_{2,1}\mathsf{A}_{2,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{2,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{5,2}^2\mathsf{A}_{1,1}\mathsf{A}_{2,1}^2\mathsf{L}_{1/2}$	2
65	$\mathbf{S}(\mathsf{A}_{8,3}\mathsf{A}_{2,1}^2)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{8,3}\mathsf{A}_{2,1}\mathsf{U}_1$	2
66	$\mathbf{S}(\mathsf{E}_{6,4}\mathsf{B}_{2,1}\mathsf{A}_{2,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{2,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{E}_{6,4}\mathsf{A}_{1,1}\mathsf{A}_{2,1}\mathsf{L}_{1/2}$	2
67	$\mathbf{S}(\mathsf{E}_{6,4}\mathsf{B}_{2,1}\mathsf{A}_{2,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{2,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{E}_{6,4}\mathsf{C}_{2,1}\mathsf{U}_1$	2
$\star$ 68	$\mathbf{S}(\mathsf{A}_{3,1}^8)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{3,1}^7\mathsf{A}_{1,1}\mathsf{U}_1$	4
69	$\mathbf{S}(\mathsf{D}_{5,2}^2\mathsf{A}_{3,1}^2)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{5,2}^2\mathsf{A}_{3,1}\mathsf{A}_{1,1}\mathsf{U}_1$	4
70	$\mathbf{S}(\mathsf{E}_{6,3}\mathsf{G}_{2,1}^3)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{E}_{6,3}\mathsf{G}_{2,1}^2\mathsf{A}_{1,3}$	4
71	$\mathbf{S}(\mathsf{A}_{7,2}\mathsf{C}_{3,1}^2\mathsf{A}_{3,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{3,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{7,2}\mathsf{C}_{3,1}^2\mathsf{A}_{1,1}\mathsf{U}_1$	4
72	$\mathbf{S}(\mathsf{A}_{7,2}\mathsf{C}_{3,1}^2\mathsf{A}_{3,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{3,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{7,2}\mathsf{C}_{3,1}\mathsf{B}_{2,1}\mathsf{A}_{3,1}\mathsf{L}_{7/10}$	4
73	$\mathbf{S}(\mathsf{D}_{7,3}\mathsf{A}_{3,1}\mathsf{G}_{2,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{G}_{2,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{7,3}\mathsf{A}_{3,1}\mathsf{A}_{1,3}$	4
74	$\mathbf{S}(\mathsf{D}_{7,3}\mathsf{A}_{3,1}\mathsf{G}_{2,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{3,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{7,3}\mathsf{G}_{2,1}\mathsf{A}_{1,1}\mathsf{U}_1$	4
75	$\mathbf{S}(\mathsf{C}_{7,2}\mathsf{A}_{3,1})/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{C}_{7,2}\mathsf{A}_{1,1}\mathsf{U}_1$	4
$\star$ 76	$\mathbf{S}(\mathsf{A}_{4,1}^6)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{4,1}^5\mathsf{A}_{2,1}\mathsf{U}_1$	6
77	$\mathbf{S}(\mathsf{C}_{4,1}^4)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{C}_{4,1}^3\mathsf{C}_{3,1}\mathsf{L}_{4/5}$	6
78	$\mathbf{S}(\mathsf{D}_{6,2}\mathsf{C}_{4,1}\mathsf{B}_{3,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{4,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{6,2}\mathsf{C}_{3,1}\mathsf{B}_{3,1}^2\mathsf{L}_{4/5}$	6
79	$\mathbf{S}(\mathsf{D}_{6,2}\mathsf{C}_{4,1}\mathsf{B}_{3,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{3,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{6,2}\mathsf{C}_{4,1}\mathsf{B}_{3,1}\mathsf{A}_{1,2}\mathsf{A}_{1,1}$	6
80	$\mathbf{S}(\mathsf{A}_{9,2}\mathsf{A}_{4,1}\mathsf{B}_{3,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{4,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{9,2}\mathsf{A}_{2,1}\mathsf{B}_{3,1}\mathsf{U}_1$	6
81	$\mathbf{S}(\mathsf{A}_{9,2}\mathsf{A}_{4,1}\mathsf{B}_{3,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{3,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{9,2}\mathsf{A}_{4,1}\mathsf{A}_{1,2}\mathsf{A}_{1,1}$	6
$\star$ 82	$\mathbf{S}(\mathsf{D}_{4,1}^6)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{4,1}^5\mathsf{A}_{1,1}\mathsf{A}_{1,1}\mathsf{A}_{1,1}$	8

No.	Theory	$(c,\mathscr{C})$	h	ℓ	Subalgebra	d
⋆ 83	$\mathbf{S}(\mathsf{A}_{5,1}^4\mathsf{D}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{5,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{5,1}^3\mathsf{A}_{3,1}\mathsf{D}_{4,1}\mathsf{U}_1$	8
⋆ 84	$\mathbf{S}(\mathsf{A}_{5,1}^4\mathsf{D}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{4,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{5,1}^4\mathsf{A}_{1,1}^3$	8
85	$\mathbf{S}(\mathsf{E}_{6,2}\mathsf{C}_{5,1}\mathsf{A}_{5,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{5,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{E}_{6,2}\mathsf{C}_{4,1}\mathsf{A}_{5,1}\mathsf{L}_{6/7}$	8
86	$\mathbf{S}(\mathsf{E}_{6,2}\mathsf{C}_{5,1}\mathsf{A}_{5,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{5,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{E}_{6,2}\mathsf{C}_{5,1}\mathsf{A}_{3,1}\mathsf{U}_1$	8
87	$\mathbf{S}(\mathsf{E}_{7,3}\mathsf{A}_{5,1})/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{E}_{7,3}\mathsf{A}_{3,1}\mathsf{U}_1$	8
⋆ 88	$\mathbf{S}(\mathsf{A}_{6,1}^4)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{6,1}^3\mathsf{A}_{4,1}\mathsf{U}_1$	10
89	$\mathbf{S}(\mathsf{D}_{8,2}\mathsf{B}_{4,1}^2)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{8,2}\mathsf{B}_{4,1}\mathsf{B}_{2,1}\mathsf{A}_{1,1}$	10
90	$\mathbf{S}(\mathsf{C}_{6,1}^2\mathsf{B}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{6,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{C}_{6,1}\mathsf{C}_{5,1}\mathsf{B}_{4,1}\mathsf{L}_{25/28}$	10
91	$\mathbf{S}(\mathsf{C}_{6,1}^2\mathsf{B}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{4,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{C}_{6,1}^2\mathsf{B}_{2,1}\mathsf{A}_{1,1}$	10
⋆ 92	$\mathbf{S}(\mathsf{A}_{7,1}^2\mathsf{D}_{5,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{7,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{7,1}\mathsf{A}_{5,1}\mathsf{D}_{5,1}^2\mathsf{U}_1$	12
⋆ 93	$\mathbf{S}(\mathsf{A}_{7,1}^2\mathsf{D}_{5,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{5,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{7,1}^2\mathsf{A}_{3,1}\mathsf{A}_{1,1}\mathsf{D}_{5,1}$	12
94	$\mathbf{S}(\mathsf{D}_{9,2}\mathsf{A}_{7,1})/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{9,2}\mathsf{A}_{5,1}\mathsf{U}_1$	12
⋆ 95	$\mathbf{S}(\mathsf{A}_{8,1}^3)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{8,1}^2\mathsf{A}_{6,1}\mathsf{U}_1$	14
96	$\mathbf{S}(\mathsf{C}_{8,1}\mathsf{F}_{4,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{8,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{C}_{7,1}\mathsf{F}_{4,1}^2 L_{14/15}$	14
97	$\mathbf{S}(\mathsf{C}_{8,1}\mathsf{F}_{4,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{F}_{4,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{C}_{8,1}\mathsf{F}_{4,1}\mathsf{C}_{3,1}$	14
98	$\mathbf{S}(\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{F}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{5,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{E}_{7,2}\mathsf{B}_{3,1}\mathsf{A}_{1,1}\mathsf{F}_{4,1}$	14
99	$\mathbf{S}(\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{F}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{F}_{4,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{C}_{3,1}$	14
⋆ 100	$\mathbf{S}(\mathsf{D}_{6,1}^4)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{6,1}^3\mathsf{D}_{4,1}\mathsf{A}_{1,1}$	16
⋆ 101	$\mathbf{S}(\mathsf{A}_{9,1}^2\mathsf{D}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{9,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{9,1}\mathsf{A}_{7,1}\mathsf{D}_{6,1}\mathsf{U}_1$	16
⋆ 102	$\mathbf{S}(\mathsf{A}_{9,1}^2\mathsf{D}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{6,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{9,1}^2\mathsf{D}_{4,1}\mathsf{A}_{1,1}$	16
103	$\mathbf{S}(\mathsf{C}_{10,1}\mathsf{B}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{10,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{C}_{9,1}\mathsf{B}_{6,1} L_{21/22}$	18
104	$\mathbf{S}(\mathsf{C}_{10,1}\mathsf{B}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{6,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{C}_{10,1}\mathsf{B}_{4,1}\mathsf{A}_{1,1}$	18
⋆ 105	$\mathbf{S}(\mathsf{E}_{6,1}^4)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{E}_{6,1}^3\mathsf{A}_{5,1}$	20
⋆ 106	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{11,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{9,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1}\mathsf{U}_1$	20
⋆ 107	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{7,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{11,1}\mathsf{D}_{5,1}\mathsf{A}_{1,1}\mathsf{E}_{6,1}$	20
⋆ 108	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{E}_{6,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{A}_{5,1}$	20
⋆ 109	$\mathbf{S}(\mathsf{A}_{12,1}^2)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{12,1}\mathsf{A}_{10,1}\mathsf{U}_1$	22
⋆ 110	$\mathbf{S}(\mathsf{D}_{8,1}^3)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{8,1}^2\mathsf{D}_{6,1}\mathsf{A}_{1,1}$	24
111	$\mathbf{S}(\mathsf{E}_{8,2}\mathsf{B}_{8,1})/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{E}_{8,2}\mathsf{B}_{6,1}\mathsf{A}_{1,1}$	26
⋆ 112	$\mathbf{S}(\mathsf{A}_{15,1}\mathsf{D}_{9,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{15,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{13,1}\mathsf{D}_{9,1}\mathsf{U}_1$	28
⋆ 113	$\mathbf{S}(\mathsf{A}_{15,1}\mathsf{D}_{9,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{9,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{15,1}\mathsf{D}_{7,1}\mathsf{A}_{1,1}$	28
⋆ 114	$\mathbf{S}(\mathsf{D}_{10,1}\mathsf{E}_{7,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{10,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{8,1}\mathsf{A}_{1,1}\mathsf{E}_{7,1}^2$	32
⋆ 115	$\mathbf{S}(\mathsf{D}_{10,1}\mathsf{E}_{7,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{E}_{7,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{10,1}\mathsf{E}_{7,1}\mathsf{D}_{6,1}$	32
⋆ 116	$\mathbf{S}(\mathsf{A}_{17,1}\mathsf{E}_{7,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{17,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{15,1}\mathsf{E}_{7,1}\mathsf{U}_1$	32
⋆ 117	$\mathbf{S}(\mathsf{A}_{17,1}\mathsf{E}_{7,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{E}_{7,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{17,1}\mathsf{D}_{6,1}$	32
⋆ 118	$\mathbf{S}(\mathsf{D}_{12,1}^2)/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{12,1}\mathsf{D}_{10,1}\mathsf{A}_{1,1}$	40
⋆ 119	$\mathbf{S}(\mathsf{A}_{24,1})/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{A}_{22,1}\mathsf{U}_1$	46
⋆ 120	$\mathsf{E}_{8,1}^2\mathsf{E}_{7,1}\cong\mathsf{E}_{8,1}^3/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{E}_{8,1}^2\mathsf{E}_{7,1}$	56
⋆ 121	$\mathbf{S}(\mathsf{D}_{16,1}\mathsf{E}_{8,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{16,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{14,1}\mathsf{A}_{1,1}\mathsf{E}_{8,1}$	56
⋆ 122	$\mathsf{D}_{16,1}^+\mathsf{E}_{7,1}\cong\mathbf{S}(\mathsf{D}_{16,1}\mathsf{E}_{8,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{E}_{8,1})$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{16,1}\mathsf{E}_{7,1}$	56
⋆ 123	$\mathbf{S}(\mathsf{D}_{24,1})/\mathsf{A}_{1,1}$	$(23,\overline{\text{Sem}})$	$3/4$	8	$\mathsf{D}_{22,1}\mathsf{A}_{1,1}$	88

A close-up of a few entries:

$\mathbf{S}(\mathsf{E}_{6,3}\mathsf{G}_{2,1}^3)/\mathsf{G}_{2,1}$	$^{106}/_5$	$^8/_5$	2	$\mathsf{E}_{6,3}\mathsf{G}_{2,1}^2$	15847
$\mathbf{S}(\mathsf{D}_{7,3}\mathsf{A}_{3,1}\mathsf{G}_{2,1})/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{G}_{2,1})$	$^{106}/_5$	$^8/_5$	2	$\mathsf{D}_{7,3}\mathsf{A}_{3,1}$	15847
$\mathbf{S}(\mathsf{D}_{6,2}\mathsf{C}_{4,1}\mathsf{B}_{3,1}^2)/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{B}_{3,1})$	$^{106}/_5$	$^3/_5$	8	$\mathsf{D}_{6,2}\mathsf{C}_{4,1}\mathsf{B}_{3,1}\mathsf{L}_{7/10}$	1
$\mathbf{S}(\mathsf{A}_{9,2}\mathsf{A}_{4,1}\mathsf{B}_{3,1})/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{B}_{3,1})$	$^{106}/_5$	$^3/_5$	8	$\mathsf{A}_{9,2}\mathsf{A}_{4,1}\mathsf{L}_{7/10}$	1
$\mathbf{S}(\mathsf{D}_{4,1}^6)/\mathsf{G}_{2,1}$	$^{106}/_5$	$^3/_5$	8	$\mathsf{D}_{4,1}^5\mathsf{L}_{1/2}\mathsf{L}_{7/10}$	2
$\mathbf{S}(\mathsf{A}_{5,1}^4\mathsf{D}_{4,1})/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{D}_{4,1})$	$^{106}/_5$	$^3/_5$	8	$\mathsf{A}_{5,1}^4\mathsf{L}_{1/2}\mathsf{L}_{7/10}$	2
$\mathbf{S}(\mathsf{D}_{8,2}\mathsf{B}_{4,1}^2)/\mathsf{G}_{2,1}$	$^{106}/_5$	$^3/_5$	8	$\mathsf{D}_{8,2}\mathsf{B}_{4,1}\mathsf{U}_1\mathsf{L}_{7/10}$	3
$\mathbf{S}(\mathsf{C}_{6,1}^2\mathsf{B}_{4,1})/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{B}_{4,1})$	$^{106}/_5$	$^3/_5$	8	$\mathsf{C}_{6,1}^2\mathsf{U}_1\mathsf{L}_{7/10}$	3
$\mathbf{S}(\mathsf{A}_{7,1}^2\mathsf{D}_{5,1}^2)/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{D}_{5,1})$	$^{106}/_5$	$^3/_5$	8	$\mathsf{A}_{7,1}^2\mathsf{D}_{5,1}\mathsf{A}_{1,2}\mathsf{L}_{7/10}$	4
$\mathbf{S}(\mathsf{C}_{8,1}\mathsf{F}_{4,1}^2)/(\mathsf{G}_{2,1} \hookrightarrow \mathsf{F}_{4,1})$	$^{106}/_5$	$^3/_5$	8	$\mathsf{C}_{8,1}\mathsf{F}_{4,1}\mathsf{A}_{1,8}$	5

Three character case, in brief

- The three-character case has been studied for vanishing Wronskian index ($\ell = 0$) in several papers [Mathur-Mukhi-Sen 1989, Hampapura-Mukhi 2015, Gaberdiel-Hampapura-Mukhi 2016, Franc-Mason 2020, Mukhi-Poddar-Singh 2020].
- Recently three groups independently completed the classification of admissible characters for this case [Kaidi-Lin-Parra-Martinez 2021, Das-Gowdigere-Santara 2021, Bae-Duan-Lee-Lee-Sarkis 2021].
- Then in [Das-Gowdigere-Mukhi 2022b] we were finally able to identify all the actual CFT within this set and rule out the rest.
- This completes the classification of three-character CFT with vanishing Wronskian index (no restriction on central charge), 33 years after it was first attempted!

#	c	(h_1,h_2)	m_1	$\mathcal{W}$	Chiral Algebra	#(primaries)
1.	$\frac{2r+1}{2}$	$(\frac{1}{2},\frac{2r+1}{16})$	$2r^2+r$	**I**	$B_{r,1}$	3
2.	r	$(\frac{1}{2},\frac{r}{8})$	$2r^2-r$	**I**	$D_{r,1}$ $(r\neq 8,16)$	4
3.	$\frac{12}{5}$	$(\frac{1}{5},\frac{3}{5})$	6	$\mathbf{III_2}$	$\mathcal{E}_3[A_{1,8}]$	4
4.	4	$(\frac{2}{5},\frac{3}{5})$	24	**I**	$A_{4,1}$	5
5.	$\frac{28}{5}$	$(\frac{2}{5},\frac{4}{5})$	28	**I**	$G_{2,1}^{\otimes 2}$	4
6.	$\frac{52}{5}$	$(\frac{3}{5},\frac{6}{5})$	104	**I**	$F_{2,1}^{\otimes 2}$	4
7.	12	$(\frac{2}{3},\frac{4}{3})$	156	**I**	$E_{6,1}^{\otimes 2}$	9
8.	$\frac{68}{5}$	$(\frac{4}{5},\frac{7}{5})$	136	$\mathbf{III_{22}}$	$\mathcal{E}_3[C_{8,1}]$	4
9.	14	$(\frac{3}{4},\frac{3}{2})$	266	**I**	$E_{7,1}^{\otimes 2}$	4
10.	15	$(\frac{7}{8},\frac{3}{2})$	255	GHM_{255}	$\mathcal{E}_3[A_{15,1}]$	4
11.	$\frac{31}{2}$	$(\frac{15}{16},\frac{3}{2})$	248	**I**	$E_{8,2}$	3
12.	17	$(\frac{9}{8},\frac{3}{2})$	221	GHM_{221}	$\mathcal{E}_3[A_{11,1}E_{6,1}]$	4
13.	$\frac{35}{2}$	$(\frac{19}{16},\frac{3}{2})$	210	GHM_{210}	$\mathcal{E}_3[C_{10,1}]$	3
14.	18	$(\frac{5}{4},\frac{3}{2})$	198	GHM_{198}	$\mathcal{E}_3[D_{6,1}^{\otimes 3}]$	4
15.					$\mathcal{E}_3[A_{9,1}^{\otimes 2}]$	4
16.	$\frac{92}{5}$	$(\frac{6}{5},\frac{8}{5})$	92	$\mathbf{III_{37}}$	$\mathcal{E}_3[E_{6,3}G_{2,1}]$	4
17.	$\frac{37}{2}$	$(\frac{21}{16},\frac{3}{2})$	185	GHM_{185}	$\mathcal{E}_3[E_{7,2}F_{4,1}]$	3
18.	19	$(\frac{11}{8},\frac{3}{2})$	171	GHM_{171}	$\mathcal{E}_3[A_{7,1}^{\otimes 2}D_{5,1}]$	4
19.	$\frac{39}{2}$	$(\frac{23}{16},\frac{3}{2})$	156	GHM_{156}	$\mathcal{E}_3[B_{4,1}D_{8,2}]$	3
20.					$\mathcal{E}_3[C_{6,1}^{\otimes 2}]$	3
21.	20	$(\frac{4}{3},\frac{5}{3})$	80	$\mathbf{V_{39}}$	$\mathcal{E}_3[A_{2,1}^{\otimes 10}]$	9
22.					$\mathcal{E}_3[A_{5,2}^{\otimes 2}C_{2,1}]$	9
23.					$\mathcal{E}_3[A_{8,3}]$	9

24.	20	$(\frac{7}{5}, \frac{8}{5})$	120	GHM_{120}	$\mathcal{E}_3[A_{4,1}^{\otimes 5}]$	5
25.					$\mathcal{E}_3[A_{9,2}B_{3,1}]$	5
26.	$\frac{41}{2}$	$(\frac{3}{2}, \frac{25}{16})$	123	GHM_{123}	$\mathcal{E}_3[D_{6,2}C_{4,1}B_{3,1}]$	3
27.					$\mathcal{E}_3[A_{9,2}A_{4,1}]$	3
28.	21	$(\frac{3}{2}, \frac{13}{8})$	105	GHM_{105}	$\mathcal{E}_3[A_{3,1}^{\otimes 7}]$	4
29.					$\mathcal{E}_3[A_{3,1}D_{5,2}^{\otimes 2}]$	4
30.					$\mathcal{E}_3[A_{7,2}C_{3,1}^{\otimes 2}]$	4
31.					$\mathcal{E}_3[D_{7,3}G_{2,1}]$	4
32.					$\mathcal{E}_3[C_{7,2}]$	4
33.	$\frac{43}{2}$	$(\frac{3}{2}, \frac{27}{16})$	86	GHM_{86}	$\mathcal{E}_3[C_{2,1}^{\otimes 3}D_{4,2}^{\otimes 2}]$	3
34.					$\mathcal{E}_3[A_{5,2}^{\otimes 2}A_{2,1}^{\otimes 2}]$	3
35.					$\mathcal{E}_3[A_{2,1}E_{6,4}]$	3
36.	22	$(\frac{3}{2}, \frac{7}{4})$	66	$\mathbf{III_{45}}$	$\mathcal{E}_3[A_{1,1}^{\otimes 22}]$,	4
37.					$\mathcal{E}_3[A_{3,2}^{\otimes 4}A_{1,1}^{\otimes 2}]$	4
38.					$\mathcal{E}_3[A_{5,3}D_{4,3}A_{1,1}]$	4
39.					$\mathcal{E}_3[A_{7,4}A_{1,1}]$	4
40.					$\mathcal{E}_3[D_{5,4}C_{3,2}]$	4
41.					$\mathcal{E}_3[D_{6,5}]$	4
42.	$\frac{45}{2}$	$(\frac{3}{2}, \frac{29}{16})$	45	GHM_{45}	$\mathcal{E}_3[A_{1,2}^{\otimes 15}]$	3
43.					$\mathcal{E}_3[A_{3,4}^{\otimes 3}]$	3
44.					$\mathcal{E}_3[A_{5,6}C_{2,3}]$	3
45.					$\mathcal{E}_3[D_{5,8}]$	3
46.	23	$(\frac{3}{2}, \frac{15}{8})$	23	$\mathbf{III_{50}}$	$\mathcal{E}_3[D_{1,1}^{\otimes 23}]$	4
47.	$\frac{47}{2}$	$(\frac{3}{2}, \frac{31}{16})$	0	**IV**	Baby Monster	3

- We also found a nice trick that produces new non-lattice meromorphic theories at $c \geq 32$ [Das-Gowdigere-Mukhi 2022a]. This uses the uniqueness of rank-3 MTCs together with transitivity.

- The idea is as follows. Take a Schellekens theory at $c = 24$ and coset by a known theory $\mathcal{V}$. Call the result $\mathcal{V}' = \mathcal{S}/\mathcal{V}$.

- Now take a known CFT $\mathcal{V}''$ whose modular representation is conjugate to $\mathcal{V}'$ such that:

$$c_{\mathcal{V}'} + c_{\mathcal{V}''} = 32$$

- From the previous argument, it follows that there must be a meromorphic theory at $c = 32$.

- In this way we wrote down entire families of new (non-lattice) meromorphic CFT at $c = 8N$ for arbitrarily large N.

- With two primaries, can we go beyond the $c < 25$ bound?
- Most of the work in this field has been done for $\ell < 6$ because in this case the poles of the Wronskian arise at known points. It has been thought that $\ell \geq 6$ is intractable. However, [Das-Gowdigere-Mukhi-Santara, in progress] we have recently found arguments to show that the MLDE for Wronskian index $\ell \geq 6$ is quite tractable.
- Beyond two primaries: there is a recent follow-up paper by my collaborator [Rayhaun: "Bosonic Rational Conformal Field Theories in Small Genera ..."] that addresses the case of 3 and 4 primaries.

- There is a construction based on generalised Hecke operators [Harvey-Wu 2018] that provides an alternate way to find admissible characters. It would be useful to compare it with our construction.
- Relation to penumbral moonshine – relation between VVMF's and certain types of finite groups [Duncan-Harvey-Rayhaun 2021].

Thank you