Update on Supermembrane Theory

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based on: *O. Lechtenfeld and HN, JHEP02 (2022) 114*

Also relevant to this particular work:

HN, Nucl. Phys. B176 (1980) 419;

B. de Wit, J. Hoppe and HN, Nucl. Phys. B305 (1988) 545;

S. Ananth, O. Lechtenfeld, H. Malcha, HN, C. Pandey, S. Pant, JHEP10(2020)199;

H. Malcha, HN: JHEP06(2021)001;

O. Lechtenfeld, M. Rupprecht, Phys. Lett. B819(2021)13613;

The relativistic p-brane

Original idea due to Dirac: a (relativistic) membrane model of the electron [Dirac(1962)]

Nambu-Goto action for p-brane with $X^\mu \equiv X^\mu(\tau, \sigma^r)$:

$$\mathcal{L}(X) = T_p \sqrt{-\det g_{ij}(X)} \qquad \textbf{with} \qquad g_{ij}(X) \equiv \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}$$

or equivalently, Polyakov-type action:

$$\mathcal{L}(g, X) = \frac{1}{2} T_p \sqrt{-g} \Big(g^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} - (p-1) \Big)$$

Simplifying feature: for string ($p = 1$) there exists a gauge for which equations of motion become *linear* !

Equivalently: gauge away world volume metric by diffeomorphisms and conformal symmetry $\rightarrow$ gravitational path integral reduced to *finite-dimensional* integral, *Gaussian theory* for target space coordinates X^μ.

However, no such simplifications for $p > 1$!

Why Supermembranes?

Unique maximally supersymmetric supermembrane theory in space-time dimension $D = 11$ [Bergshoeff,Sezgin,Townsend(1987)] is a candidate theory for *non-perturbative* formulation of string theory. (Theories also exist for $D = 4, 5, 7$.)

- can be obtained as the $N \to \infty$ limit of maximally supersymmetric $SU(N)$ matrix theory [deWit,Hoppe,HN(1988)].

Nevertheless progress has been slow because there remains the main unsolved problem = quantization:

- There is no gauge which linearizes equations of motion $\Rightarrow$
- Determination of correlation functions can *not* be reduced to free field theory computations, unlike for string theory
- $\Rightarrow$ no membrane analog of the Veneziano formula
- In view of intrinsic non-linearities of membrane theory, covariant quantization *à la* Polyakov looks hopeless

Only realistically feasible approach (so far, at least)= light-cone gauge quantization with flat (Minkowski) background? But again there remain difficulties which are mirrored in M theory matrix model:

- **Existence and properties of $N \to \infty$ limit?** ☺
- ***Quantum* target space Lorentz invariance?** ☹
  [see also: deWit,Marquard,HN(1990); Ezawa,Matsuo,Murakami(1997)]
- **Is $D\!=\!11$ the critical dimension of supermembrane?**
- **Vertex operators and correlators?** ☹

Present approach based on the fact that supermembrane theory $\equiv$ *one-dimensional supersymmetric gauge theory of area preserving diffeomorphisms* (APDs)

$\to$ in principle would allow to skip $SU(N)$ approximation for $N < \infty$ and to deal *directly* with $SU(\infty)$ theory!

Supermembrane basics

Target superspace coordinates $\{X^\mu(\xi^i), \theta(\xi^i)\}$ **with** $i, j, ... = 0, 1, 2$ **and** $\xi^i = (\tau, \sigma^r)$ $(r, s = 1, 2)$ $\rightarrow$ **vielbein and metric:**

$$E_i{}^\mu = \partial_i X^\mu + \bar{\theta}\Gamma^\mu \partial_i \theta \quad \Rightarrow \quad \mathrm{g}_{ij} = E_i{}^\mu E_j{}^\nu \eta_{\mu\nu}$$

With target space light-cone coordinates

$$X^\mu = (X^+, X^-, X^a) \quad \textbf{with} \quad X^\pm = \frac{1}{\sqrt{2}}(X^{10} \pm X^0)$$

and transverse coordinates $\{X^a\} \equiv \mathbf{X}$ $\rightarrow$ **light-cone gauge**

$$X^+(\tau, \boldsymbol{\sigma}) = X_0^+ + \tau \quad , \qquad \Gamma_+ \theta(\tau, \boldsymbol{\sigma}) = 0$$

$\Rightarrow$ **induced metric on membrane world volume**

$$\begin{aligned}
\mathrm{g}_{rs} &\equiv \bar{\mathrm{g}}_{rs} = \partial_r \mathbf{X} \cdot \partial_s \mathbf{X} \qquad\qquad (\bar{\mathrm{g}}^{rs}\bar{\mathrm{g}}_{st} = \delta_t^r) \\
\mathrm{g}_{00} &= 2\partial_0 X^- + (\partial_0 \mathbf{X})^2 + \bar{\theta}\Gamma_- \partial_0 \theta \\
\mathrm{g}_{0r} &\equiv u_r = \partial_r X^- + \partial_0 \mathbf{X} \cdot \partial_r \mathbf{X} + \bar{\theta}\Gamma_- \partial_r \theta
\end{aligned}$$

Metric determinant on world volume:

$$\mathsf{g} \equiv \det \mathsf{g}_{ij} = -\Delta \bar{\mathsf{g}}$$

with $\bar{\mathsf{g}} \equiv \det \bar{\mathsf{g}}_{rs}$ **and** $\Delta \equiv -\mathsf{g}_{00} + u_r \bar{\mathsf{g}}^{rs} u_s \;\Rightarrow$ **Lagrangian**

$$\mathcal{L} = -T_3 \Big(-\sqrt{\bar{\mathsf{g}}\Delta} + \epsilon^{rs} \partial_r X^a \, \bar{\theta} \Gamma_- \Gamma_a \partial_s \theta \Big)$$

Hence the *membrane tension* T_3 is the *only* parameter of the theory, of mass dimension three $\rightarrow$ render dimensionless by rescaling w.r.t. some reference mass.

For *double dimensional reduction* [Duff,Howe,Inami,Stelle(1987)] **use $T_s \equiv \ell_s^{-2} = T_3 R_{10}$ and $g_s^2 = T_3 R_{10}^3$** [Witten(1995)] **to obtain**

$$\boxed{T_3 \, = \, T_s^{3/2} g_s^{-1} \, \equiv \, (\alpha')^{-3/2} g_s^{-1}}$$

$\rightarrow$ **after reduction from $D = 11$ to $D = 10$ this formula ties together the two key parameters of string theory!**

Or: perturbative expansion in T_3 'entangles' α'-expansion with string loop expansion.

Canonical analysis for bosonic membrane: [Goldstone,Hoppe(1982)]

Analogous analysis for supermembrane: [BST,dWHN(1988)]

$$P^+ \;=\; T_3\sqrt{\frac{\bar{\rm g}}{\Delta}} \quad , \qquad S = -T_3\sqrt{\frac{\bar{\rm g}}{\Delta}}\Gamma_-\theta \equiv -P^+\theta$$

$$\mathbf{P} \;=\; T_3\sqrt{\frac{\bar{\rm g}}{\Delta}}\big(\partial_0\mathbf{X} - u_r\bar{\rm g}^{rs}\partial_s\mathbf{X}\big) \equiv P^+\big(\partial_0\mathbf{X} - u_r\bar{\rm g}^{rs}\partial_s\mathbf{X}\big)$$

$\Rightarrow$ **spatial diffeomorphism constraint**

$$\phi_r \;=\; \mathbf{P}\cdot\partial_r\mathbf{X} \;+\; P^+\partial_r X^- \;+\; \bar{S}\partial_r\theta \;\approx\; 0$$

Hamiltonian density: $P_0^- \equiv -\int {\rm d}^2\sigma\,\mathcal{H}$ **(stable for** $T>0$**)**

$$\mathcal{H} = \frac{\mathbf{P}^2 + T_3^2\bar{\rm g}}{P^+} - T_3\epsilon^{rs}\partial_r X^a\,\bar\theta\Gamma_-\Gamma_a\partial_s\theta$$

With $P^+(\tau,\boldsymbol{\sigma}) = P_0^+\sqrt{w(\boldsymbol{\sigma})}$ **we obtain '(mass)**^{2} **operator'**

$$\mathcal{M}^2 \;=\; -2P_0^+P_0^- - \mathbf{P}_0^2 \;=\; \int d^2\sigma\Big([\mathbf{P}^2]' + T_3^2\bar{\rm g} - 2T_3\epsilon^{rs}\partial_r X^a\,\bar\theta\Gamma_-\Gamma_a\partial_s\theta\Big)$$

(with rescaled fermionic variables $\theta \to \sqrt{P_0^+}\,\theta$**).**

Kinematics of zero modes

Supermembrane zero modes X_0^μ and θ_0 decouple:

$$\left\{Q_{0\alpha}^{(+)}\,,\,Q_{0\beta}^{(+)}\right\} \,=\, (\Gamma_+)_{\alpha\beta}\,\mathbf{P}_0^2$$

with $44 \oplus 84$ bosonic and 128 fermionic states = massless multiplet of $D = 11$ supergravity $\Rightarrow$ groundstate

$$\{D = 11 \ \textbf{SUGRA multiplet}\} \otimes \Psi$$

$\to$ requires *normalizable* Ψ obeying $\mathcal{M}^2\Psi = 0$.

For superstring: same zero mode kinematics, but now $\Psi = |0\rangle_b \otimes |0\rangle_f$ is infinite product of supersymmetric harmonic oscillator groundstate wave functions, excited states by applying raising operators $(a_n^i)^\dagger$ and $d_{n\alpha}^\dagger$ to Ψ.

This accounts for the very simple structure of the string spectrum and its factorization into left-moving and right-moving states (modulo $L_0 = \bar{L}_0$ constraint).

***Alas*, life is not so simple with supermembranes!**

For uncompactified supermembrane $\mathcal{M}^2$ has *continuous* spectrum starting at zero $\rightarrow$ no excited massive one-particle excitations [deWit,Lüscher,HN(1989);Smilga(1996)]

$\rightarrow$ not a first quantizable theory!

Different results (interpretation?) for compactified supermembrane with winding [Boulton,Garcia del Moral,Restuccia(2003)]

Yet a different question concerns the existence of a *normalizable* groundstate $\rightarrow \Psi$ is definitely not a factorized state of bosons and fermions!

For $N < \infty$ matrix model there are numerous results:

[dWHN(1987);Hoppe(1997);Fröhlich,Hoppe(1997); Halpern,Schwartz(1997); Hoppe,Yau(1997);Yi(1997); Sethi,Stern(1998);Porrati,Rosenberg(1998); Moore,Nekrasov,Shatashvile(1998);Boulton,Garcia del Moral,Restuccia(2011)&(2021)]

but $N = \infty$ theory is much more difficult.

Interlude: Area Preserving Diffeomorphisms

Are generated by vector fields $\delta\xi^r \equiv \delta\xi^r(\boldsymbol{\sigma})$ which satisfy

$$\partial_r \left(\sqrt{w}\,\delta\xi^r\right) = 0 \qquad \Rightarrow \qquad \delta\xi^r = \epsilon^{rs}\partial_s(\delta\xi)$$

with parameter $\delta\xi \equiv \delta\xi(\boldsymbol{\sigma})$ $\Rightarrow$ scalar Φ transforms as

$$\delta\Phi = \delta\xi^r\partial_r\Phi = \epsilon^{rs}\partial_r(\delta\xi)\,\partial_s\Phi \equiv \left\{\delta\xi\,,\,\Phi\right\}$$

with APD bracket

$$\left\{A\,,\,B\right\}(\boldsymbol{\sigma}) := \epsilon^{rs}\,\partial_r A(\boldsymbol{\sigma})\,\partial_s B(\boldsymbol{\sigma})$$

$\rightarrow$ satisfies all properties of a Lie bracket

In addition there are APDs generated by harmonic vector fields $\delta\xi^r$ for which $\delta\xi$ does not exist globally.

APD_0 for which $\delta\xi$ exist globally form a *normal subgroup* within the group of all APDs on the membrane.

APD Gauge Theory and Matrix Model

Partial gauge fixing $u_r = 0 \ \to$ **residual gauge invariance**

$$\phi = \epsilon^{rs}\Big(\partial_r \mathbf{P}\cdot\partial_s\mathbf{X} + \partial_r\bar\theta\Gamma_-\partial_s\theta\Big) \approx 0$$

generates $\mathbf{APD}_0$ **transformations** $\to \phi = 0$ **necessary to solve for longitudinal coordinate** $\partial_r X^-(\tau,\boldsymbol{\sigma}) = \ldots$

$$X^-(\tau,\boldsymbol{\sigma}) = -\int d^2\sigma'\, G^r(\boldsymbol{\sigma},\boldsymbol{\sigma}')\Big(\partial_0\mathbf{X}\cdot\partial_r\mathbf{X}(\tau,\boldsymbol{\sigma}') + \bar\theta\Gamma_-\partial_r\theta(\tau,\boldsymbol{\sigma}')\Big)$$

Rewrite 'potential' in terms of APD bracket

$$\bar{\mathrm{g}} = \det\big(\partial_r\mathbf{X}\cdot\partial_s\mathbf{X}\big\} \equiv \big(\{X^a,X^b\}\big)^2$$

$\Rightarrow \mathcal{M}^2$ **can be equivalently obtained from Lagrangian**

$$\mathcal{L}_{\rm APD} = \frac{1}{2}(D_t\mathbf{X})^2 + \bar\Theta\Gamma_- D_t\Theta - \frac{1}{4}g^2\{X^a,X^b\}^2 + g\bar\Theta\Gamma_-\Gamma_a\{X^a,\Theta\}$$

with covariant derivative $D_t f := \partial_t f + \{\omega, f\}$**. Putting back dimensions and rescaling we identify**

$$T_3 \propto g^2 \qquad \left(\Rightarrow g \propto \ell_s^{-3/2} g_s^{-1/2} \right)$$

Small (large) tension limit of (super-)membrane $\equiv$
weak (strong) coupling limit of APD gauge theory!

Approximate APD$_0$ $= \lim_{N\to\infty} SU(N)$ [Goldstone,Hoppe(1982)]

$$X^a(t, \boldsymbol{\sigma}) = X_a^{(0)}(t) + \sum_{A=1}^{\infty} X_a^A(t) Y^A(\boldsymbol{\sigma}) \quad etc.$$

by truncating sum at dim $SU(N) = N^2 - 1$**, such that**

$$f_{\rm APD}^{ABC} \equiv \int d^2\sigma \sqrt{w(\boldsymbol{\sigma})}\, Y^A(\boldsymbol{\sigma}) \big\{ Y^B(\boldsymbol{\sigma}), Y^C(\boldsymbol{\sigma}) \big\} = \lim_{N\to\infty} f_{SU(N)}^{ABC}$$

to end up with supermatrix model Lagrangian

$$\mathcal{L}_{SU(N)} = \frac{1}{2}(D_t X_a^A)^2 - i\bar{\theta}^A D_t \theta^A - \frac{1}{4} g^2 (f^{ABC} X_b^B X_c^C)^2 - \frac{1}{2} f^{ABC} g \theta^A \gamma^a X_a^B \theta^C$$

now with 16-component real SO(9) spinors $\theta_\alpha^A(t)$ **and 16-by-16** γ**-matrices** $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$**.**

If supplemented with $SU(N)$ gauge constraint ($\omega^A = 0$)

$$\phi^A = f^{ABC}\big(X_a^B \partial_t X_a^C + \theta_\alpha^B \theta_\alpha^C\big) \approx 0$$

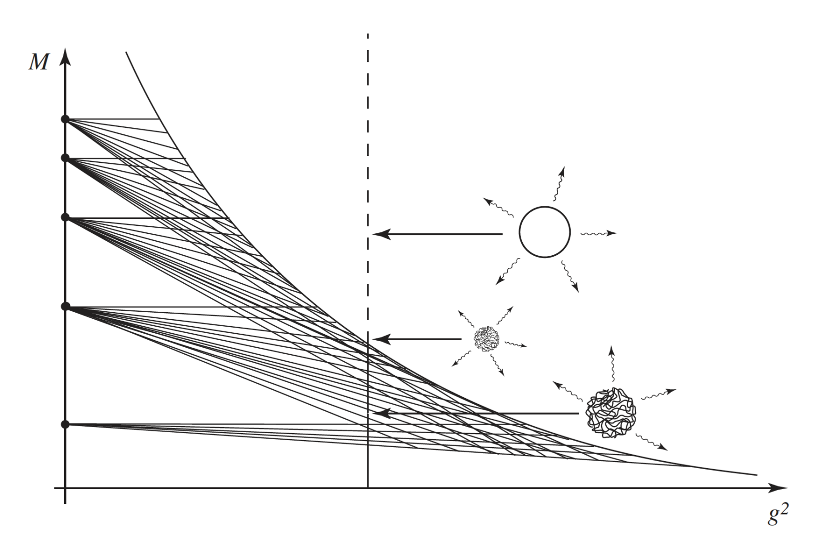
this is the very same Lagrangian that underlies the M-theory proposal of [Banks,Fischler,Shenker,Susskind(1996)].

Their picture is based on assuming *D0*-branes as basic M theory constituents. However: limiting theory for $N \to \infty$ (if it exists) *is* nothing but the supermembrane!

Interpretation: supermembrane is not a first quantizable (one-particle or N-particle for any N) theory, but a fully non-perturbative description. [e.g.Helling,HN:hep-th/9809103]

***No* massive excitations, but in double dimensional reduction get *all* string states from 'spikes' emanating from membrane $\rightarrow$ supermembrane should contain *multistring* Fock space! Details to be worked out....**

Non-perturbative superstring $\equiv$ supermembrane?



Picture from: T.Damour: 'The entropy of black holes: a primer', hep-th/0401160

As $g_s \to \infty$, extra dimension opens up ($R_{10} \to \infty$ [Witten])

$\to$ discrete string states merge into a continuum

This agrees with fact that supermembrane Hamiltonian has *continuous* spectrum!

Setting up the Path Integral

Main goal is to evaluate correlators [Lechtenfeld,HN(2022)]

$$\Big\langle \mathcal{O}_1[\mathbf{X},\theta]\cdots\mathcal{O}_n[\mathbf{X},\theta]\Big\rangle_g \;=\; \int\prod \mathcal{D}X_a(t,\sigma)\,\mathcal{D}\theta_\alpha(t,\sigma)\,\mathcal{D}\omega(t,\sigma)\,\mathcal{D}C(t,\sigma)\,\mathcal{D}\bar{C}(t,\sigma)$$
$$\times\;\mathcal{O}_1[\mathbf{X},\theta]\cdots\mathcal{O}_n[\mathbf{X},\theta]\;\exp\big(i\,\mathrm{S}_{\rm tot}\big)$$

for physical 'vertex operators' $\mathcal{O}(\mathbf{X},\theta)$.

Action $\mathrm{S}_{\rm tot}$ contains gauge fixing part (Lorenz gauge)

$$\mathcal{L}' = \frac{1}{2\alpha}(\partial_t\omega)^2 + \bar{C}\partial_t D_t C$$

with Faddeev-Popov ghosts $C(t,\boldsymbol{\sigma}),\bar{C}(t,\boldsymbol{\sigma})$, such that limit $\alpha\to 0$ puts theory on gauge hypersurface.

Present formulation uses *bosonic functional measure* obtained by integrating out all anti-commuting quantities $\to$ MSS determinant (Pfaffian) [Matthews,Salam(1954);Seiler(1975)]

$$\Delta_{MSS} = \Big[\det\big(\delta^{AB}\delta_{\alpha\beta}\delta(t_1-t_2) + gK^{AB}_{\alpha\beta}(t_1,t_2)\big)\Big]^{1/2}$$

with integral kernel

$$K^{AB}_{\alpha\beta}(t_1, t_2) := \varepsilon(t_1 - t_2) f^{ACB} \gamma^a_{\alpha\beta} X^C_a(t_2) \qquad (*)$$

and fermion propagator

$$\varepsilon(t) := \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{ip}{p^2 - i\epsilon} e^{-ipt} = -\varepsilon(-t) \quad , \qquad \varepsilon(0) = 0$$

→ well defined (Fredholm) determinant for $N < \infty$.
For APD one must replace $(*)$ by integral kernel

$$K^{\mathrm{APD}}_{\alpha\beta}(t_1, t_2; \boldsymbol{\sigma}, \boldsymbol{\sigma}') = \varepsilon(t_1 - t_2) \gamma^a_{\alpha\beta} \epsilon^{rs} \partial_r X_a(t_1, \boldsymbol{\sigma}) \delta(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \partial'_s \quad \Rightarrow$$

$$(K \circ \phi)_\alpha(t, \boldsymbol{\sigma}) = \int \mathrm{d}s \int \mathrm{d}^2\sigma' \, K^{\mathrm{APD}}_{\alpha\beta}(t, s; \boldsymbol{\sigma}, \boldsymbol{\sigma}') \phi_\beta(s, \boldsymbol{\sigma}')$$

in order to deal with derivative interactions. However, there is a subtlety: when computing $\log \Delta_{MSS}$ by means of "log det = Tr log" there appear divergent factors involving $\delta(\boldsymbol{\sigma}, \boldsymbol{\sigma})$ which make determinant ill-defined.

But before taking $N \to \infty$ need to take into account bosonic and fermionic contributions to path integral:

There exist a linearizing transformation $\mathcal{T}_g$ obeying

$S_{bos}[\mathcal{T}_g X_a; g=0] = S_{bos}[X_a; g]$ **and** [HN,1981]

$$\det\left(\frac{\delta\mathcal{T}_g X}{\delta X}\right) \;=\; \Delta_{\mathrm{MSS}}[\omega,\mathbf{X}]\;\Delta_{\mathrm{FP}}[\omega,\mathbf{X}]$$

for which the $N\to\infty$ limit exists [Lechtenfeld,HN:2109.00346]

$$\begin{aligned}
\mathcal{T}_g X_a(t) \;=\;& X_a(t) \;-\; \frac{1}{2}g^2\int \mathrm{d}s\,\mathrm{d}u\;\varepsilon(t{-}s)\,\varepsilon(s{-}u)\left\{X_b(s)\,,\,\{X_b(u),X_a(u)\}\right\}\\
&+\;\frac{1}{8}g^4\int \mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\;\varepsilon(t{-}s)\,\varepsilon(s{-}u)\,\varepsilon(u{-}v)\,\varepsilon(v{-}w)\,\Big[\\
&\qquad 6\left\{X_b(s)\,,\left\{X_c(u)\,,\,\{X_{[a}(v)\,,\,\{X_b(w),X_{c]}(w)\}\}\right\}\right\}\\
&\quad+\;2\left\{X_b(s)\,,\left\{X_{[b}(u)\,,\,\{X_{|c|}(v)\,,\,\{X_{a]}(w),X_c(w)\}\}\right\}\right\}\\
&\quad+\;2\left\{X_a(s)-X_a(t)\,,\left\{X_b(u)\,,\,\{X_c(v)\,,\,\{X_b(w),X_c(w)\}\}\right\}\right\}\Big]\\
&+\;\tfrac{1}{8}g^4\int \mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\;\varepsilon(t{-}s)\,\varepsilon(s{-}u)\,\varepsilon(s{-}v)\,\varepsilon(v{-}w)\;\times\\
&\qquad\left\{\{X_a(u),X_b(u)\}\,,\left\{X_c(v)\,,\,\{X_b(w),X_c(w)\}\right\}\right\} \;+\; \mathcal{O}(g^6)\;.
\end{aligned}$$

$\rightarrow$ **divergences of fermionic determinants are absorbed by Jacobian to give a finite result for $\mathcal{T}_g$.**

NB: divergent factor also appears in matrix model because $SU(N)$ Cartan-Killing metric $f^{ACD}f^{BCD} = N\delta^{AB}$ likewise diverges in $N \rightarrow \infty$ limit (and thus for APD).

Conclusion: for supermembrane and for finite N matching divergent factors keep path integral measure well-defined in the limit $N \rightarrow \infty$

$\Rightarrow$ ***renormalizability* of the supermembrane??**

No such matching for the bosonic membrane $\Rightarrow N \rightarrow \infty$ limit probably does *not* exist for bosonic membrane. This might settle an old question: is the bosonic membrane *non-renormalizable*?

[Related difficulties were already pointed out by G. Savvidy(1990)]

Furthermore: evidence that expansion of $\mathcal{T}_g$ in g has non-zero radius of convergence for $\|X_a^A\|_{\mathrm{L}^1} < \infty$.

Physical Correlators?

(Classical analogs of) vertex operators for emission of massless states from supermembrane [Dasgupta,Plefka,HN(2000)]

Must satisfy stringent consistency requirements:

- **world volume and target space gauge invariance**
- **linear and non-linear supersymmetry**
- **reduce to correct point particle limit** [Green,Gutperle,Kwon(1999)]
- **factorization under double dimensional reduction into type II superstring vertex operators** [Green,Schwarz(1982)]

For instance, for transverse graviton polarizations

$$\begin{aligned} V_h[\mathbf{X},\theta] &= h_{ab}\Big[D_tX^aD_tX^b - \{X^a,X^c\}\{X^b,X^c\} - i\bar\theta\,\gamma^a\{X^b,\theta\} \\ &\quad - \frac{1}{2}D_tX^a\,\bar\theta\gamma^{bc}\theta\,k_c - \frac{1}{2}\{X^a,X^c\}\,\bar\theta\gamma^{bcd}\theta\,k_c + \frac{1}{2}\bar\theta\gamma^{ac}\theta\,\bar\theta\gamma^{bd}\theta\,k_ck_d\Big]e^{-i\mathbf{k}\cdot\mathbf{X}+ik^-t} \end{aligned}$$

where h_{ab} is the transverse graviton polarization tensor, and $\{k_a\} = \mathbf{k}$ transverse components of the target-space momentum. Also: set $k^+ = 0$ to avoid $e^{ik^+X^-(t,\boldsymbol{\sigma})}$.

Idem for longitudinal graviton polarizations, gravitino and 3-form 'photon' of $D = 11$ supergravity.

But: due to complexity of path integral and vertices no concrete computations of amplitudes available.

For instance, 4-graviton correlator might provide perturbatively (in $T \propto g$) calculable supermembrane analog of the Virasoro-Shapiro amplitude formula.

Furthermore: there do not appear to exist analogs of massive superstring vertex operators $\Rightarrow$ confirms that supermembrane is not a first quantizable theory?

Many further questions:

- **Should one sum over APD groups for all genera?**
- **How to recover notions of modularity for supermembrane?**
- **$\rightarrow$ if successful this might furnish further hints on how IIA and IIB superstrings 'embed' into quantum supermembrane.**

The one million dollar question

If the $D = 11$ supermembrane is really a candidate for a mathematical formulation of M theory, where are all the dualities known from supergravity and superstring theory that are widely expected to play a key role? This would seem to require much more, *viz.*

- **3-form $A_3 \;\leftrightarrow\; M2$-brane**
- **6-form $A_6 \;\leftrightarrow\; M5$-brane**
- **dual graviton $A_{8,1} \;\leftrightarrow\; KK$-monopole**

... as well as an *infinite* tower of M-theory extended objects that can serve as sources for the fields (excitations) appearing in various level expansions of E_{10}.

Outlook

- **Due to its uniqueness properties $D = 11$ supermembrane theory is a prime candidate for a (partial?) non-perturbative unification and M theory.**
- **... but much harder than string theory!**
- **$N \to \infty$ limit unlikely to exist for quantized *bosonic* matrix model, while there is evidence that quantization might work for the supermembrane $\leftrightarrow$ existence of supersymmetric $SU(\infty)$ matrix model.**
- **Need to develop new computational tools to make quantum supermembrane *more computationally accessible*, by concentrating on quantites for which APD expressions remain well-defined.**

THANK YOU