Grothendieck lines in 3d SQCD and the quantum K-theory of the Grassmannian

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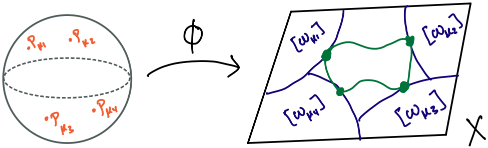
Introduction

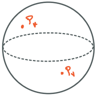
In the 90's, Witten taught us the following lessons:

- A 2d $\mathcal{N} = (2,2)$ GLSM can flow to a NLSM with target space X which is the Higgs branch of the theory (e.g. for $U(N_c)$ coupled with n_f $\square_{N_c}$ we get $X = \mathrm{Gr}(N_c, n_f)$).
- To put the theory on a Riemann surface Σ_g, one needs to *A-twist* the theory. Focus on the *twisted chiral ring* $\mathcal{R}_{\mathrm{2d}} = \{\mathcal{P}_\mu\}_{\mu \in I}$.
- $\mathcal{R}_{\mathrm{2d}} \cong (\mathrm{small})\mathrm{QH}^\bullet_{\mathrm{eq}}(X)$:
  - $\mathcal{P}_\mu \longleftrightarrow [\omega_\mu] \in \mathrm{H}^\bullet(X)$.
  - $\langle \mathcal{P}_{\mu_1} \mathcal{P}_{\mu_2} \cdots \rangle_{\mathbb{P}^1} \longleftrightarrow \mathrm{GW}_0([\omega_{\mu_1}], [\omega_{\mu_2}], \cdots)$.

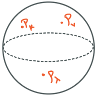


For example, taking $g = 0$, we have the following data:

- The topological metric:

$$\eta_{\mu\nu}(\mathcal{P}) = \Big\langle$$

$$\Big\rangle$$

- The structure constants:

$$\mathcal{C}_{\mu\nu\lambda}(\mathcal{P}) = \Big\langle$$

$$\Big\rangle$$

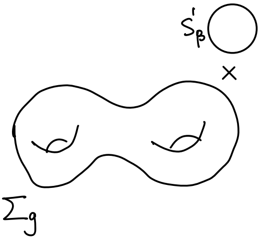
From the correspondence, the ring relations of $\mathrm{QH}^{\bullet}_{\mathrm{eq}}(X)$ are given as:

$$[\omega_\mu] \star [\omega_\nu] = \mathcal{C}_{\mu\nu}{}^{\lambda}(q_{2\mathrm{d}}, m, \omega)[\omega_\lambda] \ .$$

Here,

$$\mathcal{C}_{\mu\nu}{}^{\lambda} = \eta^{\lambda\rho}\mathcal{C}_{\mu\nu\rho} \ , \qquad \eta^{\mu\rho}\eta_{\rho\nu} = \delta^{\mu}_{\nu} \ .$$

In this talk, we are interested in the 3d (K-theoretic) uplift of this set up. [Kapustin-Willet'13, Jockers-Mayer'18, $\cdots$].



Doing a topological A-twist along Σ_g, we get the twisted chiral ring $\mathcal{R}_{3\text{d}} = \{\mathcal{L}_\mu\}_{\mu \in I}$. This consists of half-BPS line operators $\mathcal{L}_\mu$ wrapping the S^1-fibres.

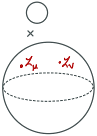
The Witten correspondence in this case becomes

- $\mathcal{L}_\mu \longleftrightarrow [\mathcal{O}_\mu] \in \mathsf{K}_{\mathsf{eq}}(X)$.
- $\langle \mathcal{L}_{\mu_1} \mathcal{L}_{\mu_2} \cdots \rangle_{\mathbb{P}^1 \times S^1_\beta} \longleftrightarrow$ K-theoretic $\mathsf{GW}_{g=0}([\mathcal{O}_{\mu_1}], [\mathcal{O}_{\mu_2}], \cdots)$.

For the $g = 0$ we compute the following data:

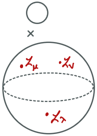
- The topological metric:

$$g_{\mu\nu}(\mathcal{L}) = \Bigg\langle \quad \Bigg\rangle$$



- The structure constants:

$$\mathcal{N}_{\mu\nu\lambda}(\mathcal{L}) = \Bigg\langle \quad \Bigg\rangle$$



And the ring relations from $\mathsf{QK}_{\mathsf{eq}}(X)$ are given by:

$$[\mathcal{O}_\mu] \star [\mathcal{O}_\nu] = \mathcal{N}_{\mu\nu}{}^{\lambda}(q_{\mathsf{3d}}, y, \mathcal{O})[\mathcal{O}_\lambda]$$

We will focus on the case where the gauge group is $U(N_c)$ and the theory is coupled with n_f multiplets in representation $\square_{N_c}$.

- Recall that in 3d, we can have CS levels as an input of the theory:

$$U(N_c)_{k,k+\ell N_c} \cong \frac{SU(N_c)_k \times U(1)_{N_c(k+\ell N_c)}}{\mathbb{Z}_{N_c}} \ .$$

  So, one question that we will discuss here is: how can we tune the levels (k, ℓ) such that we have a "3d GLSM"?

- A standard basis for $\mathrm{K}(\mathrm{Gr}(N_c, n_f))$ is given by the *Schubert classes* $[\mathcal{O}_\mu]$ with $\mathcal{O}_\mu$ being the structure sheaf of the Schubert subvariety X_μ (the closure Schubert cell $\mathrm{C}_\mu \subseteq \mathrm{Gr}(N_c, n_f)$). The index μ here is an N_c-partition:

$$n_f - N_c \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{N_c} \geq 0 \ .$$

  Another natural question is: what is the set of line operators that flow to the elements of this basis?

Outline

- 3d moduli space of vacua
- 3d A-model: lightning review
- Grothendieck lines
- 2d limit and Schubert lines
- Work to appear/in progress

3d moduli space of vacua

So, we have 3d $\mathcal{N} = 2$ CS theory with gauge group $U(N_c)_{k,k+\ell N_c}$ coupled with n_f chiral multiplets in fundamental representation. Recall that:

$$\mathcal{V} = (\sigma, A_\mu, \lambda, \overline{\lambda}, D) \ ,$$
$$\Phi = (\phi, \psi, F) \ .$$

Semi-classically [Intriligator-Seiberg'13], the SUSY vacua are solutions to:

$$(\sigma_a - m_\alpha)\phi^a_\alpha = 0 \ , \quad a = 1, \cdots, N_c \ , \quad \alpha = 1, \cdots, n_f \ ,$$
$$\sum_{\alpha=1}^{n_f} \phi^{\alpha\ \dagger}_a \phi^b_\alpha = \frac{\delta^b_a}{2\pi} F_a(\sigma, m) \ , \quad a, b = 1, \cdots, N_c \ ,$$

up to $U(N_c)$. Here

$$F_a(\sigma, m) = \xi + k\sigma_a + \ell \sum_{b=1}^{N_c} \sigma_b + \frac{1}{2} \sum_{\alpha=1}^{n_f} |\sigma_a - m_\alpha| \ .$$

Taking $m_\alpha = 0$ and $\xi \neq 0$, we have the following possible solutions (depending on k and ℓ for fixed N_c and n_f) [Closset-OK'2305]:

- **Higgs vacua:** $\sigma_a = 0 \ , \forall a$. For $\xi > 0$, the equations describe $\mathrm{Gr}(N_c, n_f)$.

- **Topological vacua:** $\sigma_a \neq 0 \ , \forall a$. The matter multiplets become massive and we integrate them out leaving us with pure CS theory with gauge group:

$$U\underbrace{(p)_{k_{\text{eff}}} \times U(N_c - p}_{k_{\text{mix}}})_{\tilde{k}_{\text{eff}}} \ .$$

- **Hybrid vacua:** $\sigma_a = 0 \ ,$ for some a. This is a hybrid of the above two cases where the vacua are described by:

$$\mathrm{Gr}(p, n_f) \times U(N_c - p)_{k_{\text{eff}}} \ .$$

- **Strongly-coupled vacua:** this the case where a non-compact Coulomb branch opens up. Usually happens at $k = \pm \frac{n_f}{2}$.

$U(2)_{k,k+2\ell}$ with 4 $\square_2$

$k \backslash l$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
0	11	10	9	8	7	**6**	**6**	**6**	7	8	9	10	11
1	15	13	11	9	7	**6**	**6**	**6**	7	9	11	13	15
2	18	15	12	9	**6**	**6**	**6**	**9**	**12**	**15**	**18**	**21**	**24**
3	20	16	12	8	**6**	**6**	10	14	18	22	26	30	34
4	21	16	11	**6**	**6**	10	15	20	25	30	35	40	45
5	21	15	9	**6**	10	15	21	27	33	39	45	51	57
6	20	13	**6**	10	14	21	28	35	42	49	56	63	70
7	18	10	10	14	20	28	36	44	52	60	68	76	84
8	15	10	14	18	27	36	45	54	63	72	81	90	99
9	15	14	18	25	35	45	55	65	75	85	95	105	115
10	14	18	22	33	44	55	66	77	88	99	110	121	132

$U(2)_{k,k+2\ell}$ with 4 $\square_2$

(k,l)	$\xi > 0$ phase	$\xi < 0$ phase
$(0,10)$	$\mathrm{Gr}(2,4) \oplus U(2)_{-2,18}$	$U(2)_{2,22} \oplus \underbrace{U(1)_{12} \times U(1)_{8}}_{10}$
$(1,3)$	$\mathrm{Gr}(2,4) \oplus \underbrace{U(1)_{6} \times U(1)_{2}}_{3}$	$U(2)_{3,9}$
$(3,-2)$	$\mathrm{Gr}(2,4)$	$\mathbb{CP}^3 \times U(1)_{-1} \oplus U(2)_{5,1}$
$(4,7)$	$\mathrm{Gr}(2,4) \oplus \mathbb{CP}^3 \times U(1)_{9} \oplus U(2)_{2,16}$	$U(2)_{6,20}$
$(5,-6)$	$\mathrm{Gr}(2,4) \oplus U(2)_{7,-5}$	$\mathbb{CP}^3 \times U(1)_{-3} \oplus U(2)_{3,-9}$
$(6,-4)$	$\mathrm{Gr}(2,4)$	$U(2)_{4,-4}$
$(7,-9)$	$\mathrm{Gr}(2,4) \oplus U(2)_{9,-9}$	$\mathbb{CP}^3 \times U(1)_{-4} \oplus U(2)_{5,-13}$
$(8,8)$	$\mathrm{Gr}(2,4) \oplus \mathbb{CP}^3 \times U(1)_{14} \oplus U(2)_{6,22}$	$U(2)_{10,26}$
$(9,10)$	$\mathrm{Gr}(2,4) \oplus \mathbb{CP}^3 \times U(1)_{17} \oplus U(2)_{7,77}$	$U(2)_{11,31}$
$(10,5)$	$\mathrm{Gr}(2,4) \oplus \mathbb{CP}^3 \times U(1)_{13} \oplus U(2)_{8,18}$	$U(2)_{12,22}$

3d A-model: lightning review

Let us now put this theory on $\Sigma_g \times S^1_\beta$. We do so by performing a topological A-twist along Σ_g. Effectively, this is a 2d $\mathcal{N} = (2,2)$ theory on Σ_g with massive KK modes on the fibres. The path integral computes the twisted index [Nekrasov-Shatashvili'09, Benini-Zaffaroni'15, Closset-Kim'16]:

$$\mathcal{I}_{\text{3d}}(q, y) = \sum_{d\in\mathbb{Z}} q^d \mathrm{Tr}_{\mathcal{H}_{\Sigma_g;d}} \left((-1)^{\mathsf{F}} \prod_{i=1}^{n_f} y_\alpha^{Q_f^\alpha} \right) ,$$

where $q \sim e^{-2\pi\beta\xi}$ and $y_\alpha \sim e^{-2\pi\beta m_\alpha}$.
Upon SUSY localization, the correlation function of a collection of half-BPS line operators reduces to:

$$\langle \prod_p \mathcal{L}_{\mu_p} \rangle_{\Sigma_g \times S^1_\beta} = \sum_{\hat{x}\in\mathcal{S}_{\text{BE}}} \mathcal{H}^{g-1}(\hat{x}) \prod_p \mathcal{L}_{\mu_p}(\hat{x}) .$$

Where these ingredients are defined in terms of the effective twisted superpotential $\mathcal{W}$ and the effective dilaton potential Ω.

$$\mathcal{W} = \frac{1}{2\pi i}\sum_{\alpha=1}^{n_f}\sum_{a=1}^{N_c}\mathrm{Li}_2(x_a y_\alpha^{-1}) + \tau\sum_{a=1}^{N_c} u_a + \frac{k+\frac{n_f}{2}}{2}\sum_{a=1}^{N_c} u_a(u_a+1)$$
$$+\frac{\ell}{2}\left(\left(\sum_{a=1}^{N_c} u_a\right)^2 + \sum_{a=1}^{N_c} u_a\right),$$
$$e^{2\pi i\Omega} = \prod_{a=1}^{N_c}\prod_{i=1}^{n_f}(1 - x_a y_\alpha^{-1})\prod_{a\neq b}\left(1-\frac{x_a}{x_b}\right)^{-1} \quad .$$

Here we introduced $x_a = e^{2\pi i u_a} \sim e^{-2\pi\beta\sigma_a}$.
In terms of these, the handle-gluing operator is given by: [Vafa' 91, Nekrasov-Shatashvili' 14]

$$\mathcal{H}(x) = e^{2\pi i\Omega}\det\left(\frac{\partial^2\mathcal{W}}{\partial u_a\partial u_b}\right)$$

As for the sum, it is taken over the *Bethe vacua* [Hori-Tong'06, Nekrasov-Shatashvili'09]:

$$\mathcal{S}_{\mathrm{BE}} = \left\{ \hat{x} : e^{2\pi i \partial \mathcal{W}} \,|_{\hat{x}} = 1 \ , \quad \hat{x}_a \neq \hat{x}_b \ , \forall a \neq b \right\} / S_{N_c}$$

In our case, the BAE take the following form:

$$q(-\det x)^{\ell}(-x_a)^{k+\frac{n_f}{2}} \prod_{\alpha=1}^{n_f} (1 - x_a y_\alpha^{-1})^{-1} = 1 \ , \qquad \forall a = 1, \cdots, N_c \ .$$

Here $\det x = \prod_{b=1}^{N_c} x_b$.

For the case

$$k = N_c - \frac{n_f}{2} \ , \qquad \ell = -1 \ ,$$

we have the equivalence [Givental et al, Mihalcea et al]:

$$\mathcal{R}_{\mathrm{3d}} \cong \mathrm{QK}_{\mathrm{eq}}(\mathrm{Gr}(N_c, n_f)) \ .$$

For example, in this ring, the equivariant Schubert classes $[\mathcal{O}_\mu]$ are represented by the double Grothendieck polynomials $\mathfrak{G}_\mu(x,y)$ [Fulton-Lascoux'94, Ikeda-Naruse'13]

$$\mathfrak{G}_\mu(x,y) = \frac{\det_{1\leq a,b\leq N_c}\left(x_a^{b-1}\prod_{\alpha=1}^{\mu_b+N_c-b}(1-x_a y_\alpha^{-1})\right)}{\prod_{1\leq a<b\leq N_c}(x_a-x_b)} \ .$$

Therefore, assuming we know the half-BPS line operator $\mathcal{L}_\mu$ corresponding to $[\mathcal{O}_\mu]$, we can compute the ring structure of $\mathrm{QK}_{\mathrm{eq}}(\mathrm{Gr}(N_c,n_f))$:

$$g_{\mu\nu}(q,y) = \sum_{\hat{x}\in\mathcal{S}_{\mathrm{BE}}}\mathcal{H}^{-1}(\hat{x})\mathfrak{G}_\mu(\hat{x})\mathfrak{G}_\nu(\hat{x})$$

and,

$$\mathcal{N}_{\mu\nu\lambda}(q,y) = \sum_{\hat{x}\in\mathcal{S}_{\mathrm{BE}}}\mathcal{H}^{-1}(\hat{x})\mathfrak{G}_\mu(\hat{x})\mathfrak{G}_\nu(\hat{x})\mathfrak{G}_\lambda(\hat{x}) \ .$$

Indeed one can perform these sums efficiently using the Gröbner basis techniques [Jiang-Zhang'17, Closset-OK'2301].

As an example, one can do this computation for $\mathbb{CP}^2$. In this case, the topological metric has the following components:

$$g_{\mu,\nu} = \begin{cases} 1 - \frac{y_3}{y_1} + \frac{q}{1-q} \ , & (\mu,\nu) = (1,2),(2,1) \ , \\ \left(1 - \frac{y_3}{y_1}\right)\left(1 - \frac{y_3}{y_2}\right) + \frac{q}{1-q} \ , & (\mu,\nu) = (2,2) \ , \\ \frac{1}{1-q} \ , & \text{otherwise} \ . \end{cases}$$

Meanwhile, the ring structure of $\mathrm{QK}_{\mathrm{eq}}(\mathbb{CP}^2)$ is given by:

$$\mathcal{O}_1 \star \mathcal{O}_2 = \left(1 - \frac{y_2}{y_1}\right)\mathcal{O}_1 + \frac{y_2}{y_1}\mathcal{O}_2$$

$$\mathcal{O}_1 \star \mathcal{O}_2 = \left(1 - \frac{y_3}{y_2}\right)\mathcal{O}_2 + \frac{y_3}{y_2} q \ ,$$

$$\mathcal{O}_2 \star \mathcal{O}_2 = \left(1 - \frac{y_3}{y_1}\right)\left(1 - \frac{y_3}{y_2}\right)\mathcal{O}_2 + \frac{y_3}{y_1} q \mathcal{O}_1 + \left(1 - \frac{y_3}{y_2}\right)\frac{y_3}{y_1} q$$

These match the calculations of [Buch-Mihalcea'11].

Grothendieck lines

Now we come to answering the second question concerning the construction of the lines $\mathcal{L}_\mu$ that flow to $[\mathcal{O}_\mu]$. What are we looking for exactly?

We are looking for is a 1d $\mathcal{N} = 2$ theory that we can couple to our 3d theory such that:

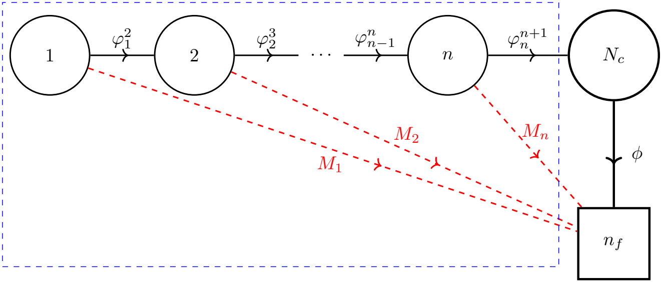
- The insertion of these lines (1d theories) at a point $z \in \mathbb{P}^1$ should restrict the target space X to the support of $[\mathcal{O}_\mu]$ which is the Schubert cell C_μ:

$$\phi(z) \in \mathsf{C}_\mu \ .$$

- If we compute the index of this 1d theory, we need to get the representative of $[\mathcal{O}_\mu]$:

$$\mathcal{I}_{1\mathsf{d}}[\mathcal{L}_\mu] = \mathfrak{G}_\mu(x, y) \ .$$

We propose that the 1d theory is the following quiver:[Closset-OK'2309]



The coupling to the 3d theory is established by introducing the 1d J-potential:

$$J^{(l)}_{\alpha^{(l)}} = \varphi_l^{l+1} \cdot \cdots \cdot \varphi_n^{n+1} \cdot \phi_{\alpha^{(l)}} \ ,$$

by adding the following term to the Lagrangian:

$$\int d\theta \sum_{l=1}^{n} \sum_{\alpha^{(l)} \in I_l} J^{(l)}_{\alpha^{(l)}}(\varphi, \phi)\, \Lambda^{(l)}_{\alpha^{(l)}}$$

Here $\Lambda^{(I)}_{\alpha^{(I)}}$ are 1d $\mathcal{N}=2$ Fermi multiplets. The number of these multiplets coupled at each 1d gauge node is determined in terms of the partition:

$$\mu = [\mu_1, \cdots, \mu_n, \underbrace{0, \cdots, 0}_{N_c - n}]$$

as follows:

$$M_I = \begin{cases} \mu_I - \mu_{I+1} + 1 \ , & I = 1, \cdots, n-1 \ , \\ \mu_n - n + N_c \ , & I = n \ . \end{cases}$$

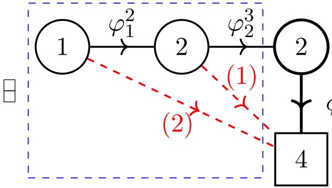
In terms of these numbers, the $SU(n_f)$ indices $\alpha^{(I)}$ live in the index set:

$$I_I = \{1 + \sum_{q=I+1}^{n} M_q, 2 + \sum_{q=I+1}^{n} M_q, \cdots, \sum_{q=I}^{n} M_q\}$$

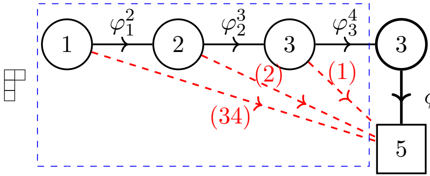
Along with the 1d D-term equations, the insertion of J-potential imposes the following constraints on ϕ:

$$J^{(I)}_{\alpha^{(I)}} = 0 \ .$$

Examples



$$\phi = \begin{pmatrix} 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{pmatrix}$$



$$\phi = \begin{pmatrix} 0 & 1 & 0 & \star & 0 \\ 0 & 0 & 1 & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Computing the 1d index

Following the localization analysis of [Hori-Kim-Yi'14], the index of the 1d theory is given by:

$$\mathcal{L}_\mu(x,y) = \oint_{\mathrm{JK}} \left[\prod_{l=1}^{n} \frac{1}{l!} \prod_{i_l=1}^{r_l} \frac{-dz_{i_l}^{(l)}}{2\pi i z_{i_l}^{(l)}} \prod_{1\le i_l \neq j_l \le l} \left(1 - \frac{z_{i_l}^{(l)}}{z_{j_l}^{(l)}}\right)\right] \mathrm{Z}^{\mathrm{1d}}_{\mathrm{matter}}(z,x,y)\,,$$

where,

$$\mathrm{Z}^{\mathrm{1d}}_{\mathrm{matter}}(z,x,y) \equiv \prod_{l=1}^{n-1}\prod_{i_l=1}^{l} \frac{\prod_{\alpha^{(l)}\in I_l}\left(1 - \frac{z_{i_l}^{(l)}}{y_{\alpha^{(l)}}}\right)}{\prod_{j_{l+1}=1}^{l+1}\left(1 - \frac{z_{i_l}^{(l)}}{z_{j_{l+1}}^{(l+1)}}\right)} \prod_{i_n=1}^{r_n} \frac{\prod_{\alpha^{(n)}\in I_n}\left(1 - \frac{z_{i_n}^{(n)}}{y_{\alpha^{(n)}}}\right)}{\prod_{a=1}^{N_c}\left(1 - \frac{z_{i_n}^{(n)}}{x_a}\right)}\,.$$

Taking the 1d FI parameters to be positive, the JK prescription instructs us to only consider the poles coming from the matter contribution. Doing so, indeed we get $\mathcal{L}_\mu(x,y) = \mathfrak{G}_\mu(x,y)$.

2d limit and $\mathrm{QH}^\bullet_{\mathrm{eq}}(\mathrm{Gr}(N_c, n_f))$

To move back to the 2d theory, we look at the $\beta \to 0$ limit. We recall that:

$$x_a \sim e^{-2\pi\beta\sigma_a} \ , \quad y_\alpha \sim e^{-2\pi\beta m_\alpha} \ , \quad q_{\mathrm{3d}} \sim (-2\pi\beta)^{n_f} q_{\mathrm{2d}} \ .$$

In this limit,

$$\mathfrak{G}_\mu(x, y) \to (2\pi\beta)^{|\mu|}\mathfrak{S}_\mu(\sigma, m) \ ,$$

where the double Schubert polynomials

$$\mathfrak{S}_\mu(\sigma, m) = \frac{\det_{1\le a,b\le N_c}\left(\prod_{\alpha=1}^{\mu_a+N_c-b}(\sigma_b - m_\alpha)\right)}{\prod_{1\le a<b\le N_c}(\sigma_a - \sigma_b)}$$

These polynomials are known to represent the equivariant Schubert classes in $\mathrm{QH}^\bullet_{\mathrm{eq}}(\mathrm{Gr}(N_c, n_f))$.

We can consider a 0d-2d coupled system similar to the 1d-3d above to construct the point defects $\mathcal{P}_\mu$ that correspond to Schubert classes $[\omega_\mu] \in \mathrm{H}^\bullet(\mathrm{Gr}(N_c, n_f))$.

The index of the gauged supersymmetric matrix model in this case is given by:

$$\omega_\lambda(\sigma, m) = \prod_{l=1}^{n} \frac{1}{l!} \oint \frac{d^l s^{(l)}}{(2\pi i)^l} \, \Delta^{(l)}(s) \, \mathrm{Z}^{\mathrm{0d}}_{\mathrm{matter}}(s, \sigma, m) \ ,$$

$$\Delta^{(l)}(s) = \prod_{i_l \neq j_l} \left(s^{(l)}_{i_l} - s^{(l)}_{j_l} \right) \ , \qquad (1)$$

and

$$\mathrm{Z}^{\mathrm{0d}}_{\mathrm{matter}}(\sigma, m) = \prod_{l=1}^{n-1} \left(\prod_{i_l=1}^{r_l} \frac{\prod_{\alpha^{(l)} \in I_l} \left(s^{(l)}_{i_l} - m_{\alpha^{(l)}} \right)}{\prod_{j_{l+1}=1}^{l+1} \left(s^{(l)}_{i_l} - s^{(l+1)}_{j_{l+1}} \right)} \right) \prod_{i_n=1}^{r_n} \frac{\prod_{\alpha^{(n)} \in I_n} \left(s^{(n)}_{i_n} - m_{\alpha^{(n)}} \right)}{\prod_{a=1}^{N_c} \left(s^{(n)}_{i_n} - \sigma_a \right)} \ .$$

Indeed, if we work out these integrals we end up with the double Schubert polynomial $\mathfrak{S}_\mu(\sigma, m)$ defined above

Work to appear/in progress

- In this talk we have focused on only one choice of (k, ℓ) from the geometric window. In a work to appear with C. Closset and H. Kim, we give an enumerative geometry interpretation for the other possible (k, l) in terms of the level structure construction of [Ruan-Zhang'19].
- In the same work, we will also give a physical realisation to the moduli space of stable maps on which GW theory acts [Bullimore-Ferrari-Kim'18].
- So far, we have been only focusing on the SQCD case. In a work in progress, we are extending this line construction to any partial flag variety.

Thank You!