Modularity of BPS black holes

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Black hole entropy in string theory

$$S_{\rm BH} = \frac{A_{\rm hor}}{4\ell_{\rm Pl}^2}$$

$$N = \log S_{\rm BH}$$

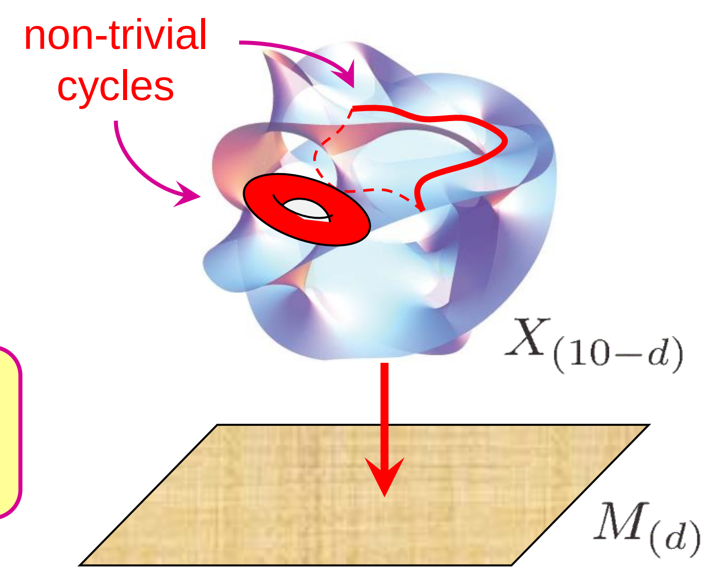
number of black hole states

What are the microstates counted by black hole entropy?

String theory solution

Black holes in string theory at small string coupling become *solitonic* objects constructed from D-branes wrapping *non-contractible cycles* on the internal manifold

BH microstates — D-brane configurations



$$X_{(10)} = M_{(d)} \times X_{(10-d)}$$

BPS indices

We are interested in string compactifications with *extended* SUSY and black hole solutions preserving a part of this SUSY → *BPS black holes*

For such black holes, we consider *BPS indices*

$$\Omega(\gamma) = \mathrm{Tr}_{\mathcal{H}(\gamma)}(-1)^F$$ ≤ number of states

Such indices are *invariant** under deformations of various parameters (string coupling, moduli, …)

* sometimes not, but in an interesting way

BPS indices have various "dual" interpretations:

- topological invariants of the compactification manifold (generalized Donaldson-Thomas invariants of Calabi-Yau threefolds)
- in certain setups they reproduce BPS indices in SUSY gauge theories
- or their topological versions (Vafa-Witten invariants of complex surfaces – count the Euler characteristic of moduli spaces of instantons in topologically twisted SU(N) SYM)

These relations open a possibility to find the indices exactly and uncover remarkable symmetry properties

Generating functions and modularity

In compactifications with N=8 (Type II/T^6) and N=4 (Type II/K3×T^2) SUSY, the BPS indices are essentially known: due to a large duality group they depend only on a few duality invariant combinations of charges and can be organized into generating functions that have been explicitly found

Example: ⅛ BPS black holes in N=8 characterized by one integer valued charge

[Moore,Maldacena,Strominger '99, Pioline '05, Shih,Strominger,Yin '05]

$$\frac{\theta_3(2\tau)}{\eta^6(4\tau)} = \sum_{n\geq -1} \Omega(n) q^n = q^{-1} + 2 + 8q^3 + 12q^4 + 39q^7 + \cdots \qquad q = e^{2\pi i \tau}$$

theta function

Dedekind eta function

For large charge, it reproduces the area law

$$\Omega(n) \sim e^{\pi\sqrt{n}} \sim e^{A_{hor}(n)/4}$$

modular functions

$$\tau \mapsto \frac{a\tau+b}{c\tau+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$h_\mu\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_\nu \rho_{\mu\nu}(g)(c\tau+d)^w h_\nu(\tau)$$

w — *modular weight*

multiplier system (phase factor)

Modular symmetry governs BH entropy

Generalize this to N=2 and use to find exact results

The plan of the talk

1. D4-D2-D0 black holes in Type IIA/CY and their BPS indices
2. (Mock) Modularity of the generating functions
3. Applications of mock modularity
4. The case of one-modulus Calabi-Yau threefolds
5. Conclusions

BPS black holes in Type IIA/CY

Type IIA string theory on
a Calabi-Yau threefold

Effective theory in 4d — N=2 SUGRA

$b_2(CY)+1$ gauge fields: from b_2 vector multiplets + graviphoton

½ BPS black holes are characterized by electro-magnetic charge vector and can be viewed as bound states of D6, D4, D2 and D0-branes wrapping 6, 4, 2 and 0-dimensional cycles

$$\gamma = (p^0, p^a, q_a, q_0) \qquad a = 1, \dots, b_2$$

BPS index $\Omega(\gamma)$ **=** **generalized Donaldson-Thomas invariant of CY**

We are interested in *D4-D2-D0* black holes (no D6-branes: $p^0 = 0$)

Natural generating function $h^{\rm DT}_{p^a,q_a}(\tau) = \sum_{q_0 > q_{0,\min}} \Omega(\gamma)\, e^{2\pi i q_0 \tau}$

But this function depends too many parameters and (secretly) on *CY moduli* and hence is *not* expected to have any nice properties

Wall-crossing

multi-centered black holes (bound states of black holes) are stable only in a region of the moduli space and decay in another

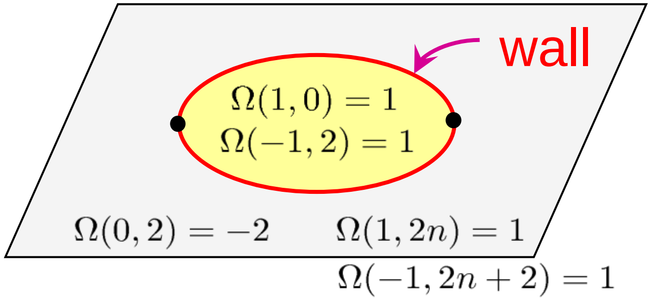
$\Omega(\gamma)$ *is only piecewise constant*

wall of marginal stability

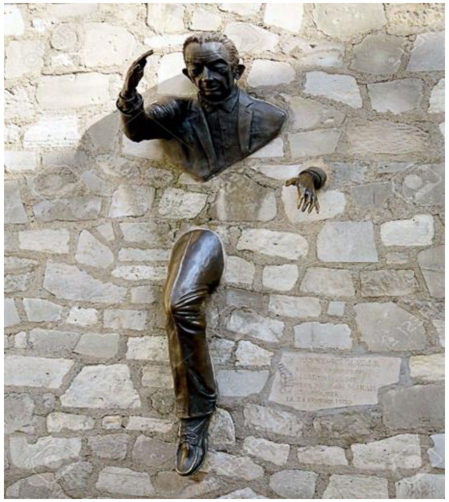
bound state $\gamma_1+\gamma_2$ does not exist | bound state $\gamma_1+\gamma_2$ exists

position of walls $$M_{\gamma_1}+M_{\gamma_2}=M_{\gamma_1+\gamma_2}$$

Example: pure SU(2) $\mathcal{N}$=2 SYM



[Seiberg,Witten '94]



How to cross a wall?

Kontsevich-Soibelman formula

allows to evaluate $\Omega(\gamma)$ on one side of a wall from their values on the other side

It is enough to find $\Omega(\gamma)$ in one chamber

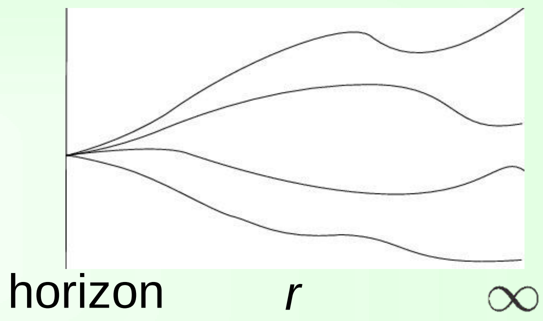
Attractor chamber

There is one special chamber in the moduli space: *attractor chamber*

Attractor mechanism [Ferrara,Kallosh,Strominger '95]

CY moduli — scalar fields in 4d which depend on the radial coordinate in the black hole background

SUGRA equations fix their values at the horizon in terms of the black hole charges and *independently* of their values at infinity $z_\star^a(\gamma)$



Maldacena,Strominger,Witten '97

$$\Omega_\gamma^{\rm MSW} = \Omega(\gamma, z_\star^a(\gamma)) = \Omega_{p,\mu}(\hat q_0)$$

- $\mu \in H_2(\mathbb{Z})/H_4(\mathbb{Z})$ – residue class of electric charge; runs over $\det \kappa_{ab}$ elements
- $\hat q_0 \equiv q_0 - \frac{1}{2}\kappa^{ab} q_a q_b$ – invariant charge bounded from above

spectral flow

$$q_a \mapsto q_a - \kappa_{ab}\epsilon^b$$

$$q_0 \mapsto q_0 - \epsilon^a q_a + \frac{1}{2}\kappa_{ab}\epsilon^a\epsilon^b$$

$\kappa_{ab} = \kappa_{abc}p^c$ – quadratic form, given by intersection numbers of 4-cycles, of *indefinite* signature $(1, b_2 - 1)$

generating function of MSW invariants

$$h_{p,\mu}(\tau) = \sum_{\hat q_0 \le \hat q_0^{\rm max}} \overline{\Omega}_{p,\mu}(\hat q_0)\, e^{-2\pi i \hat q_0 \tau}$$

Why should it be modular?

Why modularity?

S-duality

T-duality

modular group

Type IIB / $CY \times S^1$ ⟷ Type IIA / $CY \times S^1$ = M-theory / $CY \times \underbrace{S^1 \times S^1}_{T^2}$

Effective theory in 3d — SUSY non-linear sigma model with target space $\mathcal{M}$

After compactification on the circle

D4-brane / $\Gamma_4 \times S^1$ — pointlike object (*instanton*)

$\mathcal{M}$ carries an isometric action of the modular group

Affects the metric on $\mathcal{M}$ by instanton corrections

Each instanton is weighted by the BPS index $\Omega(\gamma)$

explicitly known [SA,Pioline,Saueressig,Vandoren '08 SA '09]

Restriction on (the generating function of) BPS indices $\Omega(\gamma)$

If Γ_4 is *irreducible*, $h_p(\tau)$ is a (vector valued) *modular form*.
Otherwise it has a modular *anomaly*, but one can construct an explicit expression $\widehat{h}_p(\tau,\bar\tau)$ for its *non-holomorphic modular completion*

SA,Pioline '18

Modular completion

$$\widehat{h}_{p,\mu} = h_{p,\mu} + \sum_{n=2}^{\infty} \sum_{\sum_{i=1}^n \gamma_i = \gamma} R_n(\{\gamma_i\}; \tau_2)\, e^{\pi i \tau Q_n(\{\gamma_i\})} \prod_{i=1}^{n} h_{p_i,\mu_i}$$

modular form of weight $-\frac{1}{2}\, b_2 - 1$

- $Q_n = \kappa^{ab} q_a q_b - \sum_{i=1}^{n} \kappa_i^{ab} q_{i,a} q_{i,b}$ — indefinite quadratic form on electric charges
- R_n — sum over trees weighted by (derivatives of) generalized error functions assigned to vertices of the trees, with parameters defined by charges

Example: $n = 2$

$$R_2 = -\frac{|\gamma_{12}|}{8\pi}\, \beta_{\frac{3}{2}}\left(\frac{2\tau_2 \gamma_{12}^2}{(p p_1 p_2)}\right)$$

where $\beta_{\frac{3}{2}}(x^2) = \frac{2}{|x|}\, e^{-\pi x^2} - 2\pi \mathrm{Erfc}(\sqrt{\pi}|x|)$

$\gamma_{ij} = \langle \gamma_i, \gamma_j \rangle$ — Dirac skew-symmetric product

$(p p_1 p_2) = \kappa_{abd}\, p^a p_1^b p_2^c$

holomorphic anomaly of the completion

$$\tau_2^2 \partial_{\bar\tau} \widehat{h}_{p,\mu} = \frac{\sqrt{2\tau_2}}{32\pi i} \sum_{\gamma_1+\gamma_2=\gamma} (-1)^{\gamma_{12}} \sqrt{(p p_1 p_2)}\, e^{-\frac{2\pi\tau_2\, \gamma_{12}^2}{(p p_1 p_2)} + \pi i \tau Q_2(\gamma_1,\gamma_2)}\, h_{p_1,\mu_1} h_{p_2,\mu_2}$$

Mock modular forms

Mathematically, these results imply that the functions $h_p(\tau)$ are (higher depth) *mock* modular forms

First examples of *mock theta functions* appeared in the last letter of Ramanujan to Hardy 100 years ago, but remained mysterious objects until recently….

Zwegers '02

Mock modular form — *a holomorphic* function which is "almost" modular with an anomaly controlled by another modular form (*shadow*)

$$h(\tau) \;\mapsto\; (c\tau+d)^w \left(h(\tau) - \int_{-d/c}^{-\mathrm{i}\infty} \frac{\overline{g(\bar z)}}{(\tau-z)^w}\, dz \right)$$

modular form of weight $2-w$

and (*non-holomorphic*) modular completion given by $\widehat{h}(\tau,\bar\tau) = h(\tau) - \int_{\bar\tau}^{-\mathrm{i}\infty} \frac{\overline{g(\bar z)}}{(\tau-z)^w}\, dz$

Depth n mock modularity: $\partial_{\bar\tau}\widehat{h}(\tau)$ is expressed via mock modular forms of depth $n' < n$

In our case, for $\Gamma = \sum_{i=1}^{n} \Gamma_i$, the depth is $n-1$

Applications

- New results in pure mathematics [SA,Banerjee,Manschot,Pioline '16]

Explicit construction of non-holomorphic completions of *indefinite theta series* of signature (n_+, n_-) with $n_- \geq .2$

Development of the theory of higher depth mock modular forms

- Inclusion of refinement [SA,Manschot,Pioline '19]

Refined BPS index $\Omega(\gamma, y) \sim \mathrm{Tr}_{\mathcal{H}_\gamma}(-y)^{2J_3}$

closely related to the generating function of helicity super-traces [Kiritsis '97]

Generating function

$$h^{\mathrm{ref}}_{p,\mu}(\tau, y) = \sum_{\hat{q}_0 \leq \hat{q}_0^{\max}} \frac{\overline{\Omega}_{p,\mu}(\hat{q}_0, y)}{y - y^{-1}} \, e^{-2\pi i \hat{q}_0 \tau}$$

— Jacobi mock modular form

if $z = \log y$ transforms as elliptic variable $z \mapsto \frac{z}{c\tau + d}$

The refinement simplifies the construction of the completion!

- Compactifications with higher SUSY [SA,Nampuri '20]

The refined construction allows to reproduce all known results about modular properties of BPS indices in compactifications with N=4 and N=8 SUSY

It also provides a generalization of the holomorphic anomaly equation for the immortal dyons in N=4 to the case of non-trivial torsion invariant $I(\gamma) = \gcd(Q \wedge P)$

Applications

- Non-compact CY and Vafa-Witten theory [SA,Manschot,Pioline '19, SA '20]

Consider a CY given by an elliptic fibration over a projective surface *S* and take a *local limit* where the elliptic fiber becomes large.

The only surviving divisor is the base [*S*]

All D4-brane charges are collinear

degree of reducibility of the divisor *N* = rank of the VW gauge group *U(N)*

for $b_2^+(S)=1,\ b_1(S)=0$

(refined) DT invariant of local CY = (refined) VW invariant of *S*

The formula for the completion allows to find the VW invariants themselves!

Explicit expressions for generating functions of refined VW invariants and their completions for all *N* for $S=\mathbb{P}^2$, Hirzebruch and del Pezzo surfaces

- **Challenge:** using modular properties of the generating functions, to find for *compact* CY threefolds

Explicit expressions for generating functions of black hole degeneracies ⟷ Donaldson-Thomas invariants for compact CY

Modular case: n=1

In contrast to the local CYs, in the compact case:

- the status of refined invariants is not clear
- there is no universal result for *irreducible* divisors

→ First concentrate on this *pure modular* case (n=1)

$$h_\mu(\tau) = \sum_{n \geq n_{\min}} \Omega_\mu(n)\, q^n$$

$q = e^{2\pi i \tau}$

weakly holomorphic ($n_{\min} < 0$) vector valued modular form of weight $-\frac{1}{2}\, b_2 - 1$

The space of such functions is finite dimensional!

$\dim \mathscr{M}_w(\mathfrak{Y}) \leq$ **# polar terms** — terms with $n < 0$

It is sufficient to find only the polar coefficients

We have addressed this problem for CICY with *one* Kähler modulus (13 CYs)

Previous studies (5 CYs): [Gaiotto,Strominger,Yin '06, Gaiotto,Yin '07, Collinucci,Wyder '08, Van Herck,Wyder '09]

Into the wild: space of stability conditions

Mathematically, BPS indices are defined by a *stability condition*

Physical stability is determined by the *central charge* $Z_\gamma = q_\Lambda z^\Lambda - p^\Lambda F_\Lambda$

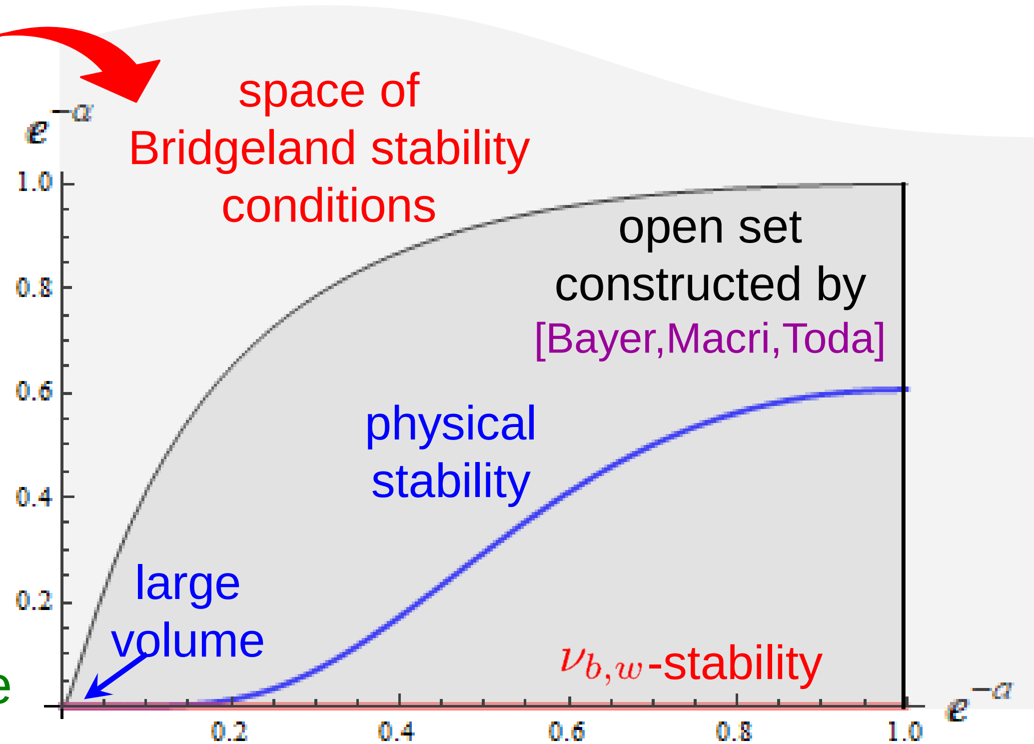
where $F_\Lambda(z) = \partial_{z^\Lambda} F(z)$

holomorphic prepotential

But one can take F_Λ to be independent parameters

With respect to $\nu_{b,w}$-stability there is a chamber where BPS indices vanish

Recipe: start in this chamber and using wall-crossing, go to the large volume chamber where the two notions of stability coincide



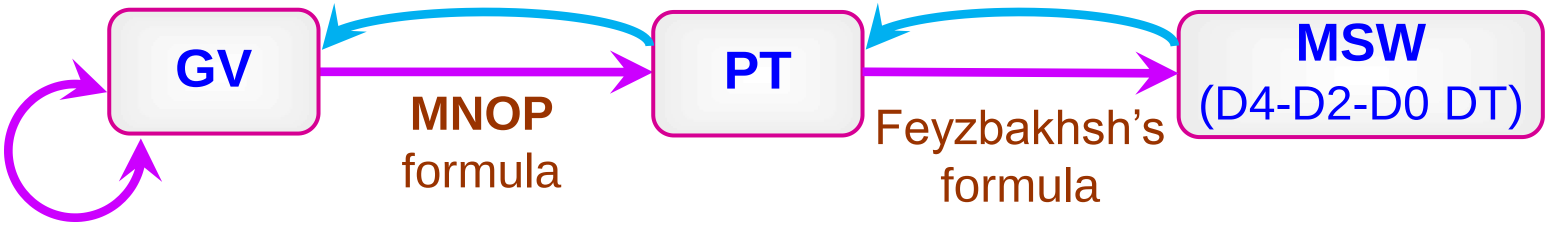
DT with $p^0 = -1$

[Feyzbakhsh '22]

Explicit formula relating Pandharipande-Thomas invariants and MSW invariants

GV/PT/DT

SA, Feyzbakhsh, Klemm,
Pioline, Schimannek ’23



“direct integration”
of topological string
[Bershadsky, Cecotti, Ooguri, Vafa ’93]

not enough conditions to fix holomorphic ambiguity of topological string

- We have improved the direct integration method to reach the *maximal* computable genus for most of CYs
- This allowed to compute sufficiently many PT invariants to get *all polar terms* for all but 2 CYs.
- In fact, in most cases we could also compute by this method several *non-polar terms* and thereby verify the modularity of the generating functions
- Expanding the generating functions, we get prediction for *infinitely* many MSW invariants (D4-D2-D0 black hole BPS indices)
- This allows to run the computation backwards and to get GV invariants *beyond* the limitations of the direct integration method

Example: quintic X_5
direct integration works up to $g_{\max} = 53$
while we achieved $g = 60$

Mock modular case: n=2

SA,Gaddam,Manschot, Pioline '22

For (*mixed*) mock modular functions, the polar terms are not enough to fix them uniquely.

What is it for 1-parameter CYs?

$$h_{2,\mu} = h^{(0)}_{2,\mu} + h^{(\mathrm{an})}_{2,\mu}$$

usual modular form fixed by polar terms

mock modular form with a specific modular anomaly

Our result for the modular completion $\widehat{h}_{2,\mu}$ implies that

$$h^{(\mathrm{an})}_{2,\mu} = \sum_{\mu_1=0}^{\kappa-1} (-1)^{\mu-2\mu_1+\kappa} G^{(\kappa)}_{\mu-2\mu_1+\kappa}\, h_{1,\mu_1}\, h_{1,\mu-\mu_1} \qquad \mu = 0, \dots, 2\kappa-1$$

where $G^{(\kappa)}_{\mu}$ is a mock modular form with the completion $\widehat{G}^{(\kappa)}_{\mu}$ satisfying the following holomorphic anomaly

$$\partial_{\bar\tau} \widehat{G}^{(\kappa)}_{\mu} = \frac{\sqrt{\kappa}}{16\pi \mathrm{i} \tau_2^{3/2}} \sum_{k\in 2\kappa\mathbb{Z}+\mu} e^{\frac{\pi \mathrm{i} \bar\tau}{2\kappa}\, k^2}$$

for $\kappa = 1$

$$G^{(1)}_{\mu} = H_{\mu}$$

generating series of Hurwitz class numbers

$$G^{(\kappa)}_{\mu} = (\mathcal{T}_{\kappa}[H])_{\mu}$$

generalized Hecke operator [Bouchard,Creutzig,Joshi '18]

The problem reduces to finding polar terms

Conclusions

Main result: *Explicit* form of the *modular completion* of the generating function of (refined) black hole degeneracies (DT invariants) for *arbitrary* divisor of CY

→ $h_p(\tau)$ – higher depth mock modular form

Numerous applications: indefinite theta series, N=4 dyons, VW invariants for arbitrary rank, explicit evaluation of BPS indices for compact CYs…

Some open problems:

- Extension of this technique to various type of *compact* CYs
  - CYs with two and more moduli
  - elliptic and K_3 fibrations
  - one modulus CY, but D4-brane charge ≥ 2
- Derivation of the modular completion from the world-sheet viewpoint
- Geometric and physical meaning of the refined construction for compact CYs

Thank you!