The AdS Virasoro-Shapiro amplitude

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February 8, 2024

Based on:

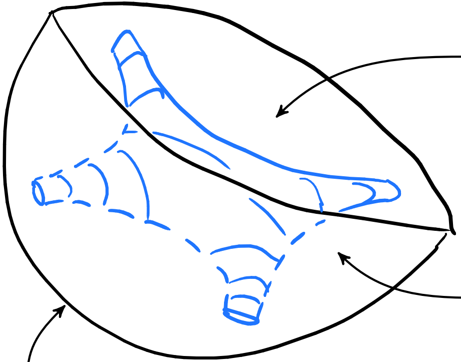
2204.07542, 2209.06223, 2303.08834, 2305.03593 with Luis F. Alday, João Silva

2306.12786 with Luis F. Alday

2308.03683 with Giulia Fardelli, João Silva

2312.02261 with Luis F. Alday, Maria Nocchi

1 process - 3 descriptions



5d bulk of AdS:
IIb string theory on $AdS_5 \times S^5$

- strings on curved background

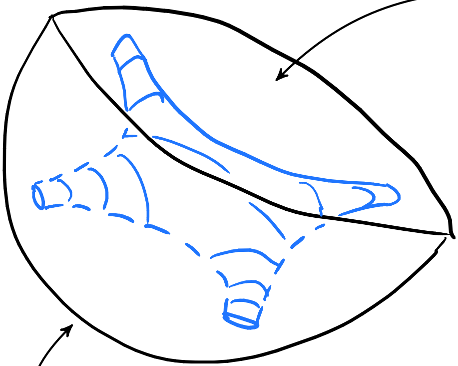
2d string world-sheet:
2d CFT???

4d boundary of AdS:
$\mathcal{N} = 4$ super Yang Mills theory

- non-abelian gauge theory
- conformal symmetry
- supersymmetry
- integrable

This talk:
Find the amplitude without quantizing the string.

Parameters



5d bulk of AdS:
IIb string theory on $AdS_5 \times S^5$

- AdS radius R_{AdS}
- string length L_s
- string coupling g_s

4d boundary of AdS:
$\mathcal{N} = 4$ super Yang Mills theory

- $SU(N)$ gauge group
- coupling $\sqrt{\lambda} = \dfrac{R_{\text{AdS}}^2}{L_s^2}$

Weakly coupled strings:

$$g_s \ll 1 \quad \Leftrightarrow \quad N \gg 1$$

Expansion around flat space:

$$\frac{R_{\text{AdS}}^2}{L_s^2} \gg 1 \quad \Leftrightarrow \quad \sqrt{\lambda} \gg 1$$

$$L_s^2 p_i \cdot p_j \text{ finite}$$

Plan for the talk

1. Review: String scattering in flat space
2. String scattering in AdS
    1. The CFT dispersion relation
    2. Single-valued functions for the world-sheet
    3. Checks: Integrability and Localization
3. High energy limit

1. Flat Space Review

The Virasoro-Shapiro amplitude (flat space)

In the beginning, there was the amplitude.
[Veneziano,1968;Virasoro,1969;Shapiro,1970]

Scattering of 4 gravitons in the type IIb superstring:

Virasoro-Shapiro amplitude

$$A^{(0)}(S,T) = -\frac{\Gamma(-S)\,\Gamma(-T)\,\Gamma(-U)}{\Gamma(S+1)\,\Gamma(T+1)\,\Gamma(U+1)}$$

$$S = -\frac{L_s^2}{4}(p_1+p_2)^2\,,\ T = -\frac{L_s^2}{4}(p_1+p_3)^2\,,\ U = -\frac{L_s^2}{4}(p_1+p_4)^2$$

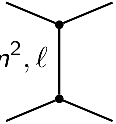
$$S+T+U=0$$

Regge boundedness (flat space)

String amplitudes have soft UV (Regge) bahaviour

$$\lim_{|S|\to\infty} A^{(0)}(S,T) \sim S^{T+\alpha_0}, \quad \mathrm{Re}(T) < 0$$

and higher spin resonances

m^2, ℓ 

$$= \frac{P_\ell(S)}{T - m^2} \qquad\qquad P_\ell(S) = S^\ell + O(S^{\ell-1})$$

Regge bahaviour places strong constraints on the coefficients $a_{\delta,\ell}$ in

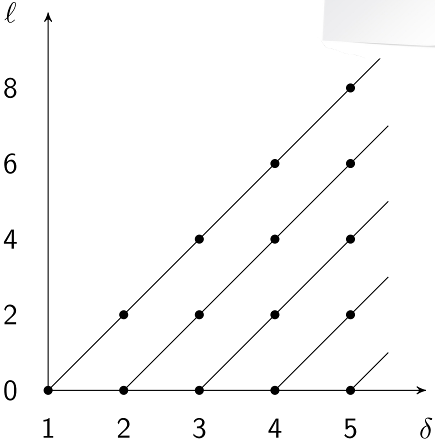
$$A^{(0)}(S,T) = \sum_{(\delta,\ell)} \frac{a_{\delta,\ell} P_\ell(S)}{T-\delta}$$

The spectrum (flat space)

The exchanged massive string spectrum is extracted via the partial wave expansion

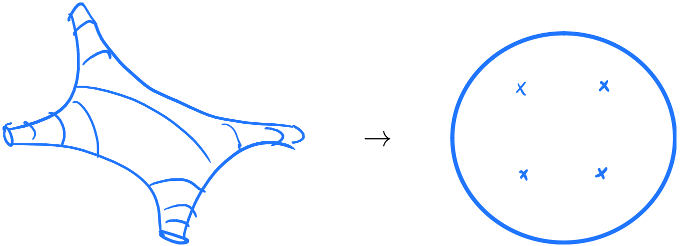
$$A^{(0)}(S,T) = \sum_{(\delta,\ell)} \frac{a_{\delta,\ell} P_\ell(S)}{T-\delta}$$

It forms linear Regge trajectories.



World-sheet integral (flat space)

The amplitude is also given by an integral over world-sheets:



$$A^{(0)}(S,T) = \int d^2z \, |z|^{-2S-2}|1-z|^{-2T-2} G^{(0)}_{\text{tot}}(S,T,z)$$

$$G^{(0)}_{\text{tot}}(S,T,z) = \frac{1}{3}\left(\frac{1}{U^2} + \frac{|z|^2}{S^2} + \frac{|1-z|^2}{T^2}\right)$$

The integrand is a single-valued function of z!

Low energy expansion (flat space)

Low energy effective action (supergravity + derivative interactions)
→ Low energy expansion:

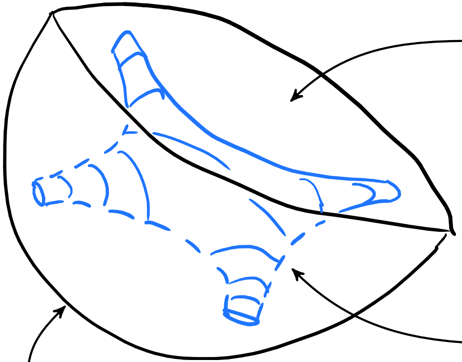
$$A^{(0)}(S,T) = \frac{1}{STU} + \sum_{a,b=0}^{\infty} (S^2+T^2+U^2)^a (STU)^b \alpha^{(0)}_{a,b}$$

$$= \underbrace{\frac{1}{STU}}_{\text{sugra}} + \underbrace{\alpha^{(0)}_{0,0}}_{R^4} + \underbrace{(S^2+T^2+U^2)\alpha^{(0)}_{1,0}}_{D^4R^4} + \underbrace{(STU)\alpha^{(0)}_{0,1}}_{D^6R^6} + \dots$$

Wilson coefficients $\alpha^{(0)}_{a,b}$ are in the ring of single-valued multiple zeta values [Stieberger;2013],[Brown,Dupont;Schlotterer,Schnetz;Vanhove,Zerbini;2018]

Example: $\alpha^{(0)}_{a,0} = \zeta(3+2a)\,, \qquad \alpha^{(0)}_{a,1} = \sum_{\substack{i_1,i_2=0 \\ i_1+i_2=a}}^{a} \zeta(3+2i_1)\zeta(3+2i_2)$

2. String scattering in AdS



5d bulk of AdS:
scattering amplitude of 4 gravitons:

$$A(S,T) = \sum_{k=0}^{\infty} \left(\frac{L_s^2}{R_{\mathrm{AdS}}^2}\right)^k A^{(k)}(S,T)$$

4d boundary of AdS:
correlator of 4 stress-tensors:

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$$

2d string world-sheet:
world-sheet integrand:

$$G_{\mathrm{tot}}(S,T,z) = \sum_{k=0}^{\infty} \left(\frac{L_s^2}{R_{\mathrm{AdS}}^2}\right)^k G_{\mathrm{tot}}^{(k)}(S,T,z)$$

Boundary correlator to bulk amplitude

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$$

$\downarrow$ superconformal Ward identity

$$H(U,V) \qquad U=\frac{(x_1-x_2)^2(x_3-x_4)^2}{(x_1-x_3)^2(x_2-x_4)^2},\; V=\frac{(x_1-x_4)^2(x_2-x_3)^2}{(x_1-x_3)^2(x_2-x_4)^2}$$

$\downarrow$ Mellin transform

$$M(s,t)$$

$\downarrow$ Borel transform (flat space limit [Penedones;2010])

$$A(S,T)=\sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{\lambda}}\right)^k A^{(k)}(S,T)$$

$\downarrow$ world-sheet integral

$$A^{(k)}(S,T)=\int d^2z\;|z|^{-2S-2}|1-z|^{-2T-2}G^{(k)}_{\text{tot}}(S,T,z)$$

The Mellin transform

Mellin transform

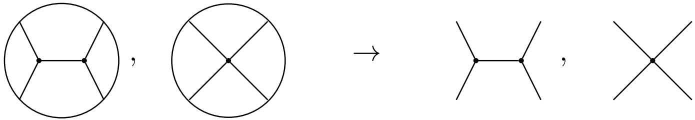
$$H(U,V) = \int_{-i\infty}^{i\infty} \frac{ds dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{t}{2}-2} \Gamma\left(2-\frac{s}{2}\right)^2 \Gamma\left(2-\frac{t}{2}\right)^2 \Gamma\left(2-\frac{u}{2}\right)^2 M(s,t)$$

The Borel transform

Borel transform

$$A(S,T) = \lambda^{\frac{3}{2}} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i}\, e^{\alpha} \alpha^{-6} M\left(\frac{2\sqrt{\lambda}S}{\alpha}, \frac{2\sqrt{\lambda}T}{\alpha}\right)$$

1. Maps Witten diagrams to Feynman diagrams for $R_{\text{AdS}} \to \infty$ [Penedones;2010]



2. Borel summation of the low energy expansion:

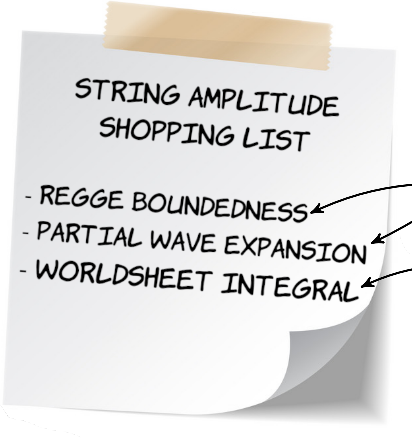
$$M(s,t) = \sum_{p,q} \frac{\Gamma(6+p+q)}{\lambda^{\frac{3}{2}}} \left(\frac{s}{2\sqrt{\lambda}}\right)^p \left(\frac{t}{2\sqrt{\lambda}}\right)^q \alpha_{p,q} \quad \Rightarrow \quad A(S,T) = \sum_{p,q} S^p T^q \alpha_{p,q}$$

3. Stringy flat space limit:

$$\sqrt{\lambda} = \frac{R_{\text{AdS}}^2}{L_s^2} >> 1\,, \qquad S \sim \frac{L_s^2}{R_{\text{AdS}}^2} s \sim L_s^2 (p_1 + p_2)^2 \text{ finite}$$

Plan of attack

We attack the problem from 2 sides:



Dispersion relation for $M(s,t)$

Single-valued ansatz for $G(S,T,z)$

Both have unfixed data.
Equating the two expressions fixes the answer!

2.1. The CFT dispersion relation

Operator product expansion

We can expand $\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$ using:

Operator product expansion (OPE)

$$\mathcal{O}_2(x)\mathcal{O}_2(0) = \sum_{\mathcal{O}_{\Delta,\ell} \text{ primaries}} C_{\Delta,\ell}\ c_{\Delta,\ell}(x,\partial_y)\mathcal{O}_{\Delta,\ell}(y)\big|_{y=0}$$

OPE data

- $\Delta =$ dimension
- $\ell =$ spin
- $C_{\Delta,\ell} =$ OPE coefficients

$M(s,t)$ has only simple poles, given by [Mack;2009], [Penedones,Silva,Zhiboedov;2019]

Poles and residues of $M(s,t)$

$$M(s,t) \sim \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s-(\Delta-\ell+2m)}$$

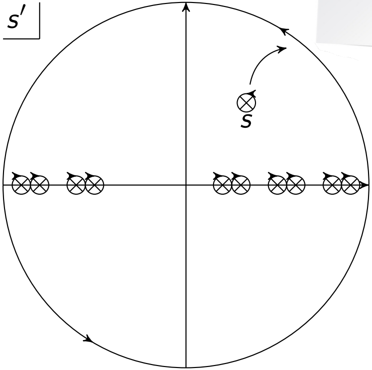
Dispersion relation

$M(s,t)$ has only OPE poles:

$$\text{poles} \;\sim\; \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s' - (\Delta - \ell + 2m)}$$

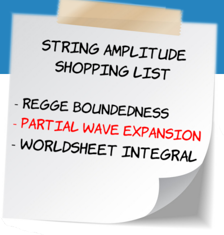
Regge bounded due to bound on chaos:
[Maldacena,Shenker,Stanford;2015]

$$\lim_{|s|\to\infty} |M(s,t)| \lesssim |s|^{-2}, \;\; \mathrm{Re}(t) < 2$$



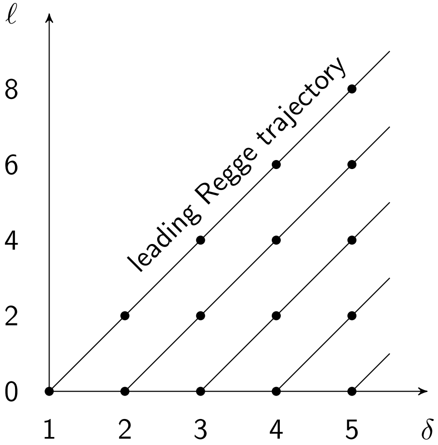
$$M(s,t) = \oint_s \frac{ds'}{2\pi i} \frac{M(s',t)}{(s'-s)} = -\sum_{\text{operators}} \left(\frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s - (\Delta - \ell + 2m)} + \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{u - (\Delta - \ell + 2m)} \right)$$

Spectrum of exchanged operators



Exchanged operators: massive string modes
$=$ unprotected single-trace operators of $\mathcal{N}=4$ SYM theory

$$\Delta(\Delta-d)=R^2m^2=R^2\frac{4\delta}{L_s^2}+O(\lambda^0) \quad\Rightarrow\quad \Delta=2\sqrt{\delta}\lambda^{\frac14}+O(\lambda^0)$$



known from flat space

$$\Delta_{\delta,\ell} = \boxed{\lambda^{\frac14}\Delta^{(0)}_{\delta,\ell}} + \boxed{\lambda^{-\frac14}\Delta^{(1)}_{\delta,\ell}} + \boxed{\lambda^{-\frac34}\Delta^{(2)}_{\delta,\ell}}$$

$$C^2_{\delta,\ell} = \boxed{C^{2(0)}_{\delta,\ell}} + \boxed{\lambda^{-\frac12}C^{2(1)}_{\delta,\ell}} + \boxed{\lambda^{-1}C^{2(2)}_{\delta,\ell}}$$

$A^{(0)}$ data	$A^{(1)}$ data	$A^{(2)}$ data

$\Delta^{(1)}_{\delta,\ell}$, $\Delta^{(2)}_{\delta,\ell}$ on leading trajectory known from integrability

Degeneracies in the spectrum

The amplitude encodes OPE data of multiple degenerate superprimaries.

Find degeneracies starting from type IIb strings in flat 10d:
[Bianchi,Morales,Samtleben;2003],[Alday,TH,Silva;2023]

$$SO(9) \rightarrow SO(4) \times SO(5) \overset{KK}{\rightarrow} SO(4) \times SO(6)$$

ℓ \ δ	1	2	3	4	5	6
						1
8					1	6
6				1	6	52
4			1	6	40	331
2		1	4	24	157	1104
0	1	2	6	22	99	547

Number of superconformal long multiplets with superprimary $\mathcal{O}_{\delta,\ell}$

- $SO(6)$ singlet
- $\Delta = 2\sqrt{\delta}\lambda^{\frac{1}{4}} + O(\lambda^0)$

Example: $\mathcal{O}_{1,0} = \text{Konishi} \sim \text{Tr}(\phi^I \phi_I)$

The counting was confirmed for $\delta \leq 3$ with quantum spectral curve.
[Gromov,Hegedus,Julius,Sokolova;2023]

Dispersion relation → Residues

Dispersion relation for $M(s,t) \rightsquigarrow A^{(k)}(S,T)$ expanded around $S=\delta=1,2,\ldots$:

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T,\delta,C^{2(0)}_{\delta,\ell})}{(S-\delta)^{3k+1}} + \ldots + \frac{R^{(k)}_{1}(T,\delta,C^{2(0)}_{\delta,\ell},\ldots,\Delta^{(k)}_{\delta,\ell},C^{2(k)}_{\delta,\ell})}{S-\delta} + \text{reg.}$$

Two lessons:

1. (OPE data)$^{(k-1)}$ fixes most residues of $A^{(k)}(S,T)$!
2. $G^{(k)}_{\text{tot}}(S,T,z)$ should have transcendentality $3k$:

$$\int d^2z\,|z|^{-2S-2}|1-z|^{-2T-2}\log^{3k}|z|^2 \propto \frac{1}{(S-\delta)^{3k+1}} + O\left((S-\delta)^0\right)$$

Next steps (order by order):

- Write world-sheet ansatz for $A^{(k)}(S,T)$.
- Compute its residues and match with the above to fix ansatz.

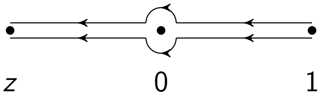
2.2. Single-valued functions for the world-sheet

Single-valued functions

Example for multivalued function:

$$\log z = \int_\sigma \frac{dt}{t}\,, \qquad \sigma = \text{path from 1 to z}$$

Depends on integration path:



$\Rightarrow \log z$ is well defined up to

$$n \oint_0 \frac{dt}{t} = 2\pi i n\,, n \in \mathbb{Z}$$

Single-valued version:

$$\mathcal{L}_0(z) \equiv 2\,\mathrm{Re}(\log z) = \log z + \log \bar{z} = \log |z|^2$$

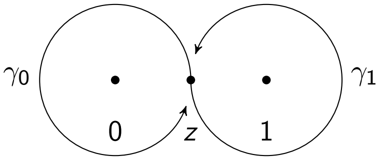
Smaller function space $\rightarrow$ constraining power of imposing single-valuedness:

$$\begin{aligned} \text{multi-valued}: &\ \log z, \log \bar{z} \\ \text{single-valued}: &\ \log |z|^2 \end{aligned}$$

Dilogarithm:

$$\mathrm{Li}_2(z) = \int_{0\le t_1\le t_2\le z} \frac{dt_1}{t_1-1}\frac{dt_2}{t_2}$$

M_x = analytic continuation along γ_x



$$M_1\mathrm{Li}_2(z) = \mathrm{Li}_2(z) + 2\pi i \log z$$
$$M_0\mathrm{Li}_2(z) = \mathrm{Li}_2(z)$$

Single-valued version:

$$\mathcal{L}_{01}(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(1-\bar{z})\log|z|^2$$
$$M_1\mathcal{L}_{01}(z) = M_0\mathcal{L}_{01}(z) = \mathcal{L}_{01}(z)$$

For arbitrary iterated integrals [Brown;2004] :

$\mathcal{L}_{abc\ldots}(z)$	single-valued multiple polylogarithms (SVMPLs)
$\mathcal{L}_{abc\ldots}(1)$	single-valued multiple zeta values (SVMZVs)

e.g. $\mathrm{Li}_2(1) = \zeta(2)$, $\mathcal{L}_{01}(1) = 0$, $\mathcal{L}_{001}(1) = -\zeta^{\mathrm{sv}}(3) = -2\zeta(3)$

Toy model for strings in AdS

Polyakov action:

$$S_P = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\,g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X)$$

$$= S_{\text{flat}} + \frac{1}{R^2_{\text{AdS}}}\underbrace{\lim_{q\to 0}\frac{\partial^2}{\partial q^\mu\partial q^\nu}V^{\mu\nu}_{\text{graviton}}(q)}_{\equiv\,\tilde V} + \dots$$

AdS metric expanded around flat space:

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{R^2_{\text{AdS}}} + \cdots$$

$$h_{\mu\nu} \sim X_\mu X_\nu \sim \lim_{q\to 0}\frac{\partial^2}{\partial q^\mu\partial q^\nu}e^{iq\cdot X}$$

Amplitude:

$$A_4(p_i) \sim \int \mathcal{D}X\mathcal{D}g\, e^{-S_P}V^4_{\text{graviton}} = \int \mathcal{D}X\mathcal{D}g\, e^{-S_{\text{flat}}}\left(1 - \frac{\tilde V}{R^2_{\text{AdS}}} + \frac{1}{2}\frac{\tilde V^2}{R^4_{\text{AdS}}} + \dots\right)V^4_{\text{graviton}}$$

$$\Rightarrow \quad A_4^{(k)}(p_i) \sim \lim_{q_i\to 0}\left(\frac{\partial}{\partial q_i}\right)^{2k} A^{(0)}_{4+k}(p_i, q_i) + \dots$$

Soft gravitons in flat space

$$A_4^{(k)}(p_i) \sim \lim_{\epsilon\to 0}\left(\frac{\partial}{\epsilon\,\partial q_i}\right)^{2k} A_{4+k}^{(0)}(p_i,\epsilon q_i) + \dots$$

Soft graviton theorem:

$$A_{n+1}(p_1,\dots,p_n,\epsilon q) = \sum_{i=1}^{n}\left(\frac{1}{\epsilon}\frac{\varepsilon_{\mu\nu}p_i^\mu p_i^\nu}{p_i\cdot q} + \frac{\varepsilon\cdot p_i\varepsilon_\mu q_\nu J_i^{\mu\nu}}{p_i\cdot q} + O(\epsilon)\right) A_n(p_1,\dots,p_n)$$

Flat space amplitude with k soft gravitons:

$$A_{4+k}^{(0)}(p_i,\epsilon q_i) \sim \frac{1}{\epsilon^k}A_4^{(0)}(p_i) + \frac{1}{\epsilon^{k-1}}\text{"}\partial_{p_i}\text{"}A_4^{(0)}(p_i) + \dots$$

$$\sim \int d^2z|z|^{-2S-2}|1-z|^{-2T-2}\left(\frac{1}{\epsilon^k} + \frac{1}{\epsilon^{k-1}}\left(\#\log|z|^2 + \#\log|1-z|^2\right) + \dots + \epsilon^{2k}\mathcal{L}_{|w|=3k}(z)\right)$$

$$\Rightarrow \quad G_{\text{tot}}^{(k)}(S,T,z) \ \sim\ \text{single-valued multiple polylogs of weight } \leq 3k$$

World-sheet correlator (ansatz)

Ansatz:

$$A^{(k)}(S,T) = B^{(k)}(S,T) + B^{(k)}(U,T) + B^{(k)}(S,U)$$

$$B^{(k)}(S,T) = \int d^2z|z|^{-2S-2}|1-z|^{-2T-2}G^{(k)}(S,T,z)$$

Assumed properties of $G^{(k)}(S,T,z)$:

- uniform transcendentality $3k$ (SVMPLs(z), SVMZVs)
- rational function in S, T with homogeneity $2k-2$
- denominator $= U^n$, $n \leq 2$
- crossing symmetry: $G^{(k)}(S,T,z) = G^{(k)}(T,S,1-z)$

Recall (flat space):

$$G^{(0)}(S,T,z) = \frac{1}{3U^2}$$

World-sheet correlator (solution)

STRING AMPLITUDE SHOPPING LIST

- REGGE BOUNDEDNESS
- PARTIAL WAVE EXPANSION
- WORLDSHEET INTEGRAL

Symmetrised single-valued multiple polylogs:

$$\mathcal{L}^{\pm}_{w}(z) = \mathcal{L}_w(z) \pm \mathcal{L}_w(1-z) + \mathcal{L}_w(\bar{z}) \pm \mathcal{L}_w(1-\bar{z})$$

$$k = 1: \quad \text{weight 3 basis} = \text{4 symmetric} + \text{3 antisymmetric functions}$$

Solution:

$$\begin{aligned} G^{(1)}(S,T,z) = & -\frac{1}{6}\mathcal{L}^{+}_{000}(z) + 0\mathcal{L}^{+}_{001}(z) - \frac{1}{4}\mathcal{L}^{+}_{010}(z) + 2\zeta(3) \\ & + \frac{S-T}{S+T}\left(-\frac{1}{6}\mathcal{L}^{-}_{000}(z) + \frac{1}{3}\mathcal{L}^{-}_{001}(z) + \frac{1}{6}\mathcal{L}^{-}_{010}(z)\right) \end{aligned}$$

$$k = 2: \quad \text{weight 6 basis} = \text{25 symmetric} + \text{20 antisymmetric functions}$$

Solution:

$$G^{(2)}(S,T,z) = \frac{ST - S^2 - T^2}{18}\mathcal{L}^{+}_{000000}(z) + \text{ too many terms}$$

2.3. Checks

OPE data

We compute $\forall \delta, \ell \qquad \# \in \mathbb{Q}$

$$k = 0: \qquad \langle C^{2(0)} \rangle_{\delta,\ell} = \#$$

$$k = 1: \qquad \sqrt{\delta}\langle C^{2(0)} \Delta^{(1)} \rangle_{\delta,\ell} = \#\,, \quad \langle C^{2(1)} \rangle_{\delta,\ell} = \#\zeta(3) + \#$$

$$k = 2: \qquad \langle C^{2(0)} (\Delta^{(1)})^2 \rangle_{\delta,\ell} = \#$$

$$\sqrt{\delta}\langle C^{2(0)} \Delta^{(2)} + C^{2(1)} \Delta^{(1)} \rangle_{\delta,\ell} = \#\zeta(3) + \#$$

$$\langle C^{2(2)} \rangle_{\delta,\ell} = \#\zeta(3)^2 + \#\zeta(5) + \#\zeta(3) + \#$$

Leading Regge trajectory:

$$\Delta\left(\tfrac{\ell}{2} + 1, \ell\right) = 2\sqrt{\tfrac{\ell}{2} + 1}\lambda^{\frac{1}{4}} - 2 + \frac{3\ell^2 + 10\ell + 16}{4\sqrt{2(\ell+2)}}\lambda^{-\frac{1}{4}}$$
$$- \frac{21\ell^4 + 144\ell^3 + 292\ell^2 + 80\ell - 128 + 96(\ell+2)^3\zeta(3)}{32(2(\ell+2))^{\frac{3}{2}}}\lambda^{-\frac{3}{4}} + O(\lambda^{-\frac{5}{4}})\,,$$

Agrees with integrability result!

[Gromov,Serban,Shenderovich,Volin;2011],[Basso;2011],[Gromov,Valatka;2011]

Wilson coefficients

$$A^{(k)}(S,T) = \mathrm{SUGRA}^{(k)} + \sum_{a,b=0}^{\infty} (S^2+T^2+U^2)^a (STU)^b \alpha_{a,b}^{(k)}$$

We compute $\forall a, b$ $\qquad \# \in \mathbb{Q}$

$$\alpha_{a,b}^{(0)} = \sum_{k_i \text{ odd}} \#\zeta(k_1)\ldots\zeta(k_n)$$

$$\alpha_{a,b}^{(1)} = \sum_{k_i \text{ odd}} \#\zeta^{\mathrm{sv}}(k_1,k_2,k_3)\zeta(k_4)\ldots\zeta(k_n) + \ldots$$

$$\alpha_{a,b}^{(2)} = \sum_{k_i \text{ odd}} \#\zeta^{\mathrm{sv}}(k_1,k_2,k_3,k_4,k_5)\zeta(k_6)\ldots\zeta(k_n) + \ldots$$

In particular:

$$\alpha_{0,0}^{(1)} = 0\,, \quad \alpha_{1,0}^{(1)} = -\frac{22}{3}\zeta(3)^2\,, \quad \alpha_{0,0}^{(2)} = \frac{49}{4}\zeta(5)\,, \quad \alpha_{1,0}^{(2)} = \frac{4091}{16}\zeta(7)$$

Agrees with localisation result!
[Binder,Chester,Pufu,Wang;2019],[Chester,Pufu;2020],[Alday,TH,Silva;2022]

The low energy expansion ($S \sim T \sim 0$)
can be computed following [Vanhove,Zerbini;2018]

$$\int d^2z|z|^{-2S-2}|1-z|^{-2T-2}\mathcal{L}_w(z)$$

$$= \text{poles} + \sum_{p,q=0}^{\infty}(-S)^p(-T)^q \int \frac{d^2z}{|z|^2|1-z|^2} \underbrace{\mathcal{L}_{0^p}(z)\mathcal{L}_{1^q}(z)\mathcal{L}_w(z)}_{=\sum_{W\in 0^p ⧢ 1^q ⧢ w}\mathcal{L}_W(z)}$$

$$= \text{poles} + \sum_{p,q=0}^{\infty}(-S)^p(-T)^q \sum_{W\in 0^p ⧢ 1^q ⧢ w} \underbrace{\mathcal{L}_{0W}(1) - \mathcal{L}_{1W}(1)}$$

$\boxed{\text{Single-valued multiple zeta values of weight } 1+p+q+|w|}$

As in flat space! [Stieberger;2013],[Brown,Dupont;Schlotterer,Schnetz;Vanhove,Zerbini;2018]

3. High energy limit

Why the high energy limit?

What is the next step towards the world-sheet theory?

Flat space [Gross,Mende;1987]:

classical solution (bosonic) of the world-sheet theory	$\rightarrow$	high energy limit ($S, T \rightarrow \infty$) of string amplitudes

An independent way to compute $\lim_{S,T\to\infty} A(S,T)$, agnostic to many details!

High energy limit via saddle point

The high energy limit of $A^{(0)}(S,T)$ is given by the saddle point $z=\bar{z}=\dfrac{S}{S+T}$

$$\lim_{S,T\to\infty}\int d^2z\,|z|^{-2S}|1-z|^{-2T}\sim e^{-2S\log|\frac{S}{S+T}|-2T\log|\frac{T}{S+T}|}$$

In AdS the limit can be computed in the same way.

Goal: Compute this exponent from the string action.

Classical solution in flat space

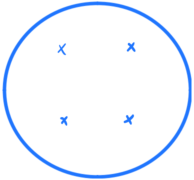
Path integral for the amplitude:

$$A_4^{\text{flat}}(S,T) \sim \int \mathcal{D}X\, e^{-\mathcal{S}(X^\mu)} = \int \mathcal{D}X \exp\left(-\int d^2\zeta \partial X^\mu(\zeta)\bar{\partial}X_\mu(\zeta)\right)\prod_{j=1}^{4}\int d^2 z_j\, e^{ip_j\cdot X(z_j)}$$

$$\Rightarrow \qquad \mathcal{S}(X^\mu) = \int d^2\zeta \left(\partial X^\mu(\zeta)\bar{\partial}X_\mu(\zeta) - i\sum_{j=1}^{4} p_j\cdot X(\zeta)\,\delta^{(2)}(\zeta - z_j)\right)$$

EOM: $$\partial\bar{\partial}X^\mu = -\frac{i}{2}\sum_j p_j^\mu \delta^{(2)}(\zeta - z_j)$$

Solution: $$X^\mu_{\text{clas}} = -i\sum_j p_j^\mu \log|\zeta - z_j|$$



This classical solution gives the correct high energy exponent:

$$\lim_{S,T\to\infty} A_4^{\text{flat}}(S,T) \sim e^{-\mathcal{S}(X^\mu_{\text{clas}})}\Big|_{z=\frac{S}{S+T}} = e^{-2S\log|\frac{S}{S+T}|-2T\log|\frac{T}{S+T}|}$$

The AdS path integral

The action for AdS:

$$\mathcal{S}(X,\Lambda) = \int d^2\zeta \left(\partial X^M \bar{\partial} X_M + \Lambda (X^M X_M + R^2) - i \sum_{j=1}^{4} P_j^M X_M \delta^{(2)}(\zeta - z_j) \right)$$

AdS_d is embedded in $\mathbb{R}^{2,d-1} \ni X^M$

$$-R^2 = X^M X_M = -X^0 X^0 + X^\mu X_\mu$$

Eliminate X^0 and expand X^μ around flat space:

$$X^\mu = X_0^\mu + \frac{1}{R^2} X_1^\mu + \ldots$$

Equation of motion for X_1^μ:

$$\partial\bar{\partial} X_1^\mu = \partial X_0 \cdot \bar{\partial} X_0 \, X_0^\mu = \frac{i}{4} \sum_{i,j,k} \frac{p_i \cdot p_j}{(\zeta - z_i)(\bar{\zeta} - z_j)} p_k^\mu \log \left| 1 - \frac{\zeta}{z_k} \right|$$

Equation of motion for X_1^μ:

$$\partial\bar{\partial}X_1^\mu = \partial X_0 \cdot \bar{\partial} X_0 \, X_0^\mu = \frac{i}{8}\sum_{i,j,k} \frac{p_i \cdot p_j}{(\zeta - z_i)(\bar{\zeta} - z_j)} p_k^\mu \mathcal{L}_{z_k}(\zeta)$$

We can "integrate" this using

$$\int d\zeta \frac{\mathcal{L}_w(\zeta)}{\zeta - z_i} \to \mathcal{L}_{z_i w}(\zeta)\,, \qquad \int d\bar{\zeta} \frac{\mathcal{L}_w(\zeta)}{\bar{\zeta} - z_j} \to \mathcal{L}_{w z_j}(\zeta) + \cdots$$

Result:

$$X_{1,\mathrm{clas}}^\mu = \frac{i}{8}\sum_{i,j,k=1}^{4} p_i \cdot p_j \; p_k^\mu \left(\mathcal{L}_{z_i z_k z_j}(\zeta) + \mathcal{L}_{z_k}(z_j)\mathcal{L}_{z_i z_j}(\zeta) - \mathcal{L}_{z_j}(z_k)\mathcal{L}_{z_i z_k}(\zeta)\right)$$

More generally:

$$X_{\mathrm{clas}}^\mu = \mathcal{L}_{|w|=1}(\zeta) + \frac{1}{R^2}\mathcal{L}_{|w|=3}(\zeta) + \frac{1}{R^4}\mathcal{L}_{|w|=5}(\zeta) + \ldots$$

Comparison with AdS Virasoro-Shapiro amplitude

$$e^{-\mathcal{S}(X^\mu_{\text{clas}})}\Big|_{z=\frac{S}{S+T}} = \exp\left(-S\mathcal{S}^{(0)}\left(\tfrac{S}{T}\right) - \frac{S^2}{R^2}\mathcal{S}^{(1)}\left(\tfrac{S}{T}\right) - \frac{S^3}{R^4}\mathcal{S}^{(2)}\left(\tfrac{S}{T}\right) - \ldots\right)$$

In the limit $S, T, R \to \infty$ with S/T and S/R fixed, $\mathcal{S}^{(2)}$ and further terms vanish!

We succesfully compare with AdS Virasoro-Shapiro at the saddle point:

$$e^{-\frac{S^2}{R^2}\mathcal{S}^{(1)}\left(\frac{S}{T}\right)} = 1 + \frac{U^2}{R^2}G^{(1)}_{\text{tot}}(z = \tfrac{S}{S+T}) + \frac{U^2}{R^4}G^{(2)}_{\text{tot}}(z = \tfrac{S}{S+T}) + \ldots$$

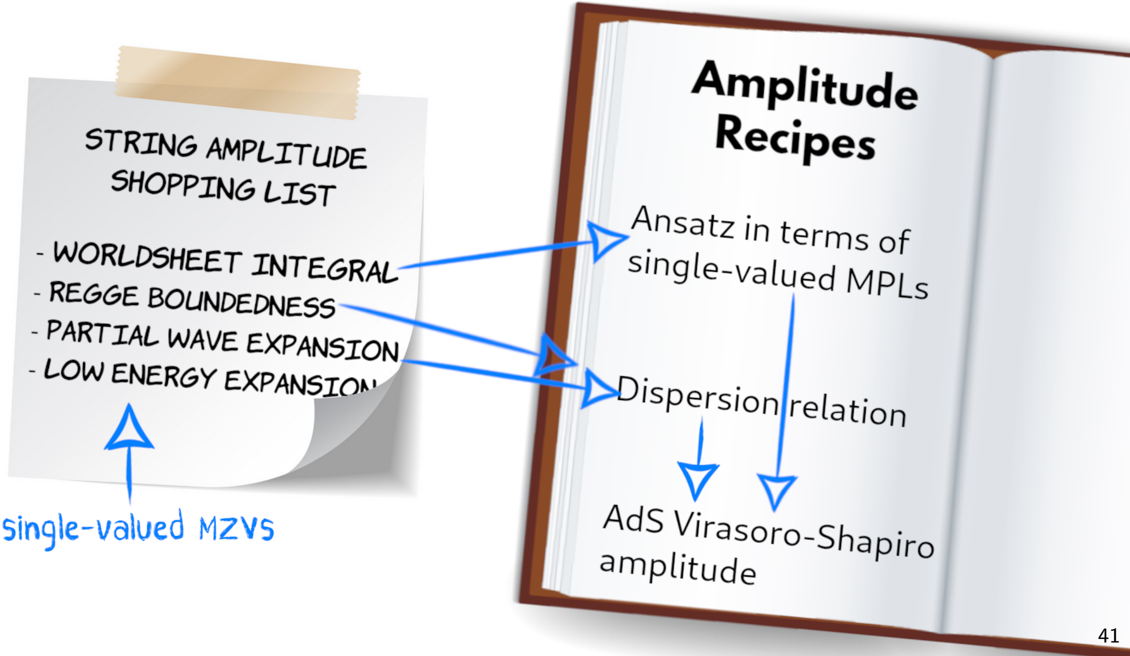
This implies

$$U^2 G^{(2)}_{\text{tot}}(z = \tfrac{S}{S+T}) = \frac{1}{2}\left(U^2 G^{(1)}_{\text{tot}}(z = \tfrac{S}{S+T})\right)^2$$

Final result to all orders in S/R:

$$\lim_{S,T,R\to\infty} A_4^{\text{AdS}}(S,T) = \left(\lim_{S,T\to\infty} A_4^{\text{flat}}(S,T)\right) e^{-\frac{S^2}{R^2}\mathcal{S}^{(1)}\left(\frac{S}{T}\right)}$$

Summary



STRING AMPLITUDE SHOPPING LIST
- WORLDSHEET INTEGRAL
- REGGE BOUNDEDNESS
- PARTIAL WAVE EXPANSION
- LOW ENERGY EXPANSION
single-valued MZVs
Amplitude Recipes
Ansatz in terms of single-valued MPLs
Dispersion relation
AdS Virasoro-Shapiro amplitude

Summary: High energy limit

We compared $A(S, T)$ to classical computation a la Gross & Mende.

- Relation to world-sheet action agnostic to fermions and prefactors
- Confirmation for formal definition of $A(S, T)$
  (Borel transform of Mellin transform of CFT correlator)
- $A(S, T)$ fixed to all orders in S/R

$$\lim_{S,T,R\to\infty} A_4^{\text{AdS}}(S, T) = \left(\lim_{S,T\to\infty} A_4^{\text{flat}}(S, T)\right) e^{-\frac{S^2}{R^2}\mathcal{S}^{(1)}\left(\frac{S}{T}\right)}$$

Future directions

- Open strings / AdS Veneziano amplitude
  - Gluon scattering on $AdS_5 \times S^3$ / $4d\ \mathcal{N} = 2$ SCFT
  - Generalizations of KLT relations / double copy?
- Other backgrounds
  - e.g. type IIA on $AdS_4 \times CP^3$ / ABJM
- Compute $A^{(k)}(S, T)$ directly from string theory?
  - Ramond-Ramond background flux...
  - String field theory?
  - Pure spinors?

Thank you!