Linear programming for hyperbolic surfaces

Joint work with Maxime Fortier Bourque, Émile Gruda-Mediavilla & Mathieu Pineault

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Séminaire Darboux

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Question: M a closed manifold, what is

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Known to be true if:

- $\dim(M) = 1$
- $\dim(M) \geq 3$ (**Colin-de-Verdière '86**)
- $\dim(M) = 2$ and $\chi(M) \geq 0$ (**Cheng '76, Besson '80, Colin-de-Verdière '87, Nadirashvili '87**)

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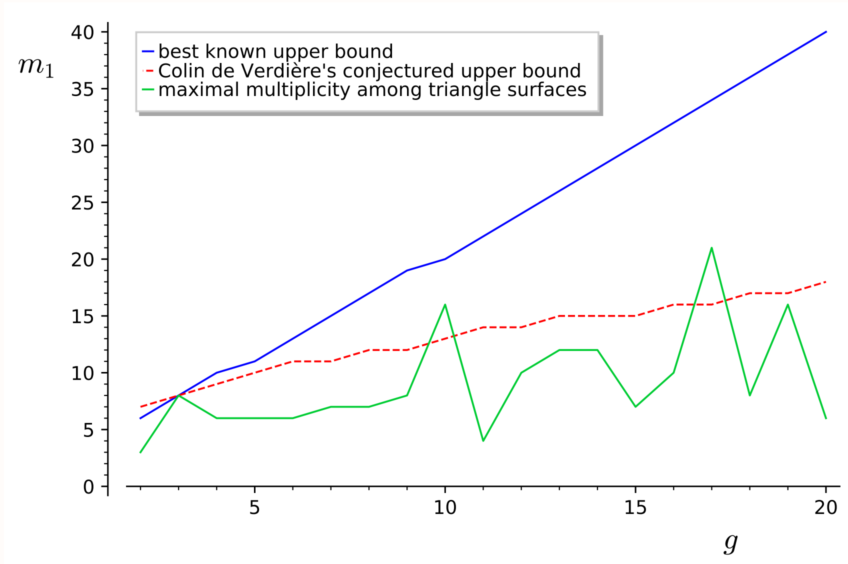
Ringel–Youngs '68: M a closed surface

$$\mathrm{chr}(M) = \left\lfloor \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)}) \right\rfloor$$

unless M is the Klein bottle, then $\mathrm{chr}(M) = 6$ (and not 7).

Counterexamples (Fortier Bourque–Gruda-Mediavilla–P.–Pineault '23)



Counterexamples to Colin-de-Verdière's conjecture: $m_1(X_{10}) = 16 > 13$ and $m_1(X_{17}) = 21 > 16$

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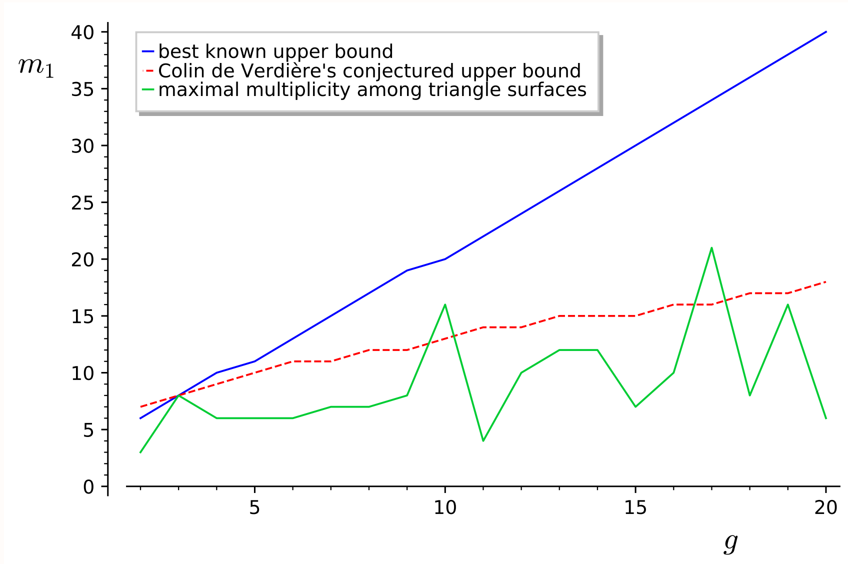

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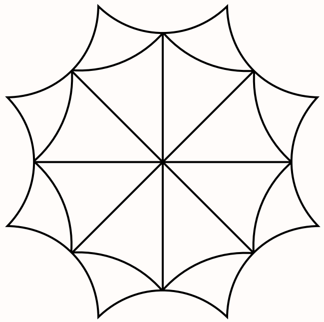
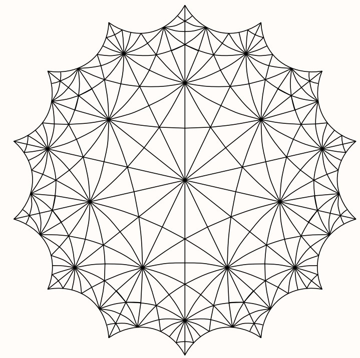
Remarks:

- If M closed orientable of genus g, $m_1(M) \leq 2g + 3$ (**Sévennec '02**)
- If M closed orientable hyperbolic of genus $g \gg 0$, $m_1(M) \leq 2g - 1$ (**Fortier Bourque–P. '23**)
- If sys $> \varepsilon$ and pinched negative curvature: sublinear upper bound (**Letrouit–Machado '23**)

Hyperbolic surfaces

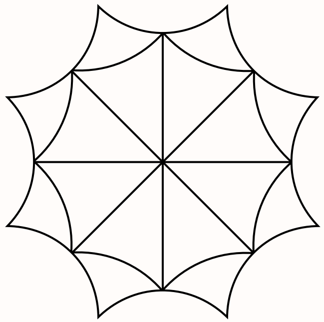


(a) The Bolza surface



(b) The Klein quartic

Hyperbolic surfaces



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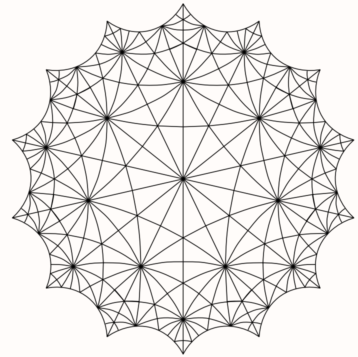

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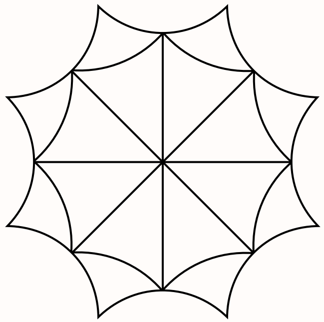

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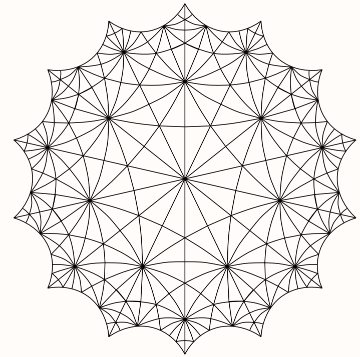

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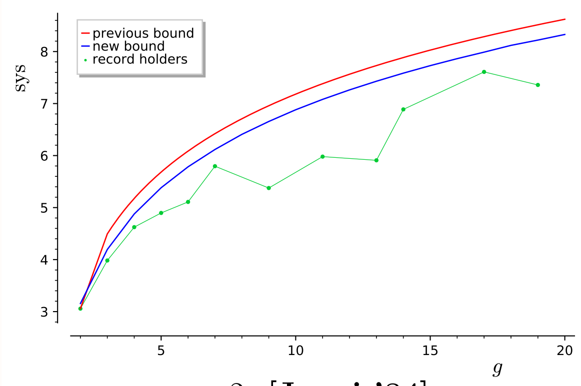
Question: Let $g \geq 2$: what are the maxima of systole(X), kiss(X), $\lambda_1(X)$ and $m_1(X)$ for $X \in \mathcal{M}_g$?

Lots of previous work: Huber '74, Cheng '75, Huber '76, Buser '77, Huber '80, Yang–Yau '80, Jenni '84, Burger Colbois '85, Brooks '88, Burger–Buser–Dodziuk '88, Colbois–Colin-de-Verdière '88, Burger '90, Schmutz '93, Schmutz '94, Buser–Sarnak '94, Bavard '96, Bavard '97, Schmutz-Schaller '97, Adams '98, Hamendstädt '01, Hamenstädt–Koch '02, Kim–Sarnak '03, Casamayou-Boucau '05, Katz–Schaps–Vishne '07, Otal '08, Gendulphe '09, Otal–Rosas '09, Parlier '13, Strohmaier–Uski '13, Fanoni–Parlier '15, Gendulphe '15, Cook '18, Petri–Walker '18, Petri '18, Hide–Magee '21, Jammes '21, Bonifacio '21, Kravchuk–Mazac–Pal '21, Wu–Xue '21, Lipnowski–Wright '21, Fortier Bourque–Rafi '22, Magee–Naud–Puder '22, Anantharaman–Monk '23, and many more.

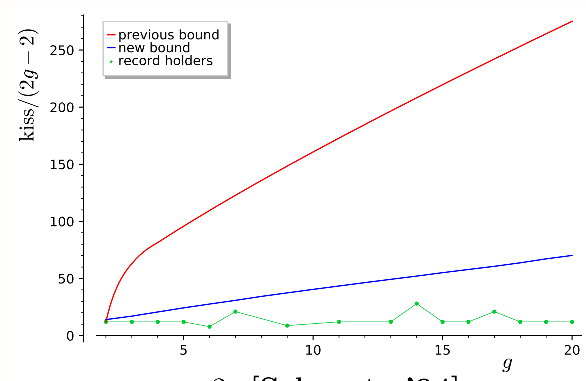
Known maximizers

	Systole	**Kissing number**	λ_1	$\mathbf{m_1}$
genus 2	Bolza surface **[Jenni '84]**	Bolza surface 24, **[Schmutz '94]**	Conjecture: Bolza surface	Conjecture: Bolza surface
genus 3	Conjecture: Picard curve	Conjecture: Picard curve	Conjecture: Klein quartic	Klein quartic **[Fortier Bourque –P. '21]**
genus ≤ 10	Local maximizers **[Schmutz '99]** **[Hamenstädt '01]**			
$g \to \infty$	$\gg g^{ag}$ local max. for some $a > 0$ **[Fortier Bourque –Rafi '22]**			

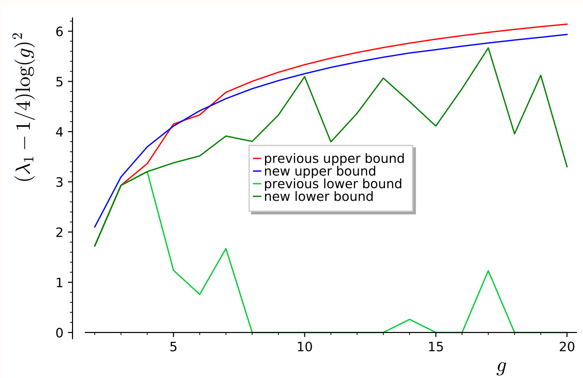
New bounds (Fortier Bourque – P. '23):



$g = 2$: [**Jenni '84**]



$g = 2$: [**Schmutz '94**]



$g = 2, 3$: [**Bonifacio '21**], [**Kravchuk–Mazac–Pal '21**],
$4, 6$: [**Yang–Yau '80**]

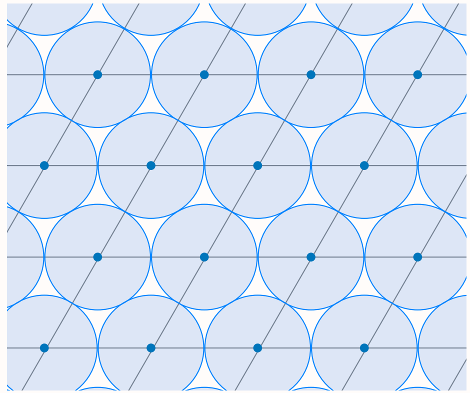
Large genus asymptotics: For $g \gg 0$:

$$\mathrm{systole}(X) < 2\log(g) + 2.409,$$

$$\mathrm{kiss}(X) < \frac{4.873 \cdot g^2}{\log(g) + 1.2045},$$

$$\lambda_1(X) < \frac{1}{4} + \left(\frac{\pi}{\log(g) + 0.7436}\right)^2,$$

Bounds before: **Bavard '96**, **Fortier Bourque–P. '22** (and **Parlier '13**), **Cheng '75** and **Huber '76**.



dim.	Density	Kissing number
1	Trivial	Trivial
2	**[Thue, Fejes-Tóth]**	Exercise
3	**[Hales]**	**[Newton,Gregory,...,Schütte–van der Waerden]**
4	?	**[Musin]**
8	**[Viazovska]**	**[Levenshteĭn,Odlyzko–Sloane]**
24	**[Cohn–Kumar–Miller–Radchenko–Viazovska]**	**[Levenshteĭn,Odlyzko–Sloane]** 196560

The Selberg trace formula:

Let X be a closed and oriented hyperbolic surface of genus g. Then

$$\sum_{n\geq 0}\widehat{f}(r_n) = (g-1)\cdot\int_{-\infty}^{+\infty}\widehat{f}(r)\cdot r\tanh(\pi r)dr + \sum_{\gamma\in\mathcal{G}(X)}\frac{\ell_0(\gamma)}{2\sinh(\ell(\gamma)/2)}f(\ell(\gamma))$$

where:

- if the eigenvalues of the Laplacian on X are $0=\lambda_0<\lambda_1\leq\lambda_2\leq\ldots$, then

$$r_n = \begin{cases} i\sqrt{\frac{1}{4}-\lambda_n} & \text{if } \lambda_n\leq\frac{1}{4} \\ \sqrt{\lambda_n-\frac{1}{4}} & \text{if } \lambda_n>\frac{1}{4}\end{cases}$$

- $\mathcal{G}(X)$ is the set of closed geodesics on X. Given $\gamma\in\mathcal{G}(X)$, $\ell(\gamma)$ is the length of γ and $\ell_0(\gamma)$ is the length of the unique primitive geodesic γ_0 for which there exists a $k\in\mathbb{N}$ such that $\gamma_0^k=\gamma$

- $(f,\widehat{f})$ is an admissible pair of functions: the function $f:\mathbb{R}\to\mathbb{C}$ is even and continuous, its Fourier transform $\widehat{f}(\xi)=\int_{\mathbb{R}}f(x)e^{i\xi x}dx$ is holomorphic and

$$\widehat{f}(\xi)=O\left((1+|\xi|^2)^{-1-\varepsilon}\right)\quad\text{on }\left\{\xi\in\mathbb{C};\ |\mathrm{Im}(\xi)|<\frac{1}{2}+\varepsilon\right\}\to\mathbb{C}$$

Triangle groups:

$$T(p,q,r) = \langle x, y, z \mid x^p = y^q = z^r = xyz = \mathrm{id} \rangle$$

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Subgroups:

$$\Lambda_{10} = \langle\langle\ (zyxz)^2 yz^{-1}xy^{-1}z^{-2}xz\ \rangle\rangle^{T(2,3,8)}$$

and

$$\begin{aligned}\Lambda_{17} = \langle\langle\ & z^{-3}xyz^{-3}, \\ & xzyxzyxy^{-1}z^{-1}xy^{-1}z^{-1}xy^{-1}z^{-1}xy^{-1}z^{-1}xy^{-1}z^{-1}xy^{-1}z^{-1}, \\ & zyxzyz^{-1}xzy^{-1}z^{-1}xy^{-1}z^{-2}xzy^{-1}z^{-1}xy^{-1}z^{-2}xz\ \rangle\rangle^{T(2,3,7)}\end{aligned}$$

Presentations found by **Conder**.

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Surfaces:

$$X_g = \Lambda_g \backslash \mathbb{H}^2, \quad g \in \{10, 17\}.$$

Fact: X_{10} is a 9-sheeted cover of the Bolza surface and X_{17} is an 8-sheeted cover of the Klein quartic.

Representation theory:

The dimensions of the real irreps of $G_{10} = T(2,3,8)/\Lambda_{10}$:

$$1, 1, 2, 3, 3, 4, 4, 4, 8, 8, 16$$

and the dimensions of the real irreps of $G_{17} = T(2,3,7)/\Lambda_{17}$:

$$1, 6, 6, 6, 7, 7, 7, 8, 14, 21, 21$$