Deriving the Simplest Gauge/String Duality

with Rajesh Gopakumar
- Part I: Open-Closed-Open Triality (arXiv: 2212.05999)
- Part II: The B-Model
- Part III: The A-Model

Today's Focus:
Moments in the Gaussian Matrix Model as Stringy Correlators via OCO-Triality

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The Big Picture

Holography as Open/Closed String Duality

Can we make
't Hooft's insight
precise?

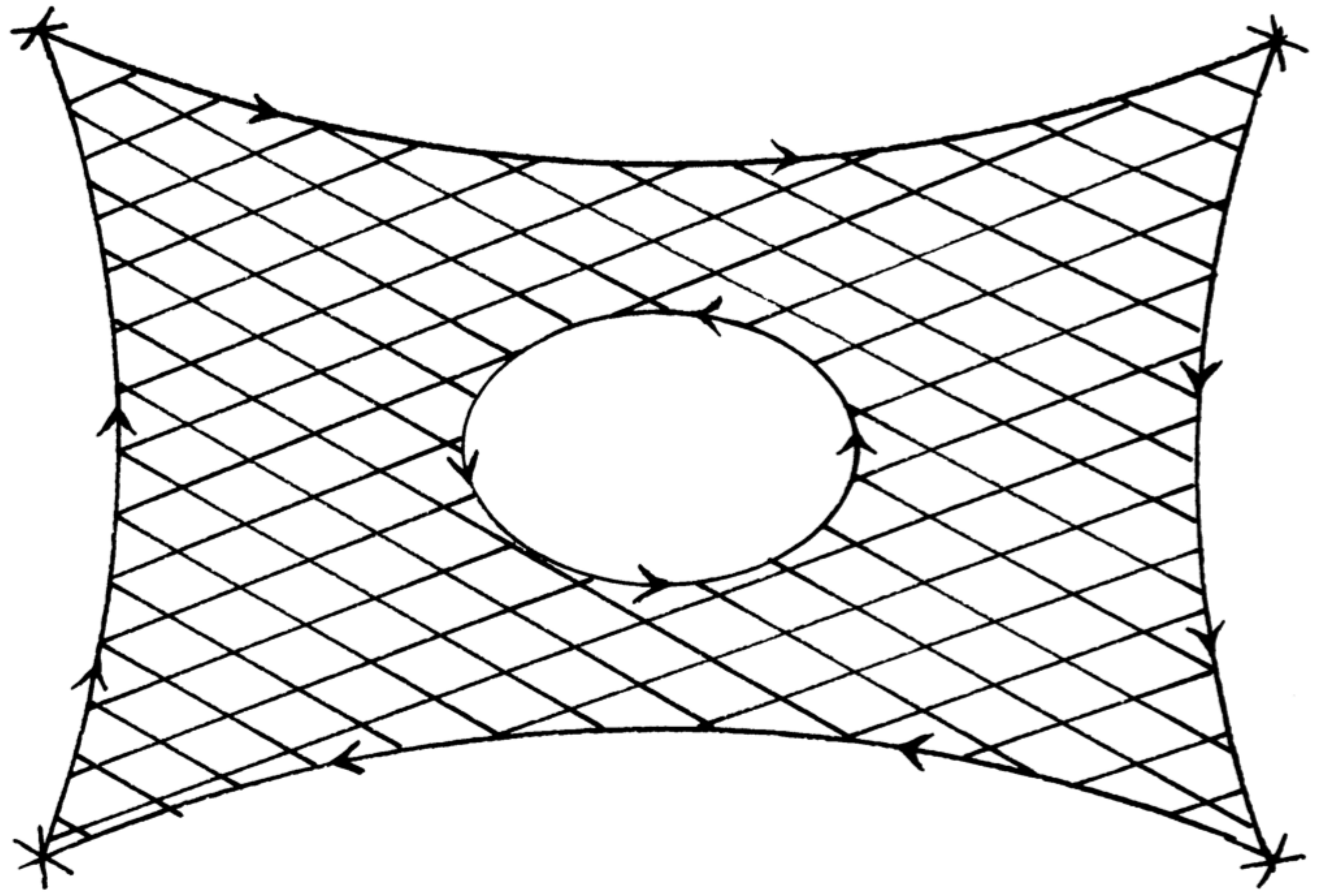
Closed String Description
Riemann surfaces with marked points
e.g. IIB on $AdS_5 \times S^5$

Open String Description
Matrices
e.g. $\mathcal{N} = 4$ *SYM*

Low-energy "Effective" Description
GR + QFT
e.g. QG in Anti-de Sitter

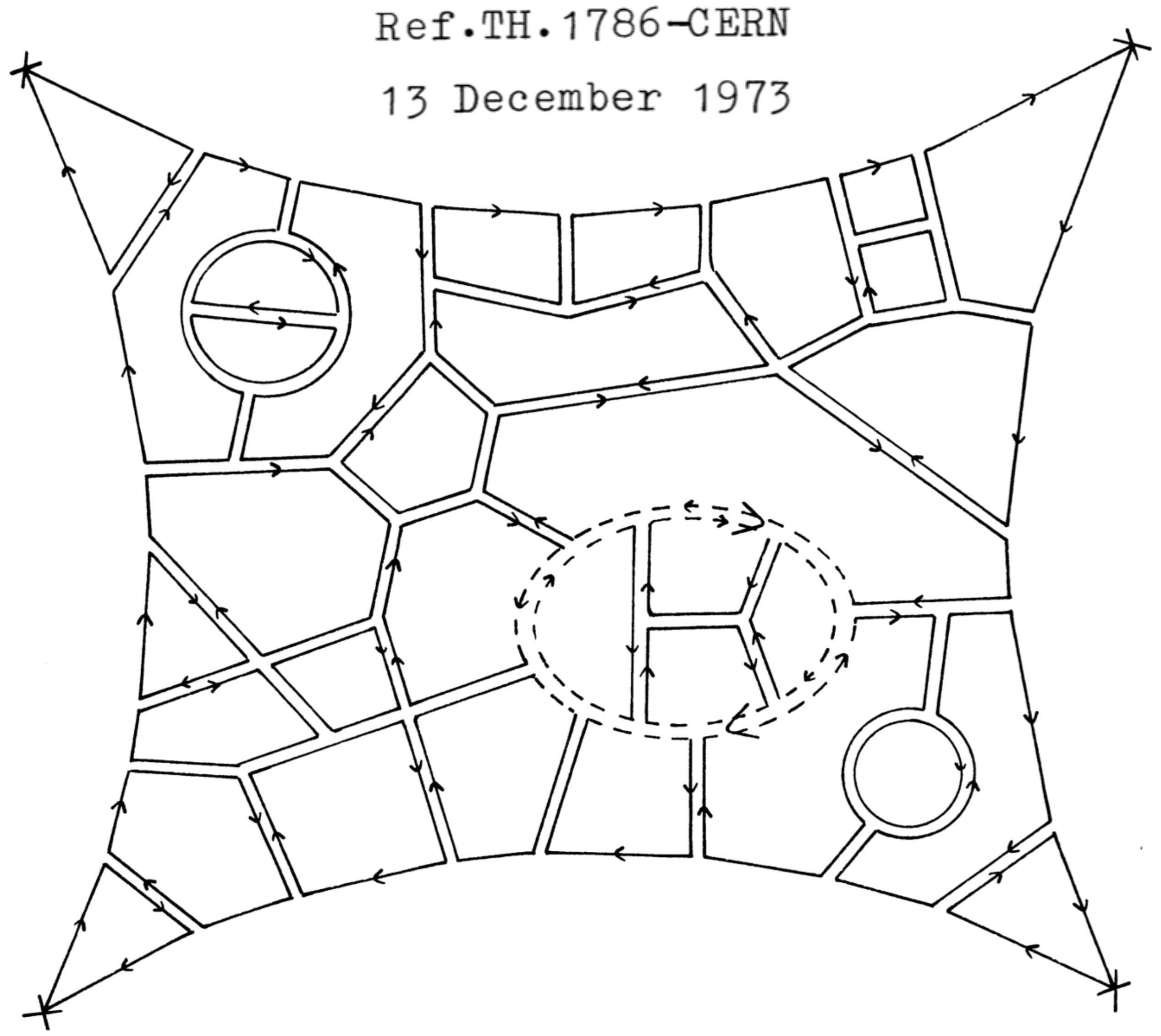
$$\int dM_{N\times N}\; e^{-\frac{N}{g}TrV(M)} \prod_{i=1}^{n} TrM^{k_i}$$

[BIPZ'80]



(a)

Ref.TH.1786-CERN
13 December 1973



- Figure 3 -

A PLANAR DIAGRAM THEORY FOR STRONG INTERACTIONS

G. 't Hooft
CERN - Geneva

Matrix Models and Strings

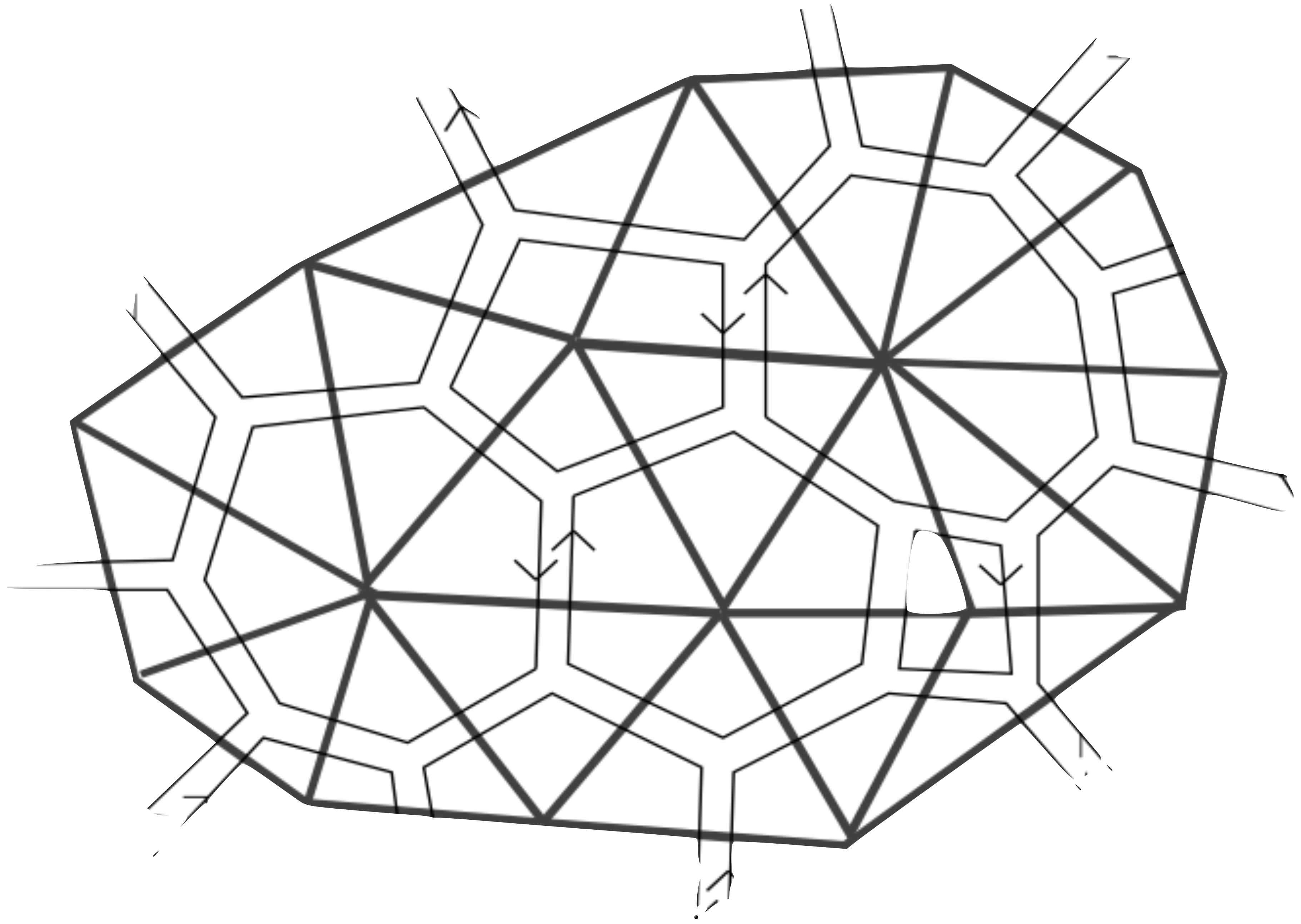
What is new here?

- **Previously focused on double-scaling limit**: Feynman diagrams as “latticization” of the worldsheet

  [Cf. Gross-Migdal; Douglas-Shenker; Brezin-Kazakov]

Not ‘t Hooft limit as in AdS/CFT!

- **Dijkgraaf-Vafa**: matrix integrals from localized holomorphic Chern-Simons Theory (i.e. open string field theory on branes in non-compact CYs)



Their proposal: closed B-model string on spectral curve of matrix model?

No worldsheet theory

No operator dictionary/correlators

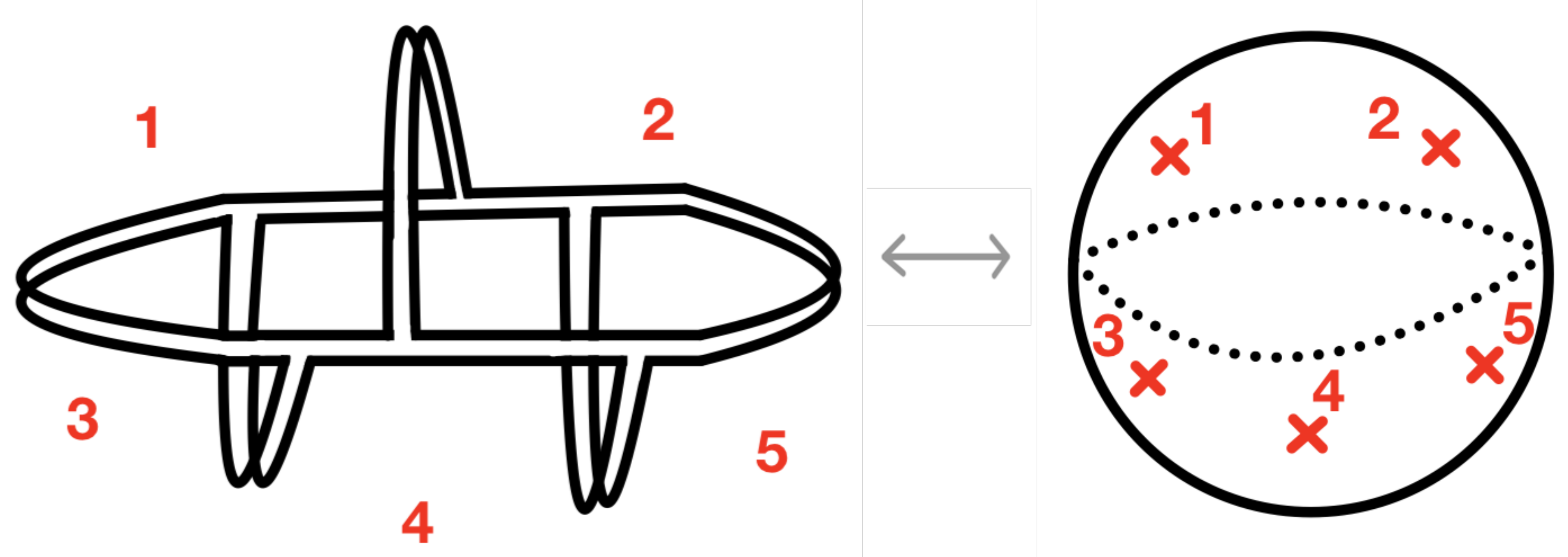
Three Main Takeaways

How do large N gauge theories reorganize themselves into closed string theories?

Use Strebel parametrization of $\mathscr{M}_{g,n} \times \mathbb{R}_+^n$ (via lengths of edges)

[Cf. Strebel, Kontsevich, Gopakumar]

"Each gauge theory FD as a string worldsheet"



What constitutes a derivation of open-closed duality?

We rewrite all correlators of traces exactly as integrals over $\mathscr{M}_{g,n}$

Open-Closed Operator dictionary: $\mathrm{Tr}M^k \leftrightarrow \mathcal{O}_k$

Can we identify a world-sheet theory which gives rise to these moduli-space integrals?

B-Model: Superpotential derived from matrix model spectral curve

A-Model: Twisted $\left(SL(2,\mathbb{R})/U(1)\right)_1$ coset model with momentum condensate

Today's Roadmap

The Proposal

An equality of matrix and string correlators

The Verification

OCO-Triality: 2 equivalent matrix models

OCO-Triality: the broader lessons

The Derivation

A-Model: Lattice points on $\mathscr{M}_{g,n}$, Belyi Maps & the BMN-limit

B-Model: Topological Recursion, Top. Matter + 2d gravity & the BMN-limit

The Proposal

Matrix Correlators as Stringy n-point Functions

Concrete String Dual to the Gaussian MM

An explicit all-genus mapping of correlators

A-Model Kazama-Suzuki

on $\frac{SL(2,\mathbb{R})_1}{U(1)}$

with momentum (+2) background

$$\langle \mathcal{V}_{k_1} \dots \mathcal{V}_{k_n} \rangle_g$$

⟷ ***Open/ Closed Duality***

Gaussian Matrix Model:

$$\left\langle \frac{1}{Nk_1} \mathrm{Tr} M^{k_1} \dots \frac{1}{Nk_n} \mathrm{Tr} M^{k_n} \right\rangle_g$$

⟷ ***Open/ Closed Duality***

B-Model top. L.G.
with superpotential

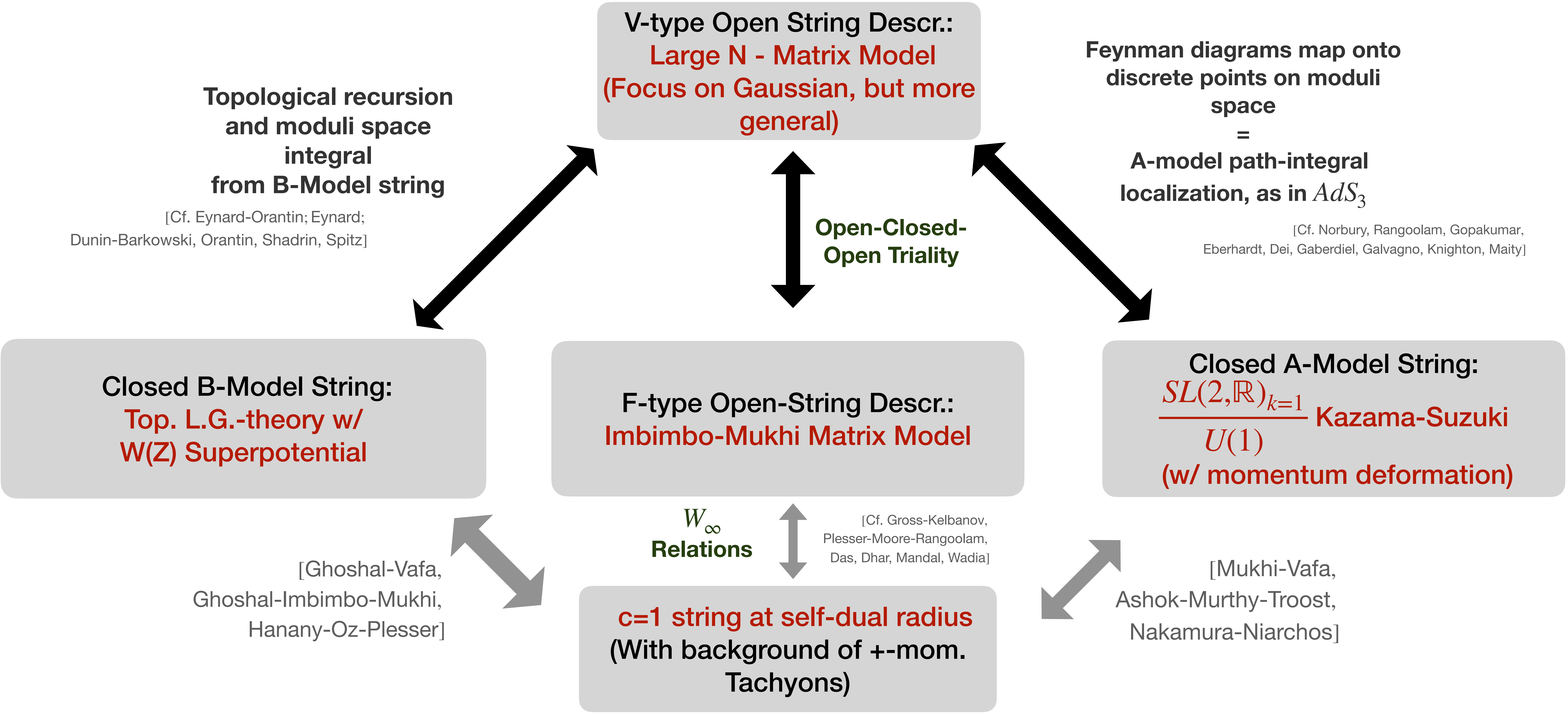
$$W(Z) = \frac{1}{Z} + t_2 Z$$

$$\langle \mathcal{T}_{-k_1}(Z) \dots \mathcal{T}_{-k_n}(Z) \rangle_g$$

(Interacting theories correspond to further deformations of background)

The Underlying Logic

A web of dualities & papers

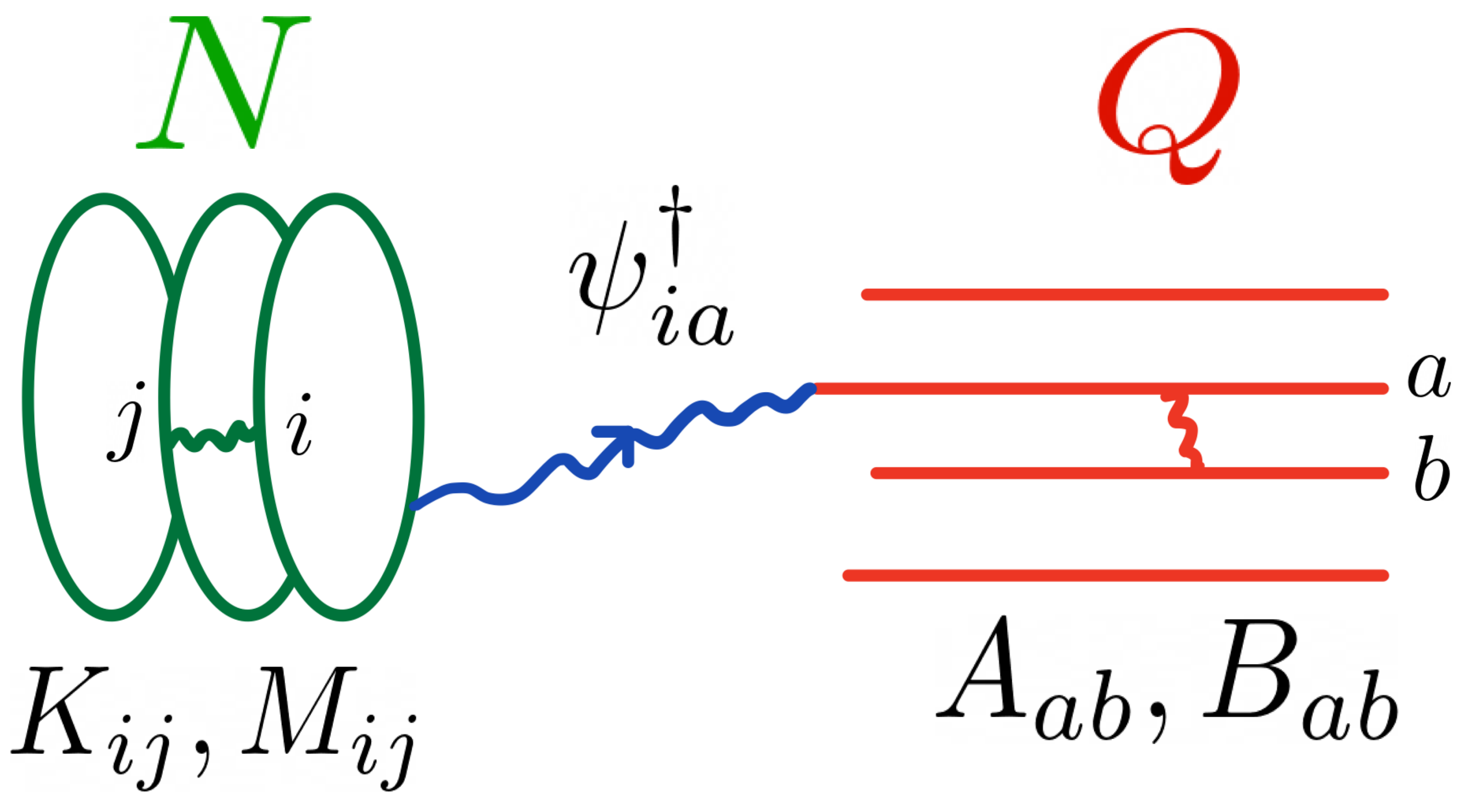


The Verification

An Equality of Matrix Integrals and the Appearance of the c=1 String

A New Equality of 2 Matrix Integrals

Open-Closed-Open Triality as Verification



$$Z(X,Y) = \frac{1}{Z_N}\int dKdM_{N\times N} e^{+\frac{1}{g}Tr\left(V(K) - K(M-Y)\right)} \prod_{a=1}^{Q} \det(x_a - M)$$
$$= \frac{(-1)^{NQ}}{Z_Q}\int dAdB_{Q\times Q} e^{-\frac{1}{g}Tr\left(V(A) + A(B-X)\right)} \prod_{i=1}^{N} \det(y_i - B)\,.$$

Key Steps

$$Z(X,Y) = \frac{1}{Z_N}\int dKd\psi d\psi^\dagger e^{\frac{1}{g}Tr_N\left(V(K)+KY\right)+\psi^\dagger_{ia}X_{ab}\psi_{ib}} \int dM e^{-\frac{1}{g}M_{ij}\left(K_{ji} - g\psi^\dagger_{ia}\psi_{ja}\right)}$$
$$= \int dKd\psi d\psi^\dagger e^{\frac{1}{g}Tr_N\left(V(K)+KY\right)+\psi^\dagger_{ia}X_{ab}\psi_{ib}}\delta\left(K_{ji} - g\psi^\dagger_{ia}\psi_{ja}\right)$$
$$= \int d\psi d\psi^\dagger e^{\frac{1}{g}Tr_N V[(-g\psi\psi^\dagger)] + \psi^\dagger_{ia}\left(X_{ab}\delta_{ij} - \delta_{ab}Y_{ij}\right)\psi_{jb}}$$

[Cf. Maldacena-Moore-Seiberg-Shih, Aganagic-Dijkgraaf-Klemm-Marino-Vafa, Goel - H. Verlinde, Altland-Sonner]

"Color-Flavor Transformation"

$$Tr_N\left[(\psi\psi^\dagger)^k\right] = (-1)^{2k-1}Tr_Q\left[(\psi^\dagger\psi)^k\right]$$

→ Reverse Steps

$$A_{ba} = -g\psi^\dagger_{ia}\psi_{ib}$$

The Imbimbo-Mukhi Matrix Model

Traces as tachyon modes in c=1 at self-dual radius

[Cf. "Kontsevich-Penner-Model", Chekhov et al., Bonora-Xiong, Moore-Plesser-Rangoolam]

$$\frac{1}{Z_N}\int dKdM_{N\times N}e^{-\frac{1}{g}\mathrm{Tr}\left(V_p(K)-K(M-Y)\right)}\prod_{a=1}^{Q}\det(x_a-M)$$
$$=$$
$$\frac{1}{Z_Q}\int dAdB_{Q\times Q}e^{+\frac{1}{g}\mathrm{Tr}\left(V_p(A)+A(B-X)\right)}\prod_{i=1}^{N}\det(y_i-B)$$

$$\frac{1}{Z_N}\int dKdM_{N\times N}e^{+NTr\left(V_p(K)-KM+\sum_{k=1}^{\infty}\bar{t}_kM^k\right)}$$
$$=$$
$$\det(X)^{-N}\int dA_{Q\times Q}e^{-NTr\left(V_p(A)-AX\right)-(N+Q)Tr\log(A)}\times\text{(Penner Model)}$$

$$Z_{IM}(t_k,\bar{t}_k)=\det(X)^{-i\mu}\int dA_{Q\times Q}e^{+i\mu\sum_{k=1}t_kTr(A^k)+i\mu Tr(AX)-(i\mu+Q)Tr\log(A)}$$

$N=i\mu$ Genus-expansion = large N expansion

$$\bar{t}_k=\frac{1}{k}Tr_Q\left(X^{-k}\right)$$

Exact Operator Dictionary:

$$\frac{1}{Nk}TrM^k\leftrightarrow\frac{\partial}{\partial\bar{t}_k}\leftrightarrow T_{-k}$$

Generating Function of "Tachyon" correlators in "c=1 2d-string theory" in large phase space

All Genus 1-pt Function

A detailed sanity check

Gaussian Matrix Model ↔ c=1 string at self-dual radius with momentum +2 tachyon-background

$$\langle \frac{1}{N} Tr M^{2n} \rangle_{Gaussian} = 2n \langle T_{-2n} \rangle_{t_2}$$

[Gopakumar-Mukhi ('95),
–unpublished]

$$= \frac{1}{N^{2n+1}} \frac{1}{2n+1} \oint dz z^{-N} e^{-\frac{N}{2} t_2 z^2} \partial_z^{2n+1} \left(z^N e^{+\frac{N}{2} t_2 z^2} \right)$$

How This Verifies Our Proposal

From c=1 to A- and B-model string

A-Model Kazama-Suzuki on $\frac{SL(2,\mathbb{R})_1}{U(1)}$

with momentum (+2) background

$$\langle \mathcal{V}_{k_1} \ldots \mathcal{V}_{k_n} \rangle_g$$

↔ ***Open/ Closed Duality*** ↔

Gaussian Matrix Model:

$$\left\langle \frac{1}{Nk_1} \mathrm{Tr} M^{k_1} \ldots \frac{1}{Nk_n} \mathrm{Tr} M^{k_n} \right\rangle_g$$

↔ ***Open/ Closed Duality*** ↔

B-Model top. L.G. with superpotential

$$W(Z) = \frac{1}{Z} + t_2 Z$$

$$\langle \mathcal{T}_{-k_1}(Z) \ldots \mathcal{T}_{-k_n}(Z) \rangle_g$$

$$T_{-k} \leftrightarrow \mathcal{V}_k = c e^{-\frac{(k-2)}{\sqrt{2}}\phi} e^{-i\frac{k}{\sqrt{2}}X}$$

$$T_{-k} \Leftrightarrow \mathcal{T}_{-k}(Z) \equiv \left(\frac{\partial}{\partial Z} W(Z,t)^k \right)_-$$

[Mukhi-Vafa, Ashok-Murthy-Troost, Nakamura-Niarchos]

$$\frac{1}{Nk} TrM^k \leftrightarrow T_{-k}$$

From duality with c=1 string

[Ghoshal-Vafa, Ghoshal-Imbimbo-Mukhi, Hanany-Oz-Plesser]

Role of double-scaling?

Imbimbo-Mukhi vs. Kontsevich

V-type
$N \times N$ Hermitian MM

"$N \to \infty$" Double-scaling

$\epsilon \to 0$

[Hashimoto-Huang-Klemm-Shih]

V-type
Double-scaled Hermitian MM

[$p = 2$: Cf. Maldacena-Moore-Seiberg-Shih]

OCO Triality

$$V_p(z) = \sum_{k=1}^{p} \frac{1}{k}(z+1)^k$$

$$g = \frac{1}{N} = \epsilon^{p+1}$$

$$A = -1 + \epsilon Z$$

$$X = \epsilon^p \tilde{X}$$

OCO Triality

F-type
$Q \times Q$
Imbimbo-Mukhi MM

F-type
$Q \times Q$
Kontsevich MM

$$Z_{IM} \propto \int dA_{Q\times Q} e^{-\frac{1}{g}Tr\left(V_p(A) - AX\right) - (N+Q)Tr\log(A)}$$

"N" Double-scaling
Q fixed!

$$Z_{Kontsevich} \propto \int dZ_{Q\times Q} e^{Tr\left(\frac{Z^{p+1}}{p+1} + Z\tilde{X}\right)}$$

[Gaiotto-Rastelli]

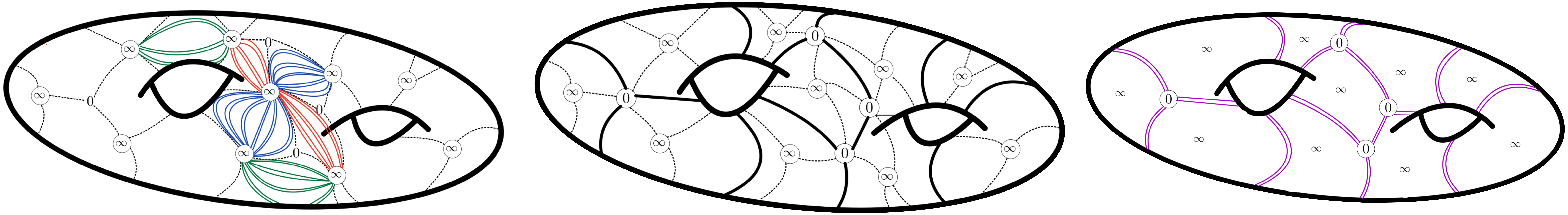
p=2: OSFT c=28 Liouville +c=-2 on Q FZZT

OCO-Triality beyond Matrix Models

2 ways to reconstruct closed strings from open ones

OCO-Triality as Graph Duality

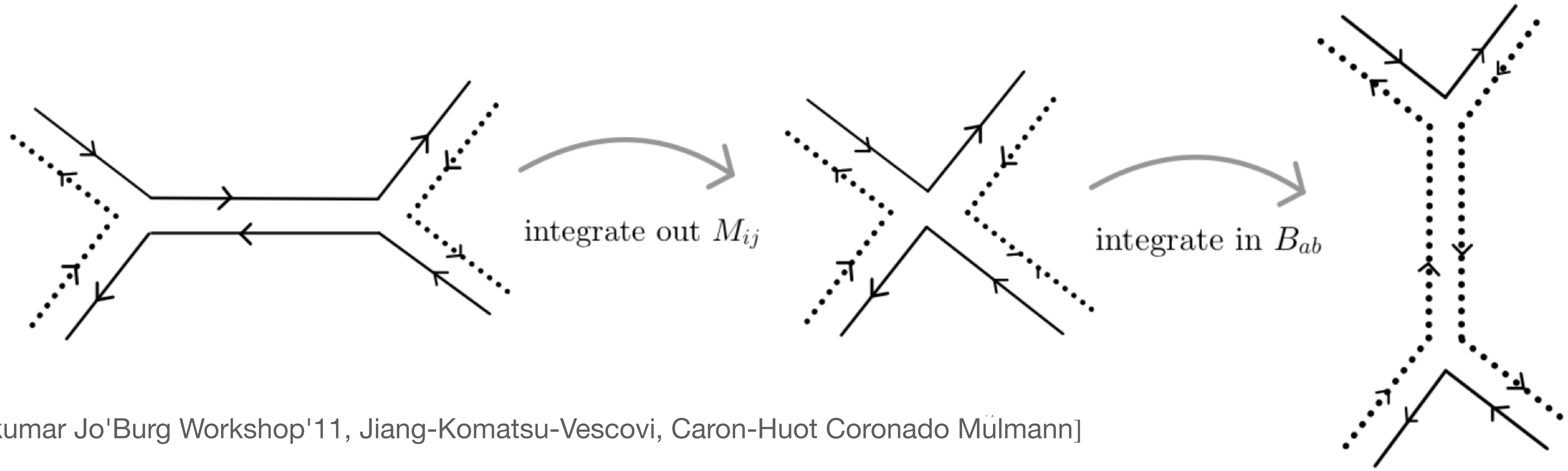
V/F-type: 2 ways to reconstruct closed strings from gauge theory FDs



V-type Duality

Closed String WS

F-type Duality



Edge → Dual Edge

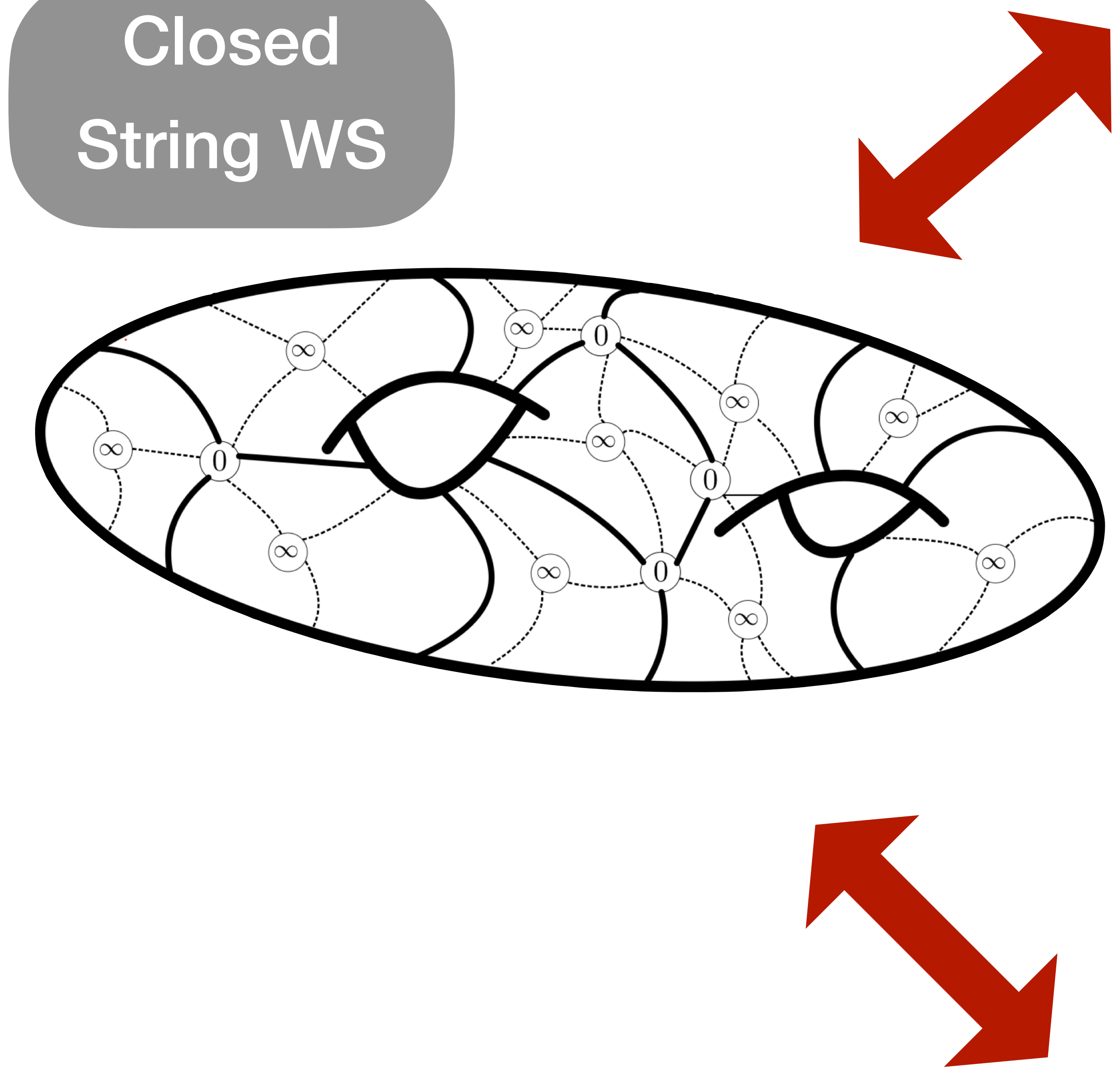
[Gopakumar Jo'Burg Workshop'11, Jiang-Komatsu-Vescovi, Caron-Huot Coronado Mülmann]

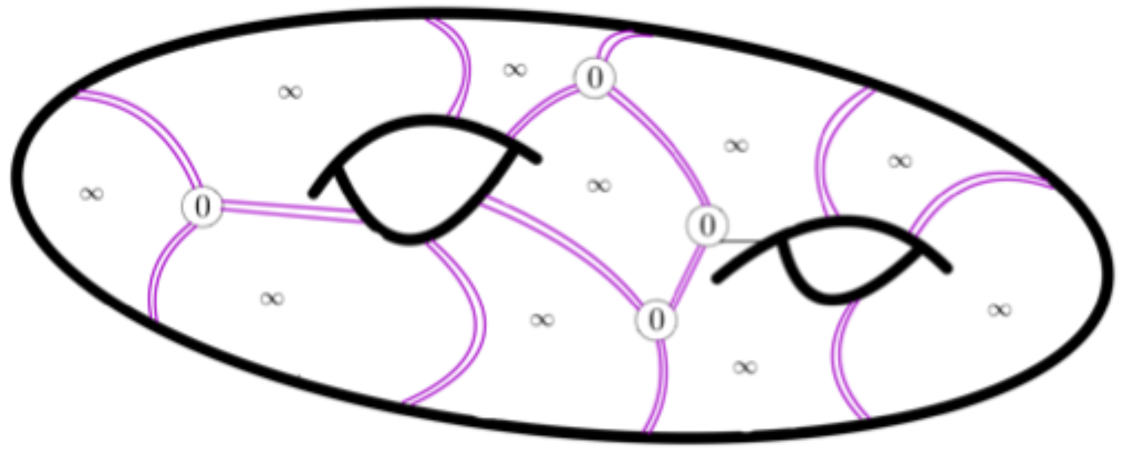
More on V- versus F-type

Reconstructing closed strings from open string strips

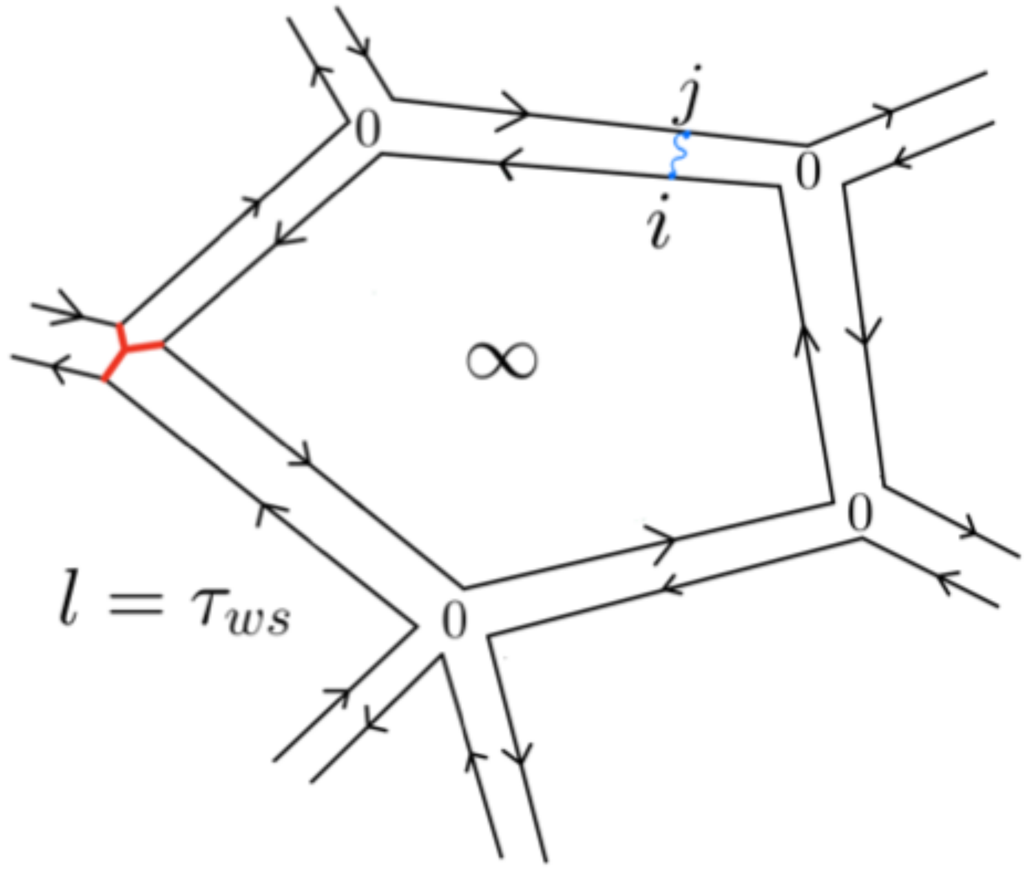
Cf. Witten, Zwiebach

Closed
String WS

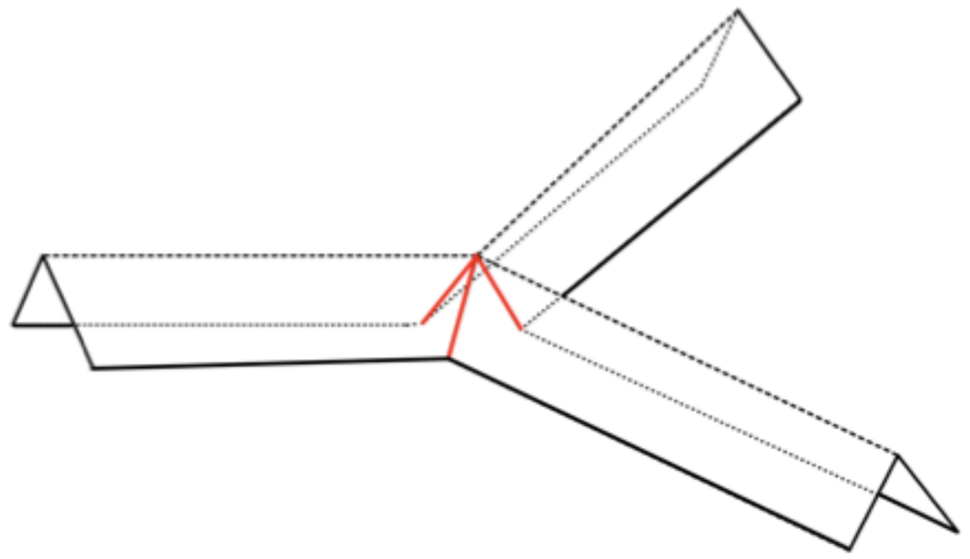




(a) F-Dual Reconstruction

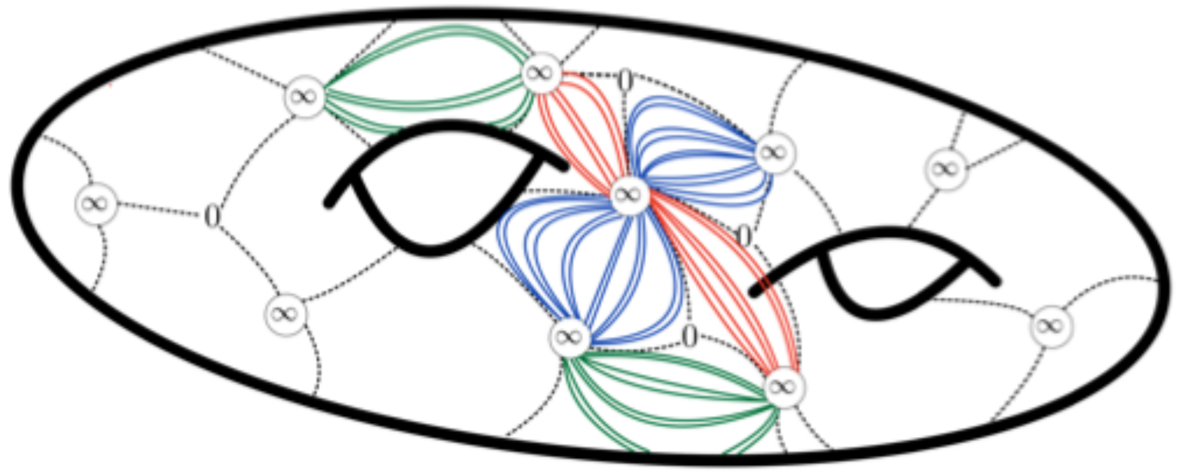


(b) Edges as open string strips

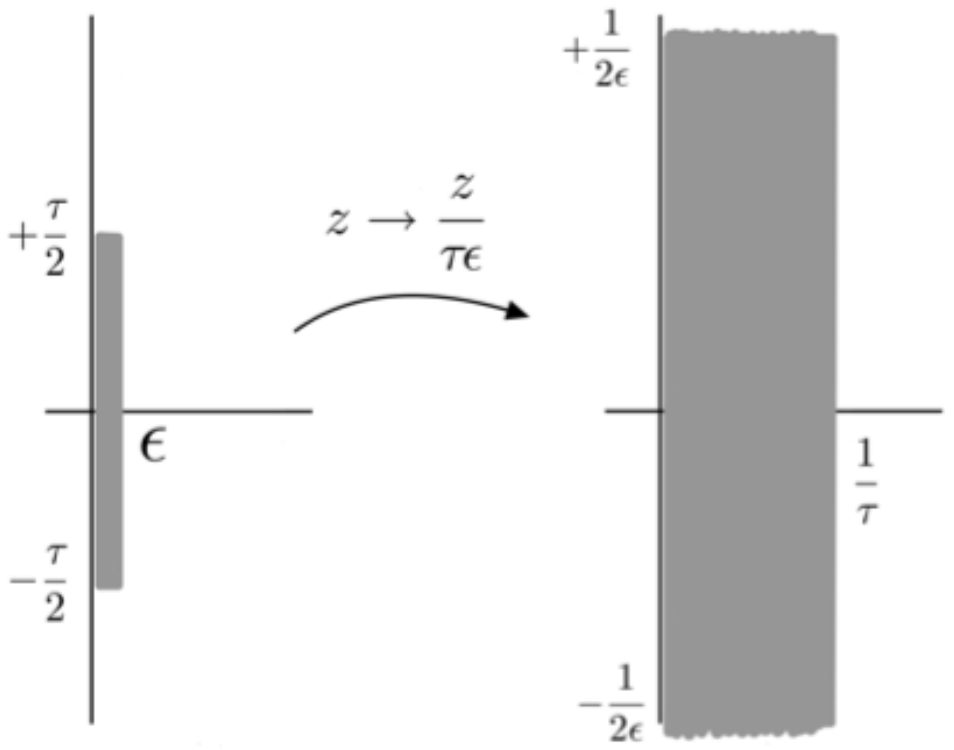


(c) Glueing 3 open strings along their midpoint

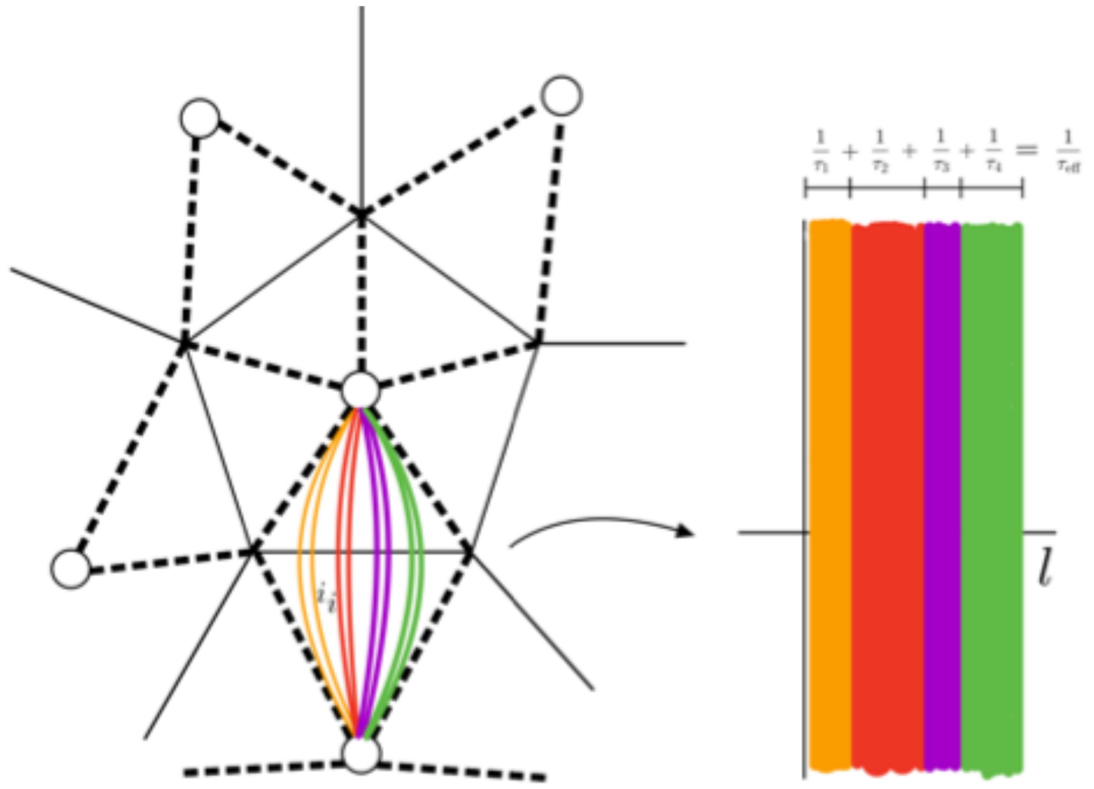
Cf. Gopakumar



(a) V-Dual Reconstruction



(b) From worldlines to open string strips



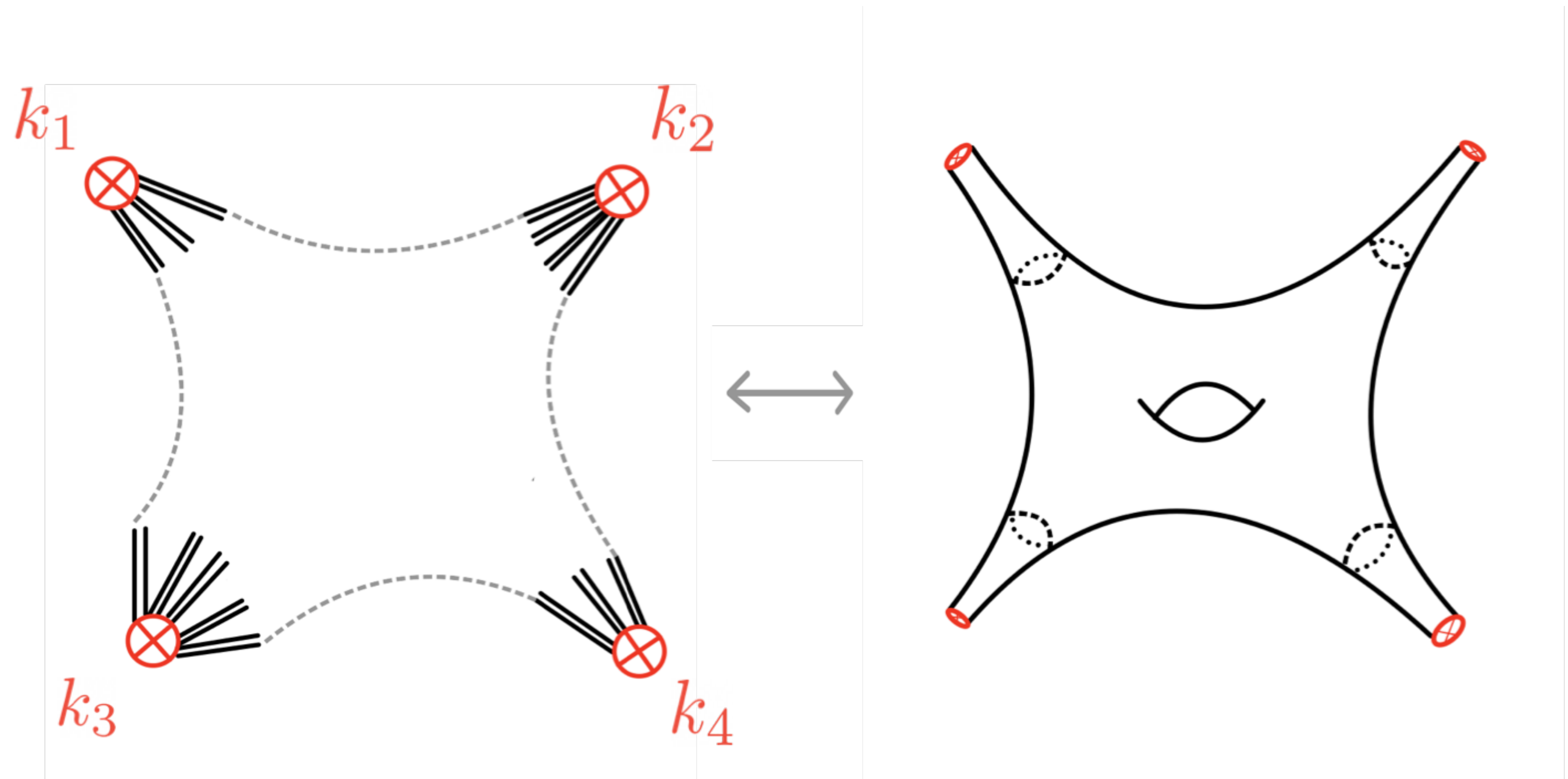
(c) Gluing homotopically equivalent ribbons (string bits)

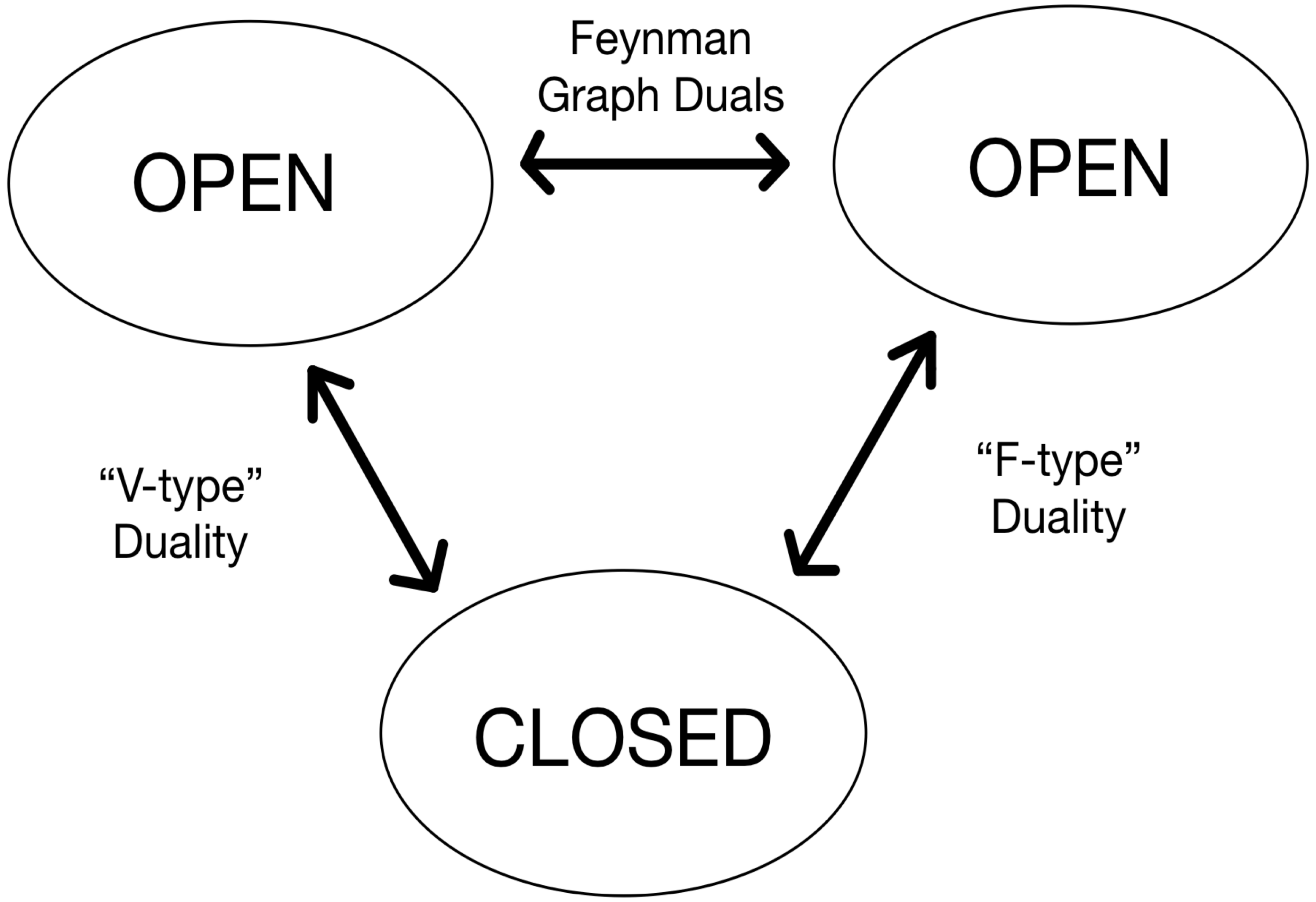
V- *and* F-type Open/Closed String duality

Extending holographic duality to a triality

Vertex Type

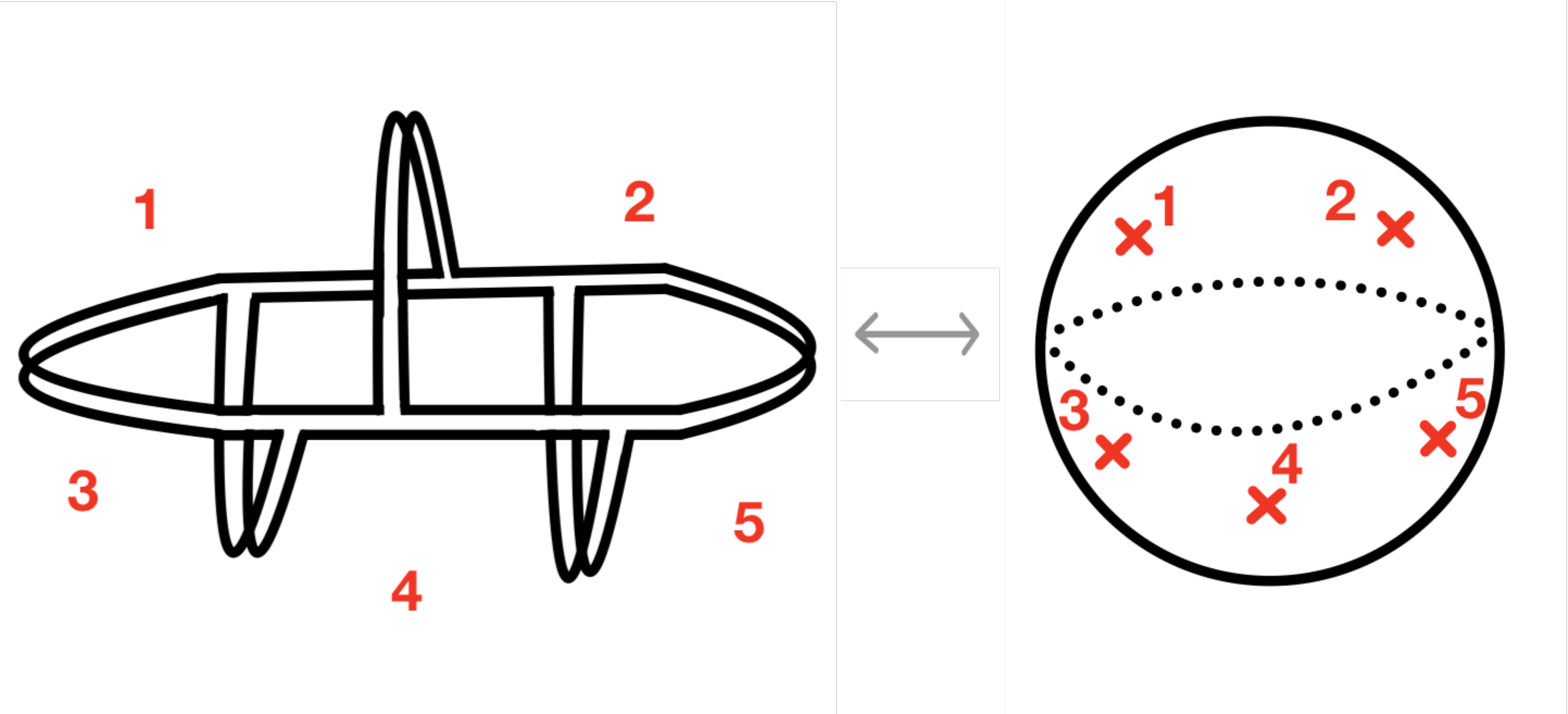
E.g.
$\mathcal{N} = 4$ SYM





Face Type

E.g.
Gopakumar/
Vafa duality



The Derivation

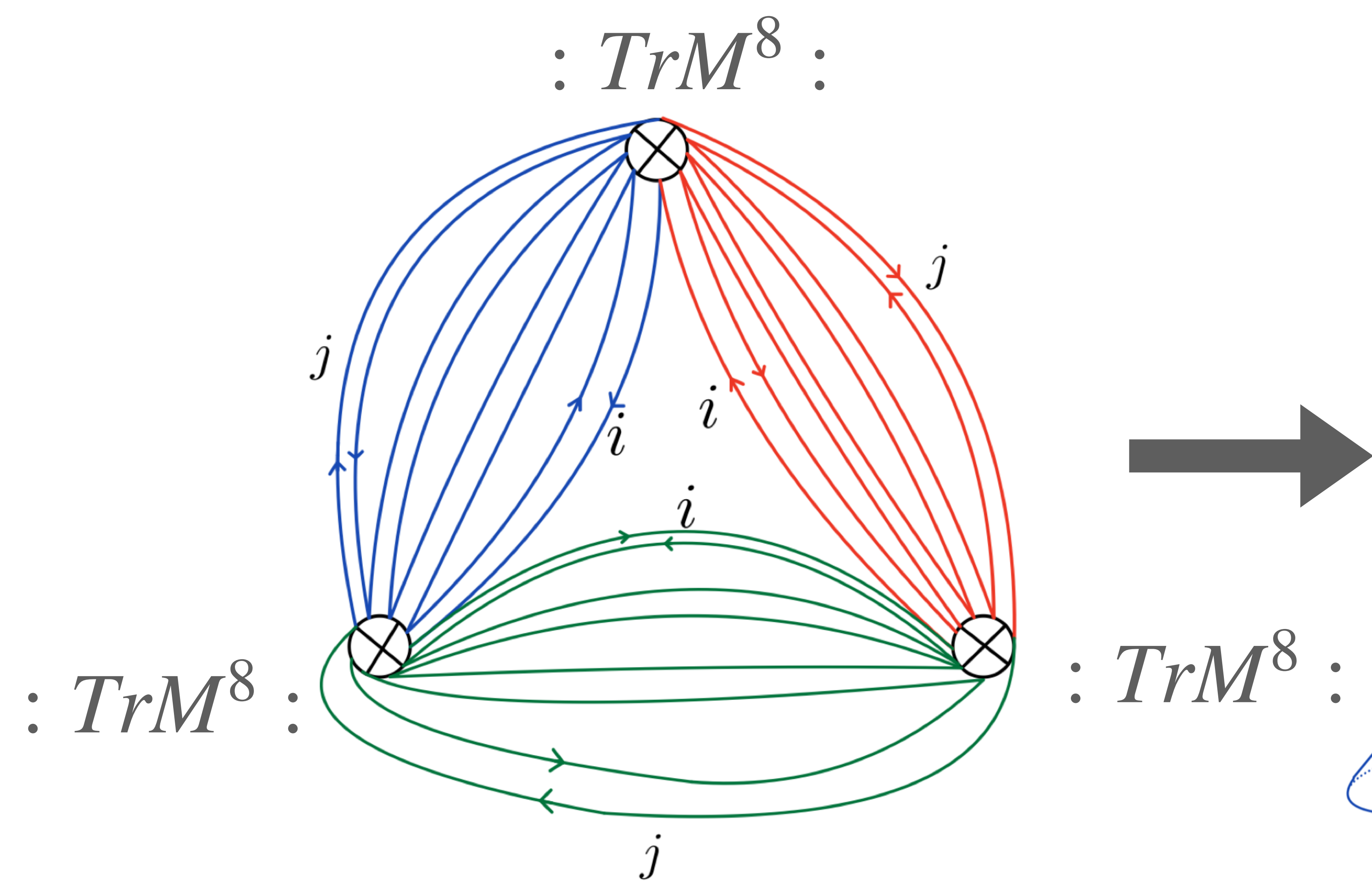
***Why* do these closed string theories appear?**

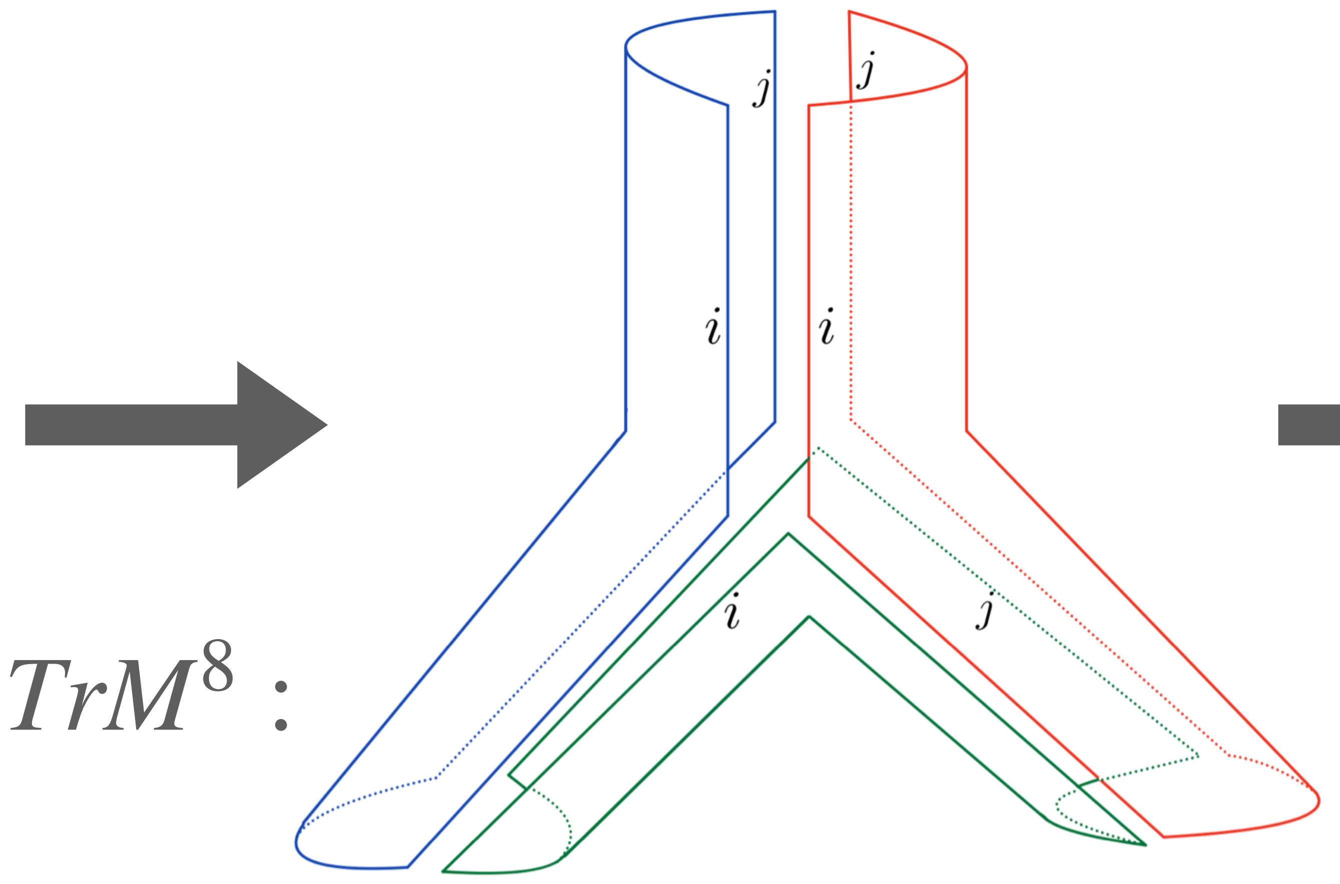
The A-Model

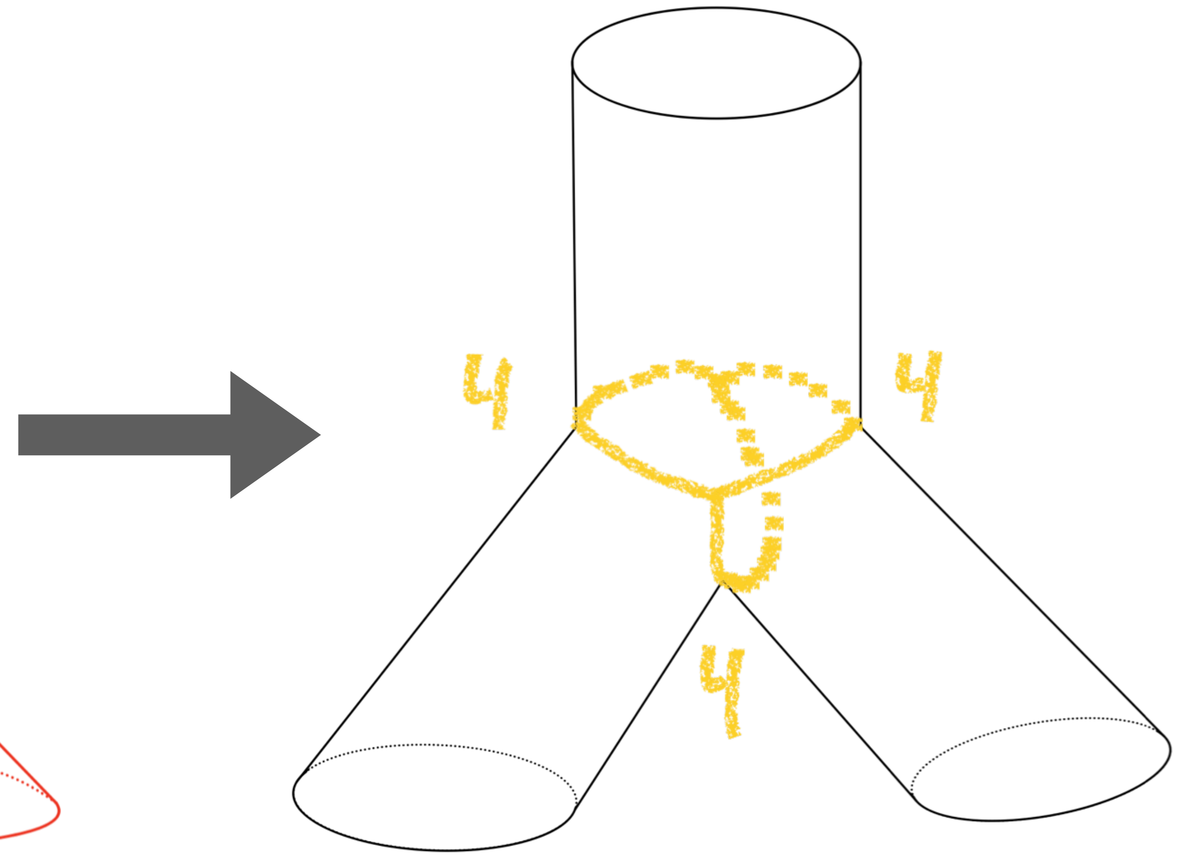
How do we see the holomorphic maps from the WS to the TS from the matrix?

Putting V-Type Open/Closed Duality to the Test

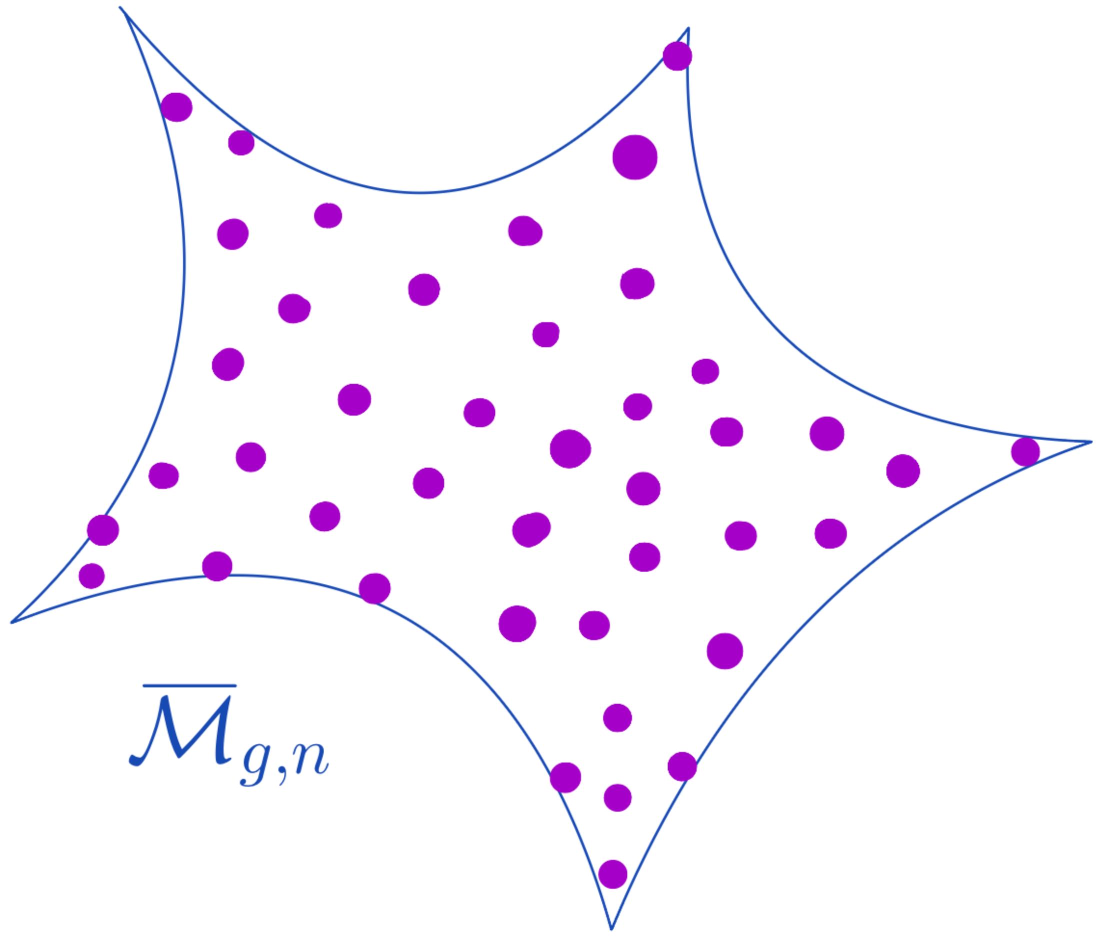
→ **Gaussian correlators count lattice points on moduli space $\mathscr{M}_{g,n}$**







$N_{0,3}(8,8,8) = 1$



[Cf. Norbury, Chekhov et al.]

$$\langle \prod_{i=1}^{n} \frac{1}{Nk_i} : TrM^{k_i} :\rangle_c^g = \text{Sum over integer length Strebel graphs} \equiv N_{g,n}(k_1, \ldots, k_n)$$

Explicit sanity checks: $\langle \prod_{i=1}^{4} \frac{1}{N2k_i} : TrM^{k_i} :\rangle_c^{g=0} = N_{g=0,4}(k_1, \ldots, k_4)$ **&** $\langle \frac{1}{N2k_1} : TrM^{2k_1} :\rangle_c^{g=1} = N_{g=1,n}(2k_1)$

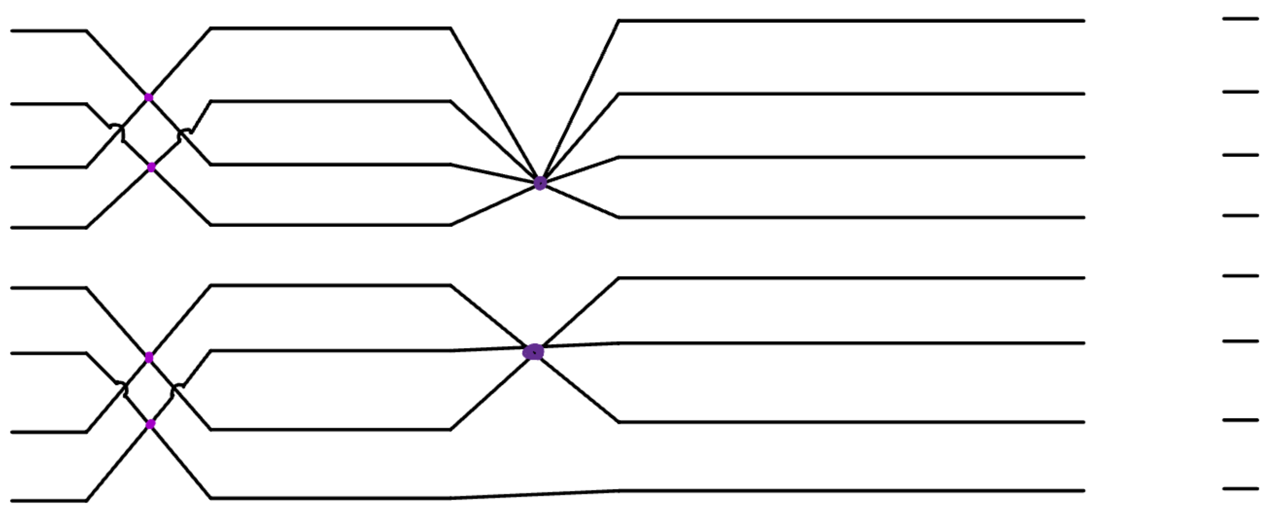
From Wick Contractions to Belyi Maps

Gaussian Correlators as Holomorphic Branched Covers

[Cf. Bauer-Itzykson; Rangoolam, de Mello Koch; Gopakumar; Dunin-Barkowski, Orantin, Popolitov, Shadrin]

$$\langle \prod_{i=1}^{n} TrM^{2k_i} \rangle = \sum_{\alpha,\gamma \in S_{2k}} \delta(\alpha \cdot \beta \cdot \gamma) N^{-k+C_\gamma}$$

$k = \sum_{i=1}^{n} k_i$



$\alpha \in (2)^k$

Wick Contractions = Edges

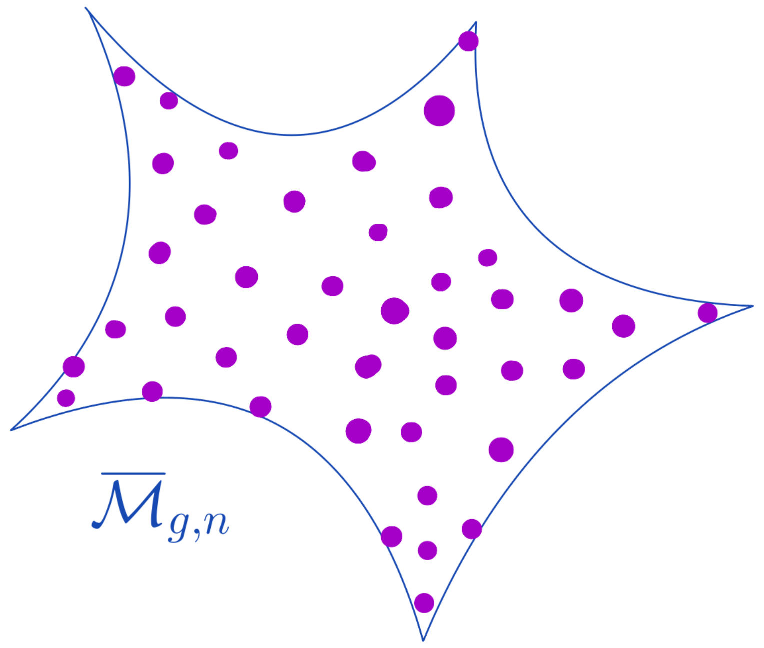
$\beta \in (2k_1)\ldots(2k_n)$

Cyclity of Traces = Vertices

$\gamma = (\alpha \cdot \beta)^{-1}$

Closed index loop post Wick = Faces

Belyi Maps: Holomorphic covering maps $\Sigma_{g,n} \to \mathbb{P}^1$ of degree $2k$ with exactly three branch points $(0,1,\infty)$ and branching profile $(2k_1,\ldots,2k_n)$ at ∞; $(2)^k$ branching at 1



Belyi Thm: Such Maps only exist for these integer points on $\mathscr{M}_{g,n}$!

[Cf. Mulase-Penkava]

Such localization on moduli space $\mathscr{M}_{g,n}$ indeed already seen in $\frac{SL(2,\mathbb{R})_1}{U(1)}$ string!

[Cf. Eberhardt, Dei, Gaberdiel, Gopakumar, Knighton, Maity]

The « BMN-Limit »

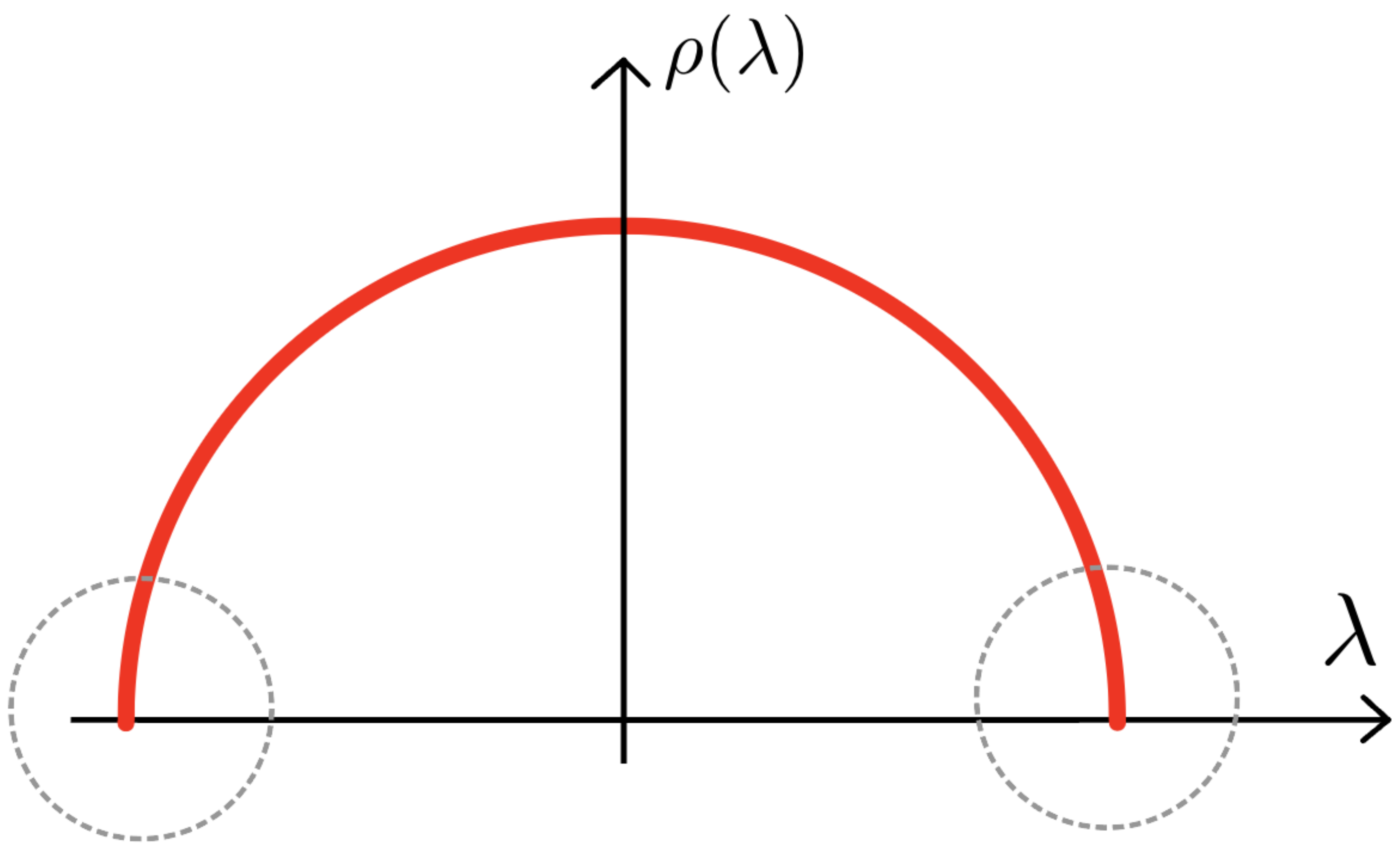
A New Perspective on Double-Scaling

$$\omega_{Kontsevich} = \sum_{i=1}^{n} k_i^2 \psi_i$$

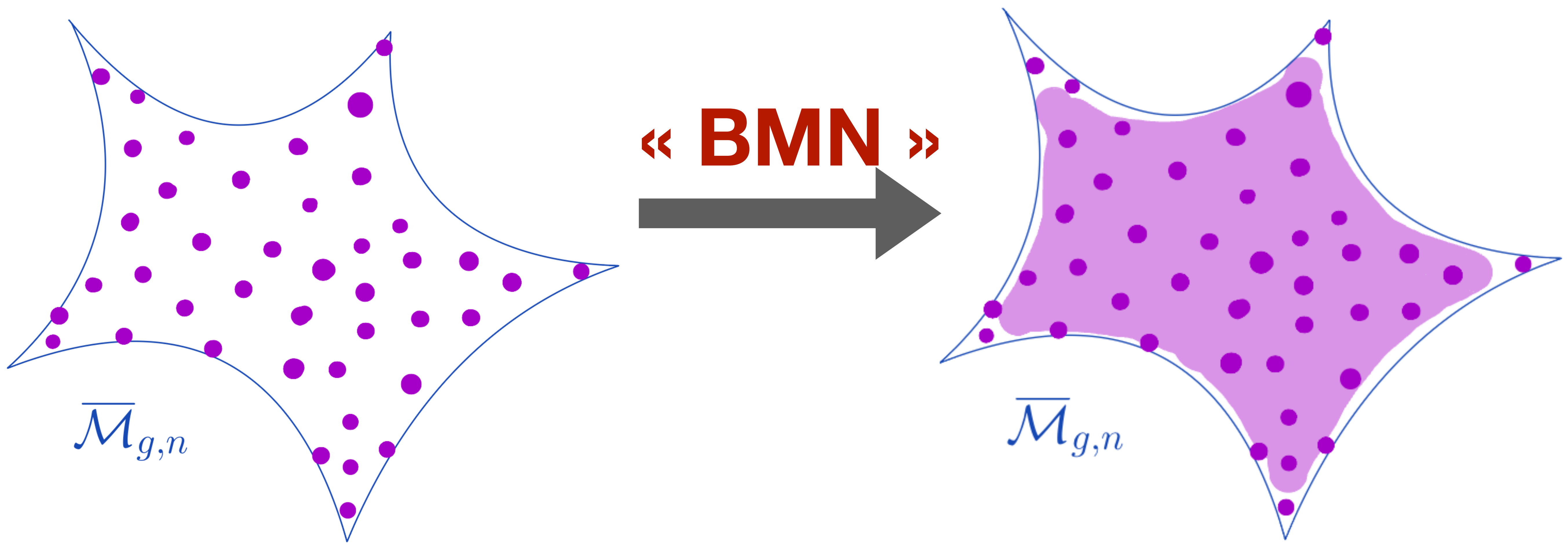
[Cf. Ehrhart, Norbury]

$$\lim_{k_i\to\infty} \langle \prod_{i=1}^{n} \frac{1}{Nk_i} : TrM^{2k_i} : \rangle_c = \lim_{k_i\to\infty} N_{g,n}(2k_1, \ldots, 2k_n) \to Vol_{Kontsevich}(2k_1, \ldots, 2k_n)$$

Pure 2d top gravity in BMN limit



Wigner Semicircle ↔ full AdS
Edge Region (Airy) ↔ pp-Wave geometry



Cover all of moduli space!

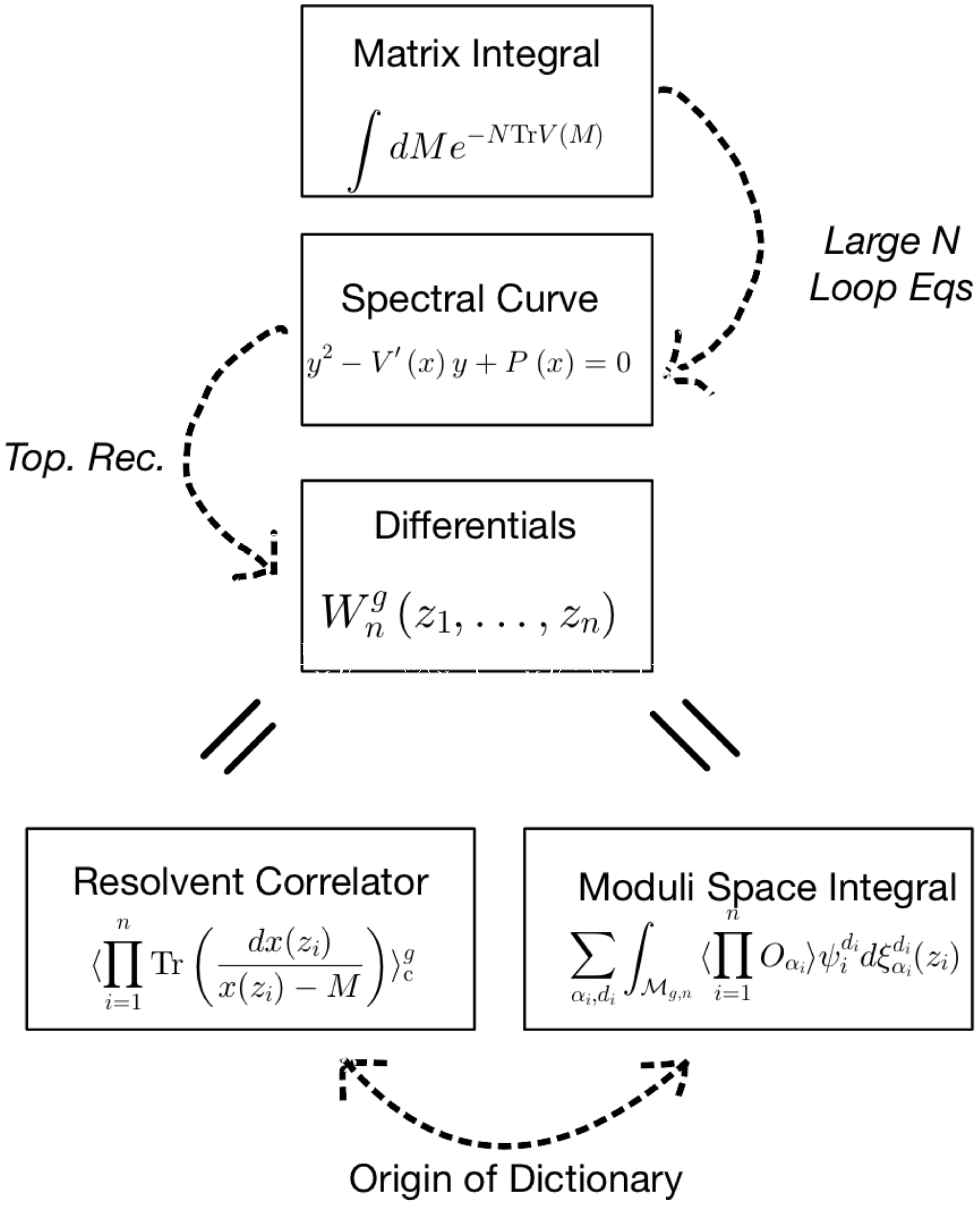
The B-Model

How do we see the constant maps from the WS to the critical points of the super potential?

Finding a B-Model in Disguise

The Many Faces of Topological Recursion

[Cf. Eynard-Orantin; Eynard; DOSS
KS : Dijkgraaf-Vafa; Post, v.d. Heijden, E Verlinde
LG : Dunin-Barkowski, Norbury, Orantin, Popolitov, Shadrin]



Gaussian Model Spectral Curve

$$x = \frac{1}{y} + ty$$

Landau-Ginsburg Superpotential

$$W(Z) = \frac{1}{Z} + tZ$$

(Cf. Dijkgraaf-Vafa)

Branchpoints of Spectral Curve

$$dx = 0$$

Critical Points of Superpotential

$$dW = 0$$

Topological Recursion: Residues at branchpoints of spec curve
B-model string: localization to constant maps into critical points of W

Integrate out matter first: moduli space integral & intersection numbers

$\updownarrow$

Integrate out « gravity » first (cf. Losev): Top. Recursion as matter residue calculus with new contact terms

CohFT Correlators

[$WS\ Sol$: Cf. E. & H. Verlinde
Dijkgraaf-Verlinde-Verlinde]

Traces as Matter Primaries + Gravitational Descendants

$$\int dM_{N\times N}\; e^{-NTrV(M)} \prod_{i=1}^{N} TrM^{k_i}$$

Potential determines matter theory

large N expansion = genus expansion

$$\sum_g N^{2-2g-n} \sum_{\alpha_i, d_i} c^{k_i}_{\alpha_i, d_i} \int_{\overline{\mathscr{M}}_{g,n}} \langle O_{\alpha_1} \ldots O_{\alpha_n} \rangle \prod_{i=1}^{n} \psi_i^{d_i}$$

Extra psi-class Insertions

$$TrM^{2k} \leftrightarrow \mathbf{O}_+ \sum_{d=0}^{k-1} \frac{(2k)!}{(k-d)!(k-1-d)!} \psi^{2d} + \mathbf{O}_- \sum_{d=0}^{k-1} \frac{(2k)!}{(k-1-d)!(k-1-d)!} \psi^{2d+1}$$

Main tool: TR as CohFT
(Eynard 2011 + DOSS 2014 + Giachetto Thesis+…)

Matter Primaries =
Edges of Eigenvalue Distribution

Sanity Checks:
g=0 3pt & 4pt, and g=1 1pt correlators from explicit moduli-space integrals

Very Explicit Universal Operator Dictionary!

The BMN Limit - Take 2

« Washing Out the Matter Theory »

Reproduce Okounkov & Okounkov-Pandharipande!

$$\lim_{\kappa \to \infty} \frac{\langle TrM^{2\kappa x_1} \dots TrM^{2\kappa x_n} \rangle^{(g)}}{2^{2\kappa |x|} \kappa^{3g-3+3n/2}} = \frac{2^g}{(\pi)^{n/2}} \sum_{d_1+\ldots+d_n = d_{g,n}} \langle \prod_{\alpha=1}^{n} \psi_\alpha^{d_\alpha} \rangle_{\mathcal{M}_{g,n}} x_\alpha^{d_\alpha + 1/2}$$

From our operator dictionary

$$\lim_{k \to \infty} \frac{1}{k} : Tr\ M^k :\sim O_a \times e^{\frac{k^2}{2}\psi}$$

Maximize ψ-class insertions

$\leftrightarrow$

Decouples matter theory

$$\langle O_{\alpha_1} \dots O_{\alpha_n} \rangle^{TFT} \times \int_{\mathcal{M}_{g,n}} \prod_{i=1}^{n} \sum_{d} c_{\mathbf{k_i},d}\ \psi_i^d$$

Pure 2d top gravity in BMN limit!

AdS/CFT

How do these lessons fit into holography?

The Even Bigger Picture

As a topological subsector of “standard” AdS/CFT

Thank You!

Happy to go into more details!

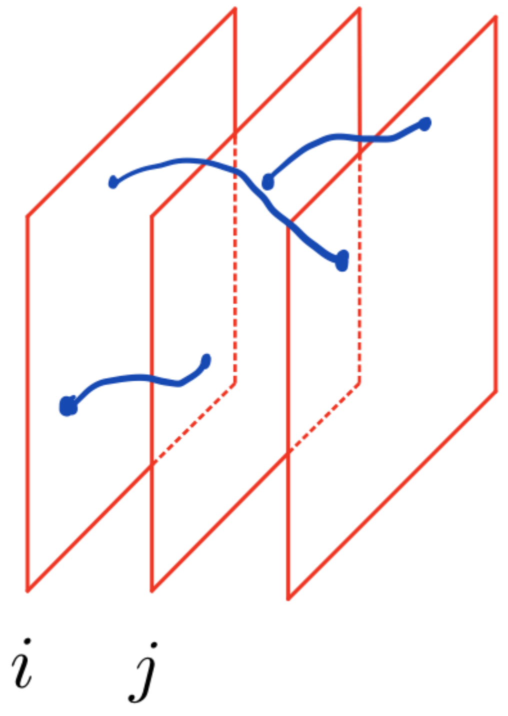
4pt Fn Example

Some Words
on Relation to JT

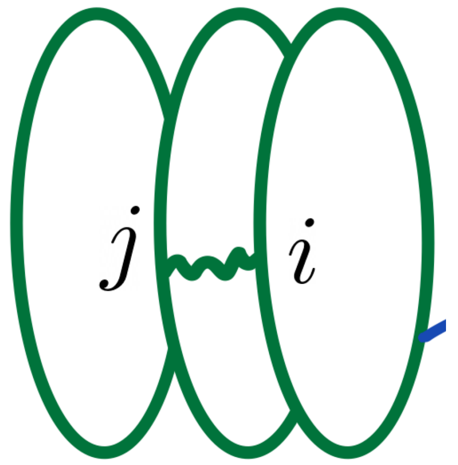
The Simplest Gauge Theory from Branes

A Comparison with AdS_5/CFT_4

BRANE SETUP



N D3s wrapping $\mathbb{R}^{3,1}$ in $\mathbb{R}^{9,1}$



N D1s wrapping $\mathbb{CP}^1 \sim S^2$ in $\mathscr{O}(0) \oplus \mathscr{O}(-2) \to \mathbb{CP}^1$

GAUGE THEORY
(Open String Description)

$SU(N), \mathcal{N} = 4 \ SYM$

DBI on worldvolume of N Branes,
+ low-energy Limit

$N \times N$ Hermitian Matrix Integral

Open string field theory on N branes,

EMERGENT GEOMETRY
(Closed String Description)

IIB on $AdS_5 \times S^5$

A Model: $\dfrac{SL(2,\mathbb{R})_{k=1}}{U(1)} \sim \mathbb{CP}^1$

Localization for Simplification

From Open String Field Theory to a Matrix Integral

Open String Field Theory

$$S = \frac{1}{2g}\int_{string\ configs} \Psi \star Q\Psi + \frac{2}{3}\Psi \star \Psi \star \Psi$$

[Witten '92]

B-Model Localization
„string → point"

6d Hol. Chern-Simons

$$S = \frac{1}{2g}\int_{Target} \Omega \wedge Tr_N\left(A \wedge \bar{\partial}A + \frac{2}{3}A \wedge A \wedge A\right)$$

Brane Setup &
$A_{\mu\in\{//,0,-2\}} = (A_{//}, \Phi_0, \Phi_1)$

[Dijkgraaf-Vafa'02]

2d Gauged $\beta\gamma$-System

$$S = \frac{1}{g}\int_{\mathbb{CP}^1} Tr_N\left[\Phi_1(\bar{\partial}_{//} + [A_{//},\ ])\Phi_0\right]$$

$\mathbb{CP}^1$ & Lagrange Multiplier

Our 2-Matrix Model

$$S = \frac{1}{g}Tr_N(KM)$$

More than what we can do in AdS/CFT !

Two B-Model Perspectives on 4-pt Fn.

"Pure Matter" vs. Intersection Theory Computations of $N_{0,4}(2k_1,\ldots,2k_4)$

Matrix Model Answer: $\langle \prod_{i=1}^{n=4} \frac{1}{N2k_i} : TrM^{2k_i} : \rangle_c^{g=0} = N_{g=0,n=4}(2k_1,..,2k_4) = k_1^2 + k_2^2 + k_3^2 + k_4^2 - 1$

"Pure Matter" LG w/ contact terms (cf. Losev)

$$W(Z) = \frac{1}{Z} + Z$$

$$\frac{1}{Nk} : TrM^k : \leftrightarrow \mathcal{O}_k \equiv \frac{1}{Z^{k+1}}$$

$$\begin{aligned} N_{g=0,n=3}(k_1,k_2,k_3) &= \langle \mathcal{O}_{k_1}\mathcal{O}_{k_2}\mathcal{O}_{k_3}\rangle \\ &= \oint \frac{1}{W'(z)}\frac{1}{z^{k1+1}}\frac{1}{z^{k2+1}}\frac{1}{z^{k3+1}} \\ &= \operatorname*{Res}_{z\to 1}\frac{z^2}{(z^2-1)}\frac{1}{z^{k1+1}}\frac{1}{z^{k2+1}}\frac{1}{z^{k3+1}} + \operatorname*{Res}_{z\to -1}\frac{z^2}{(z^2-1)}\frac{1}{z^{k1+1}}\frac{1}{z^{k2+1}}\frac{1}{z^{k3+1}} \\ &= \left(\frac{1}{2}\right) + (-1)^{k1+k2+k3}\left(\frac{1}{2}\right) \end{aligned}$$

Start with 3-pt Fn. (Cf. Vafa)

Matter Theory as Iterated Residue Calculus

B-Model "after integrating out gravity"

"Pure Matter" LG w/ contact terms (cf. Losev)

$$C_W(\mathcal{O}_{k_i}, \mathcal{O}_{k_j}) = \frac{d}{dz}\left(\frac{\mathcal{O}_{k_i}\mathcal{O}_{k_j}}{W'(z)}\right)_- = \sum_{l=1}^{k_i+k_j} 2l\,\mathcal{O}_{2l}$$

$$\langle \mathcal{O}_{2k_1}\mathcal{O}_{2k_2}\mathcal{O}_{2k_3}\mathcal{O}_{2k_4}\rangle = \frac{d}{dt}\langle \mathcal{O}_{2k_1}\mathcal{O}_{2k_2}\mathcal{O}_{2k_3}\rangle_{W+t\mathcal{O}_{2k_4}}\Big|_{t=0} + \sum_{i=1}^{3}\langle C_W(\mathcal{O}_{2_k4}, \mathcal{O}_{2k_i})\prod_{j\neq i}^{3}\mathcal{O}_{2k_j}\rangle$$

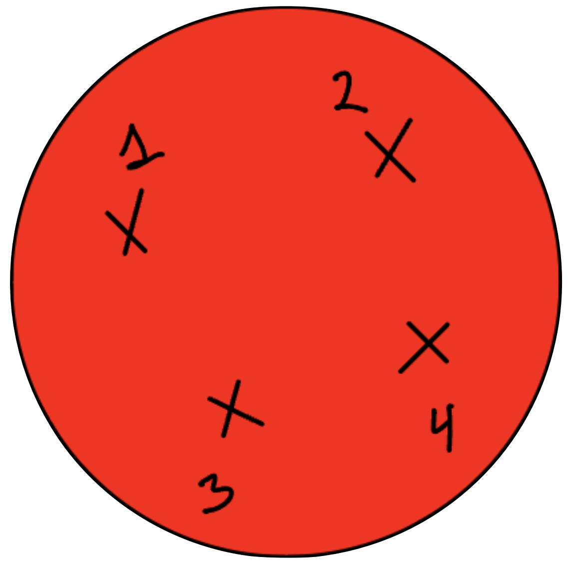
$$= -(2k_4+1)\big(k_1+k_2+k_3+k_4+1\big) \;+\; k_1(1+k_1)+k_2(1+k_2)+k_3(1+k_3)+2k_4(k_1+k_2+k_3)+3k_4(1+k_4)$$

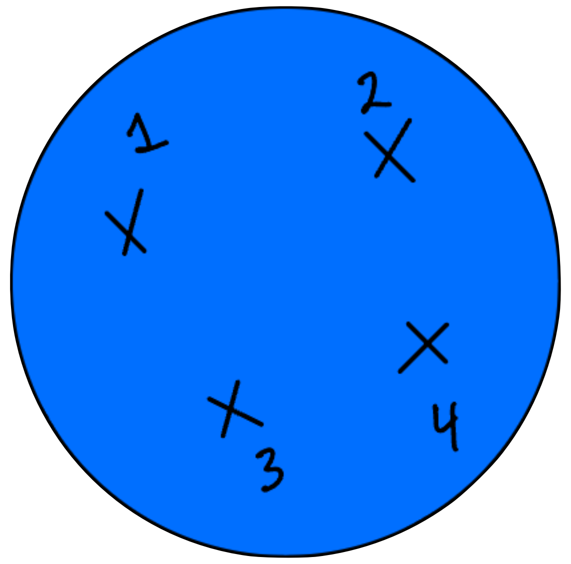
Contributions from deformed 3-pt fn | Contributions from contact terms

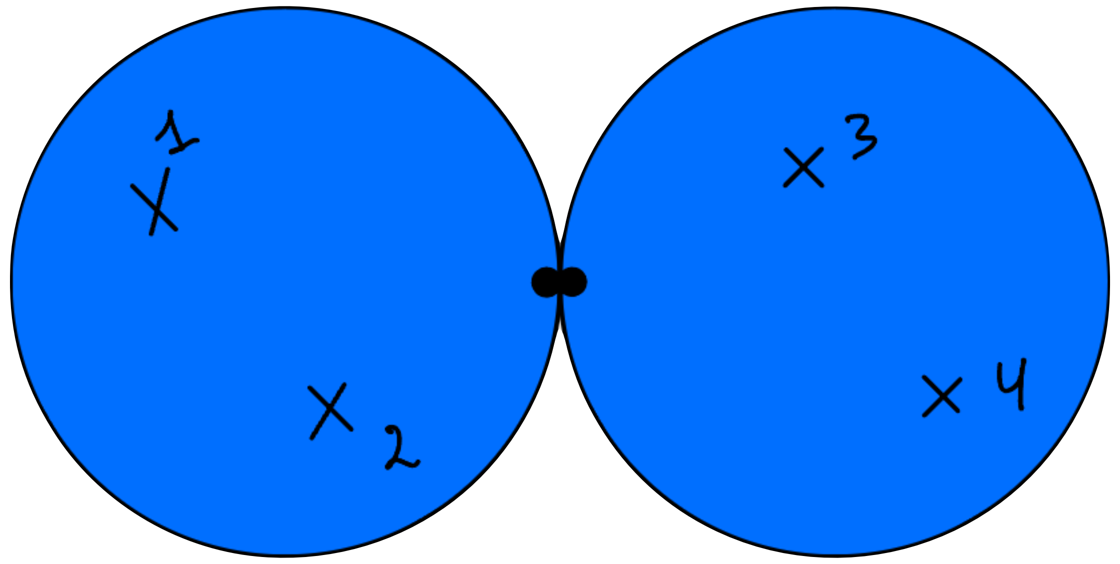
$$= k_1^2+k_2^2+k_3^2+k_4^2-1 = N_{g=0,n=4}(2k_1,\ldots,2k_4)$$

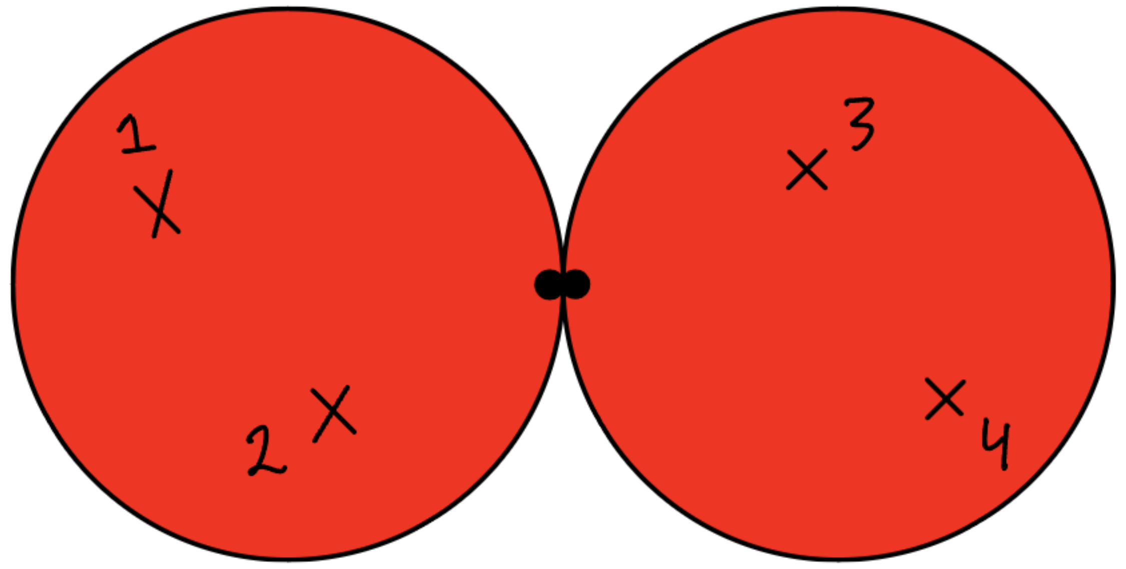
4-pt Function from Moduli Space Integral

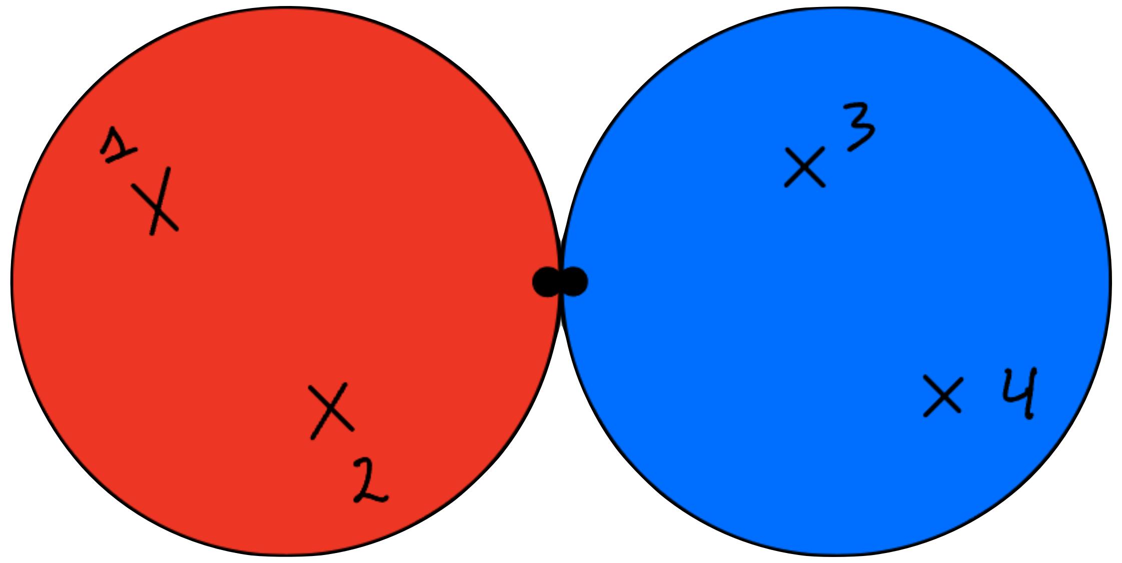
B-Model “after integrating out matter”











$\frac{1}{2}(k_1^2 - \frac{1}{16})\langle\psi_1\rangle_{\mathcal{M}_{0,4}}$ +Perm(1,2,3,4)

$\frac{1}{2}(k_1^2 - \frac{1}{16})\langle\psi_1\rangle_{\mathcal{M}_{0,4}}$ +Perm(1,2,3,4)

$-\frac{3}{64}\langle\psi_1^0\psi_2^0\psi^0\rangle_{\mathcal{M}_{0,3}}\langle\psi_3^0\psi_4^0\psi'^0\rangle_{\mathcal{M}_{0,3}}$

$-\frac{3}{64}\langle\psi_1^0\psi_2^0\psi^0\rangle_{\mathcal{M}_{0,3}}\langle\psi_3^0\psi_4^0\psi'^0\rangle_{\mathcal{M}_{0,3}}$

$-\frac{3}{32}\langle\psi_1^0\psi_2^0\psi^0\rangle_{\mathcal{M}_{0,3}}\langle\psi_3^0\psi_4^0\psi'^0\rangle_{\mathcal{M}_{0,3}}$

From operator insertions

Contributions from boundary of moduli space (cf. contact terms)

$-\frac{3}{64}\langle\kappa_1\rangle_{\mathcal{M}_{0,4}}$ $-\frac{3}{64}\langle\kappa_1\rangle_{\mathcal{M}_{0,4}}$

From “background” dual to matrix potential

Coefficients fixed both by local behavior of spectral curve & Bergmann kernel near branchpoints) <u>and</u> our new operator dictionary

$$= k_1^2 + k_2^2 + k_3^2 + k_4^2 - 1 = N_{g=0,n=4}(2k_1, \ldots, 2k_4)$$

Traces from Hodge-GUE

All genus-0 and genus-1 correlators

$$\Lambda(x) \equiv \sum_{j=0}^{g} c_j(\mathbb{E}) x^j$$

Cf. Borot & Garcia-Failde

$$\langle TrM^{2k_1} \dots TrM^{2k_a} \rangle_c^g = \sum_{h=0}^{[g/2]} \frac{2^g}{2^{3h}(2h)!} \sum_{l=0} \frac{1}{l!} \int_{\bar{\mathcal{M}}_{g-h,a+l+2h}} \Lambda(-1)\Lambda(-1)\Lambda(1/2) \prod_{i=1}^{a} \frac{1}{1-k_i\psi_i} \frac{(2k_i)!}{(k_i)!(k_i-1)!} \prod_{j=a+1}^{a+l} \frac{-\psi_j^2}{1-\psi_j}$$

Reproduce $g=0$ n-pt Matrix Model answer

$$\langle \prod_{i=1}^{n} TrM^{2k_i} \rangle_c^{g=0} = \frac{(k_{tot}-1)!}{(k_{tot}-n+2)!} \prod_{i=1}^{n} \frac{(2k_i)!}{(k_i)!(k_i-1)!}$$

Hodge bundle trivial, ψ-class intersection numbers purely combinatorial

$g=1$ n-pt (Also beyond MM answers)

Cf. Morozov-Shakirov

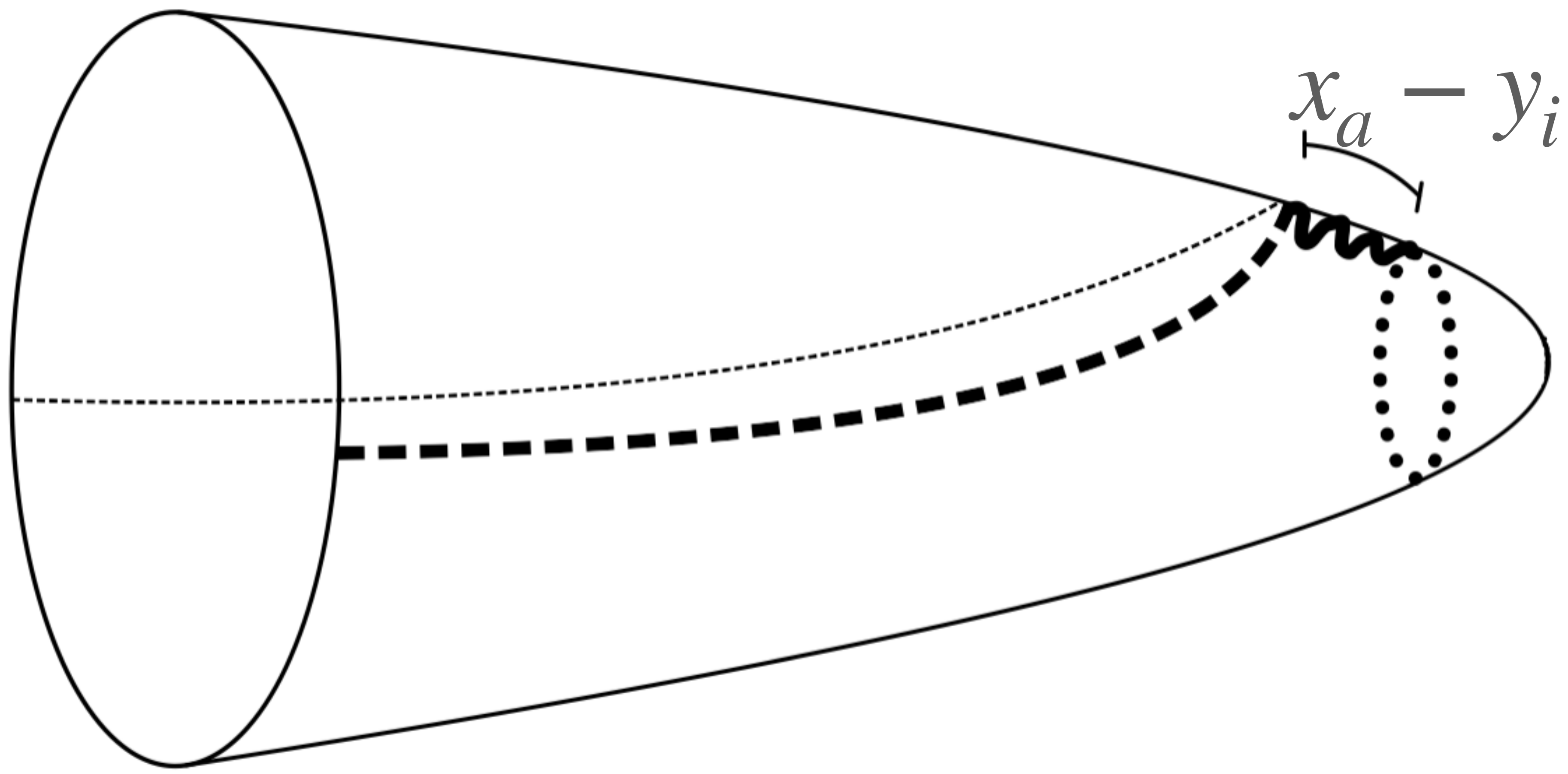
$$\langle TrM^{2k_1} TrM^{2k_2} \rangle_{conn}^{g=1} = \left(\frac{k_1^2 + k_1k_2 + k_2^2 - 2(k_1+k_2) + 1}{12} \right) \prod_{i=1}^{2} \frac{(2k_i)!}{(k_i)!(k_i-1)!}$$

Requires the λ_g-formula (Faber-Pandharipande)

$$\int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n} \lambda_g = \frac{(2g+n-3)!}{\alpha_1! \dots \alpha_n!} \times \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}$$

The A-Model: the Cigar

From c=1 at self-dual radius to the topological coset



$$\int dK dM_{N\times N} e^{-\frac{1}{g}\mathrm{Tr}\left(V_p(K) - K(M-Y)\right)} \prod_{a=1}^{Q} \det(x_a - M)$$

- Can map tachyon vertex operators in c=1 at self-dual radius to operators in coset model, giving operator dictionary for traces : $\mathrm{Tr}M^k \leftrightarrow D^k_{j=1/2}$, where the momentum background makes us consider vertex operators in the spectrally flowed $j = 1/2$ sector
- Compact ~ZZ // Non-compact~FZZT

All Genus 1-pt Function

A detailed sanity check

$$\langle \frac{1}{N} TrM^{2n} \rangle_{Gaussian} = 2n \langle T_{-2n} \rangle_{t_2}$$

Gaussian Matrix Model ↔ c=1 string at self-dual radius with momentum +2 tachyon-background

c=1 string calculation:

$$2n\langle T_{-2n}\rangle_{t_2} = \frac{1}{2n+1}\left(\frac{1}{i\mu}\right)^{2n+1} \oint dz W^{2n+1}(z)$$

W_∞ currents $W^{2n+1}(z) \equiv \bar{\psi}(z)\partial_z^{2n+1}\psi(z)$

Bosonization $\psi(z) = e^{i\mu\phi(z)}$

Background $\phi(z) = \log z + \sum_{k=1}^{\infty} \frac{1}{k} t_k z^k$

$$2n\langle T_{-2n}\rangle_{t_2} = \frac{1}{N^{2n+1}}\frac{1}{2n+1}\oint dz z^{-N} e^{-\frac{N}{2}t_2 z^2} \partial_z^{2n+1}\left(z^N e^{+\frac{N}{2}t_2 z^2}\right)$$

All Genus 1-pt Function

A detailed sanity check

$$\langle \frac{1}{N} TrM^{2n} \rangle_{Gaussian} = 2n \langle T_{-2n} \rangle_{t_2}$$

Gaussian Matrix calculation:

Orthogonal Polynomials $\int d\lambda e^{-\lambda^2} F_m(\lambda) F_n(\lambda) = h_m \delta_{mn}$

$$\langle \frac{1}{N} TrM^{2n} \rangle = \frac{1}{Z} \int dM e^{-\frac{N}{2t_2} TrM^2} TrM^{2n} = \left(\frac{2t_2}{N} \right)^n \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{h_k} \int dx e^{-x^2} F_k(x) F_k(x)$$

$$= \left(\frac{2t_2}{N} \right)^n \frac{1}{N} \frac{1}{h_{N-1}} \frac{2}{2n+1} \int dx e^{-x^2} x^{2n+1} F_N(x) F_{N-1}(x)$$

Christoffel-Darbboux $\sum_{k=0}^{N-1} \frac{1}{h_k} F_k(x) F_k(x) = \frac{1}{h_{N-1}} \left(F'_N(x) F_{N-1}(x) - F_N(x) F'_{N-1}(x) \right)$

EOM $(2x - \partial_x) F'_k(x) = 2k F_k(x)$

$$= \frac{1}{N^{2n+1}} \frac{1}{2n+1} \oint dz z^{-N} e^{-\frac{N}{2} t_2 z^2} \partial_z^{2n+1} \left(z^N e^{+\frac{N}{2} t_2 z^2} \right) = 2n \langle T_{-2n} \rangle_{t_2}$$

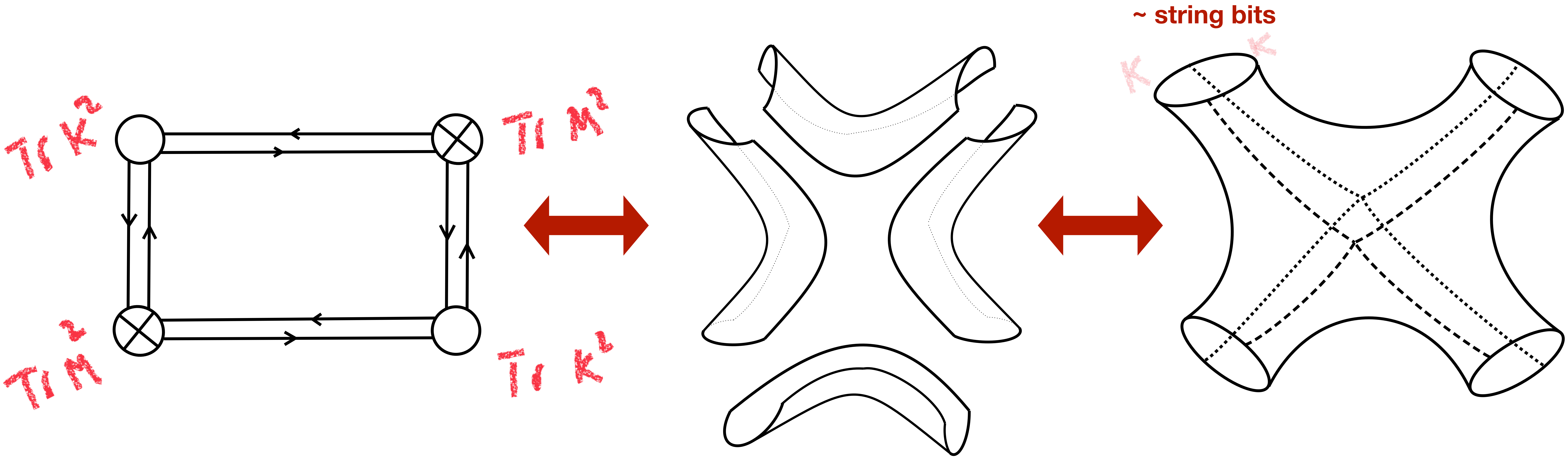
Integral Representation

$$F_N(x) = 2^{-N} (-1)^N e^{x^2} \frac{1}{2\sqrt{\pi}} \int (is)^N e^{isx - s^2/4}$$

$$F_{N-1} = 2^{-(N-1)} (N-1)! \oint \frac{e^{2ux - u^2}}{u^N}$$

An Example of V-type Reconstruction

Back to the simplest FD



Each Edge as an Infinite Strip