Integrability of the finite temperature Airy and sine kernels (and beyond)

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Based on a work with Tom Claeys

Outline

Free fermions at finite temperature

[Dean, Le Doussal, Majumdar, Schehr 2016] Consider the same model of free fermions (non-interacting, harmonic potential), but at finite temperature $T > 0$. Then the PDF of the positions $x_1, \dots, x_N$ of the fermions is instead given by

$$P_T(x_1, \dots, x_N) = \frac{1}{Z_N(T^{-1})} \sum_{\substack{\mathbf{k}=(k_1,\dots,k_N) \\ k_1<\dots<k_N}} |\Psi_{\mathbf{k}}(x_1, \dots, x_N)|^2 e^{-T^{-1}(\sum_{i=1}^N \epsilon_{k_i})}$$

with

$$Z_N(T^{-1}) = \sum_{k_1<\dots<k_N} \mathrm{e}^{-T^{-1}(\sum_{i=1}^N \epsilon_{k_i})} \text{ and } \Psi_{\mathbf{k}}(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}} \det_{1\le i,j\le N} \varphi_{k_i}(x_j).$$

Remark The positions $x_1, \dots, x_N$ are no longer in correspondence with GUE eigenvalues. However, they are in correspondence with the eigenvalues of the Moshe-Neureberg-Shapiro model (of parameter b related to $e^{-T^{-1}}$).

Limiting behaviors at finite temperature

The limiting behavior of the positions for $N \to \infty$ is described by deformations of the Airy and sine dpp's (see also [Johansson 2005 - Lietchy, Wang 2018]).

- At the edge ($T \propto N^{1/3}$), the correlation kernel is

$$K^{\mathrm{Ai}}_{w_T}(x,y) = \int_{-\infty}^{+\infty} w_T(z)\mathrm{Ai}(x+z)\mathrm{Ai}(y+z)dz, \quad w_T(z) = \frac{1}{\mathrm{e}^{-cz}+1},$$

  and the limiting distribution of the rightmost position at the edge is

$$\lim_{N\to\infty} \mathbb{P}\left(\left(x_{\max} - \sqrt{2N}\right)\sqrt{2}N^{1/6} \le s\right) = \det(1 - \mathcal{K}^{\mathrm{Ai}}_{w_T}|_{(s,+\infty)}).$$

- In the bulk ($T \propto N$), the correlation kernel is

$$K^{\sin}_{\widetilde{w}_T}(x,y) = \int_{0}^{+\infty} \widetilde{w}_T(z)\cos(\pi(x-y)z)dz, \quad \widetilde{w}_T(z) = \frac{1}{\lambda\mathrm{e}^{-z^2}+1},$$

  and the limiting bulk spacing distribution is

$$\lim_{N\to\infty} \mathbb{P}\left(\{\sqrt{N}x_i\}_{i=1}^{N} \notin \left(-\frac{s}{\pi}, \frac{s}{\pi}\right),\right) = \det\left(1 - \mathcal{K}^{\sin}_{\widetilde{w}_T}|_{(-s,s)}\right).$$

Generic deformations of the kernels

$\rightsquigarrow$ For a smooth function $w : \mathbb{R} \to [0,1]$ fast decaying to zero at $-\infty$, the deformed Airy kernel

$$K^{\mathrm{Ai}}_{w,s}(x,y) = \int_{-\infty}^{+\infty} w(z)\mathrm{Ai}(x+s+z)\mathrm{Ai}(y+z+s)dz$$

and the associated Fredholm determinant $G_w(s) = \det(1 - \mathcal{K}^{\mathrm{Ai}}_{w,s}|_{(0,+\infty)})$.

$\rightsquigarrow$ For Schwartz function $w : \mathbb{R} \to [0,1]$, we consider the deformed sine kernel

$$K^{\sin}_w(x,y) = \int_{-\infty}^{\infty} w(u)\mathrm{e}^{2\pi i(x-y)u}\mathrm{d}u$$

and the associated Fredholm determinant $F_w(s) = \det\left(1 - \mathcal{K}^{\sin}_w|_{(-s,s)}\right)$.

Question

Does the Tracy-Widom formula hold for some generalizations of the distributions G_w, F_w?

Integrable structures

$\mathcal{K}^{\sin}_{w_s}$ and $\mathcal{K}^{\mathrm{Ai}}_{w,s}$ factorize as

$$\mathcal{K}^{\sin}_{w_s} = \mathcal{F}^* w_s \mathcal{F}, \quad \text{and} \quad \mathcal{K}^{\mathrm{Ai}}_{w} = \mathcal{A}_s^* w \mathcal{A}_s,$$

where w_s denotes the multiplication operator by $w(\frac{\cdot}{2s})$, $\mathcal{F}$ is the Fourier transform and $\mathcal{A}_s$ is the s-shifted Airy transform.

$$\downarrow$$

Thanks to that, composing with the projection operators and exploiting

$$\det(1 - AB) = \det(1 - BA)$$

we have

$$F_w(s) = \det\left(1 - \sqrt{w_s}\mathcal{K}^{\sin}\sqrt{w_s}\right), \quad \text{and} \quad G_w(s) = \det\left(1 - \sqrt{w}\mathcal{K}^{\mathrm{Ai}}_s\sqrt{w}\right)$$

where $\sqrt{w_s}$ denotes the multiplication operator with a square root of the function $w(\frac{\cdot}{2s})$ and $\mathcal{K}^{\mathrm{Ai}}_s$ acts through the s-shifted Airy kernel $K^{\mathrm{Ai}}(x+s, y+s)$.

Remark This identity allows to interpret $G_w(s)$ and $F_w(s)$ respectively as gap probabilities for the w-thinned Airy and sine processes.

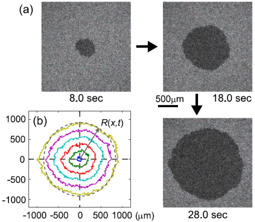
Outline

The KPZ equation

The Kardar-Parisi-Zhang equation is a stochastic PDE

$$\partial_T h(X,T) = \frac{1}{2}\partial_X^2 h(X,T) - \frac{1}{2}(\partial_X h(X,T))^2 + \xi(X,T)$$

where $\xi(X,T)$ is a Gaussian space-time white noise. It happens to govern many different random growth models.



Picture from [Takeuchi - Sano, 2018]: growing interfaces in turbolent liquid cristals.

[Amir - Corwin - Quastel, 2011] The probability distribution function of the Hopf-Cole solution of the KPZ equation $h(X,T) = -\log Z(X,T)$ with narrow wedge initial condition $Z(X,0) = \delta_0(X)$ is given by

$$\mathbb{P}\left(h(X,T) - \frac{X^2}{2T} - \frac{T}{24} \geq -s\right) = \int_\Gamma \det\left(1 - \mathcal{K}^{\mathrm{Ai}}_{w_{KPZ},0}\big|_{((T/2)^{-1/3}a(s),+\infty)}\right)\frac{e^{-\tilde{z}}}{\tilde{z}}d\tilde{z}$$

where $a(s) = s - \log\sqrt{2\pi T}$, Γ is a (properly defined) contour and

$$w_{KPZ}(z) = \frac{\tilde{z}}{\tilde{z} - e^{-(T/2)^{1/3}z}}.$$

Generalization of the Tracy-Widom formula

[Amir - Corwin - Quastel, 2011] Generalization of the Tracy-Widom formula as

$$\frac{d^2}{ds^2} \log G_{w_{KPZ}}(s) = -\int_{\mathbb{R}} \varphi^2(r;s) w'_{KPZ}(r) dr$$

where φ is solution of the so called integro-differential Painlevé II equation

$$\frac{\partial^2}{\partial s^2}\varphi(z;s) = \left(z + s + 2\int_{\mathbb{R}} \varphi^2(r;s) w'_{KPZ}(r) dr \right) \varphi(z;s).$$

with $\varphi(z;s) \sim \mathrm{Ai}(z+s)$ per $s \to +\infty$ pointwise in z.

Remark 1 This characterization of G_w for a large class of functions w has been proved through different methods [Krajenbrink, 2020], [Cafasso - Claeys - Ruzza, 2021], [Bothner, 2021].

Remark 2 [Bothner - Little, 2022] The edge limiting (weak non-hermiticity limit) behavior of the real parts of eigenvalues of the Complex Elliptic Ginibre ensemble is described by G_w with $w(r) = \Phi(s\sigma^{-1}(r+1)) - \Phi(s\sigma^{-1}(r-1))$ and $\Phi(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} \mathrm{e}^{-y^2} dy$ (and σ a parameter of the model).

Generalization of the Jimbo-Miwa-Mori-Sato formula

Theorem (Claeys - T., 2023)

For every $s > 0$, we have the identity

$$\partial_s s \partial_s \log F_w(s) = \frac{1}{\pi} \int_{\mathbb{R}} \lambda w'(\lambda) \phi(\lambda; s) \psi(\lambda; s) \mathrm{d}\lambda,$$

where ϕ, ψ solve the (Zakharov-Shabat) system of equations

$$\partial_s \phi(\lambda; s) = \mathrm{i}\lambda \phi(\lambda, s) - \frac{1}{2\pi \mathrm{i} s} \int_{\mathbb{R}} \phi^2(\mu; s) w'(\mu) \mathrm{d}\mu \; \psi(\lambda; s),$$

$$\partial_s \psi(\lambda; s) = \frac{1}{2\pi \mathrm{i} s} \int_{\mathbb{R}} \psi^2(\mu; s) w'(\mu) \mathrm{d}\mu \; \phi(\lambda; s) - \mathrm{i}\lambda \psi(\lambda; s).$$

with $\lambda \to \pm\infty$ asymptotics $\phi(\lambda; s) \sim \mathrm{e}^{\mathrm{i}s\lambda}, \psi(\lambda; s) \sim \mathrm{e}^{-\mathrm{i}s\lambda}$.

Remark If w is even, then

$$\partial_s s \partial_s \log F_w(s) = \frac{1}{\pi} \int_{\mathbb{R}} \lambda w'(\lambda) \phi(\lambda; s) \phi(-\lambda; s) \mathrm{d}\lambda,$$

where ϕ solves the integro-differential equation

$$\partial_s \phi(\lambda; s) = \mathrm{i}\lambda \phi(\lambda; s) - \frac{1}{2\pi \mathrm{i} s} \int_{\mathbb{R}} \phi^2(\mu; s) w'(\mu) \mathrm{d}\mu \; \phi(-\lambda; s).$$

Reduction to Painlevé V

Back to the case $w(r) = \chi_{\left(-\frac{1}{2},\frac{1}{2}\right)}$, $w'(r) = \delta_{-\frac{1}{2}}(r) - \delta_{\frac{1}{2}}(r)$.

Then our integro-differential equations reduce to

$$\partial_s \phi\left(\pm\frac{1}{2}; s\right) = \pm\frac{\mathrm{i}}{2}\phi\left(\pm\frac{1}{2}; s\right) - \frac{1}{2\pi \mathrm{i} s}\left(\phi^2\left(-\frac{1}{2}; s\right) - \phi^2\left(\frac{1}{2}; s\right)\right)\phi\left(\mp\frac{1}{2}; s\right)$$

and by defining

$$v(x) = \frac{1}{2\pi\mathrm{i}}\phi\left(\frac{1}{2}; \frac{x}{2\mathrm{i}}\right)\phi\left(-\frac{1}{2}; \frac{x}{2\mathrm{i}}\right), \qquad u(x) = \frac{\phi^2\left(\frac{1}{2}; \frac{x}{2\mathrm{i}}\right)}{\phi^2\left(-\frac{1}{2}; \frac{x}{2\mathrm{i}}\right)},$$

we recover the system

$$xv' = v^2\left(u - \frac{1}{u}\right), \quad xu' = xu - 2v(u-1)^2$$

implying that u solves the Painlevé V equation

$$u'' = \frac{u}{x} - \frac{u'}{x} - \frac{u(u+1)}{2(u-1)} + (u')^2\frac{3u-1}{2u(u-1)}.$$

Moreover, the JMMS formula is recovered by $\nu'(s) = -\frac{1}{\pi}\phi\left(\frac{1}{2}; s\right)\phi\left(-\frac{1}{2}; s\right)$.

Connection with the Korteweg-de Vries equation for Airy

[Cafasso - Claeys - Ruzza, 2021] For every w within the class, the function

$$u(x,t) = \partial_x^2 \log G_w(x,t) + \frac{x}{2t}$$

for $G_w(x,t)$ the deformed Fredholm determinant associated to

$$t^{2/3} K^{\mathrm{Ai}}_{w_t, xt^{-1}}(t^{2/3}\cdot, t^{2/3}\cdot),$$

with $w_t(\lambda) = w(t^{2/3}\lambda)$, solves the Korteweg-de Vries equation

$$\partial_t u + 2u\partial_x u + \frac{1}{6}\partial_x^3 u = 0.$$

The dependence on w_t of the solution of the KdV equation $u(x,t)$ is described in different asymptotic (x,t) regimes [Charlier - Claeys - Ruzza, 2022].

Remark For $w = \chi_{(0,+\infty)}$, the Tracy-Widom formula gives

$$u(x,t) = -t^{-2/3} q^2(-xt^{-1/3}) + \frac{x}{2t}$$

which is an instance of the self-similarity relation between the Painlevé II equation and the KdV equation.

Connection with a *new* PDE for sine

Let $W : \mathbb{R} \to \mathbb{C}$ be smooth and decaying fast at $+\infty$, such that the even function $w^{(y)}(\lambda) = W(\lambda^2 - y)$ is a Schwartz function.

With a change of variable we have

$$F_{w^{(y)}}(s) = \det\left(1 - \mathcal{K}^{\sin}_{w^{(y)}}|_{[-\frac{s}{2\pi}, \frac{s}{2\pi}]}\right) = \det\left(1 - \mathcal{K}^{\sin}_{w_{y,s}}|_{[-1/2,1/2]}\right) = Q_W(y,s)$$

with $w_{y,s}(\zeta) = W\left(\frac{\pi^2\zeta^2}{s^2} - y\right)$.

Theorem (Claeys - T., 2023)

The function $\sigma = \sigma_W = \log Q_W(y,s)$ *solves the PDE*

$$(\partial_s^2 \partial_y \sigma)^2 = 4\partial_s^2\sigma\left(-2s\partial_s\partial_y\sigma + 2\partial_y\sigma - (\partial_s\partial_y\sigma)^2\right).$$

Remark For the specific choice $W(r) = \frac{1}{e^{4r}+1}$, this result was found in the original paper of [Its - Izergin - Korepin - Slavnov, 1991].

Initial boundary value problem for σ

Theorem (Claeys, T. 2023)

Let $f:\mathbb{R}\to\mathbb{C}$ be C^∞ and decaying fast at $-\infty$, such that $f(y-.^2)$ is a Schwartz function for all $y\in\mathbb{R}$. Then the initial value problem for the σ-PDE with initial data

$$\lim_{s\to 0}\frac{1}{s}\sigma_W(y,s)=f(y)$$

is solved by $\sigma_W(y,s)=\log Q_W(y,s)$, with

$$W(r)=-2\int_0^\infty f'(-u^2-r)\mathrm{d}u.$$

Remark Notice that the W, which can be interpreted as *scattering data*, and the initial data f are related by a very simple integral transformation.

Outline

Summary and final remarks

Conclusion

Both F_w, G_w admit charcaterization in terms of some integro-differential analogues of the Painlevé equations and they are further related to solutions of integrable PDEs.

- The sine deformed kernel for $W = \chi_I \widetilde{w}$ appeared also in the description of the dynamical evolution for large time of a free fermionic model with delta impurity [Gouraud, Le Doussal, Schehr, 2022]. It would be interesting to understand how our results extend to this degenerate case.
- In [Bothner - Little, 2022] another characterization of G_w which generalizes the alternative version of the Jimbo-Miwa-Mori-Sato formula, but the explicit relation between these two formulas (well known in the classical setting) is not clear yet. Investigation of the Hamiltonian formulation of these equations should help.
- In the w-thinned DPPs, apart from gap probabilities (alias Fredholm determinants) we can also study the Janossy densities : for the Airy case [Claeys - Glesner - Ruzza - T., 2023].

New *finite-temperature* deformed kernels

For given Σ, ℓ specific contours in the complex plane, $N \geq 1$ and $s, \tau \in \mathbb{R}$ we consider the class of kernels

$$H^{\sigma}_{N,s,\tau}(y, y') = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{R}} \sigma(x) \int_{\Sigma} du \int_{\ell} dv \mathrm{e}^{(y+x+s)u-(y'+x+s)v} \mathrm{e}^{\frac{\tau}{2}(v^2-u^2)} \frac{W_N(v)}{W_N(u)} dx,$$

that for $\sigma(x) = \sigma_t(x) = (1 + \mathrm{e}^{-x-t})^{-1}$ and $W_N(z) = \frac{1}{\prod_{k=1}^{n} \Gamma(z-a_k)}$ are related to the partition function of the O'Connel-Yor polymer
[Borodin - Corwin, 2014] , [Imamura - Sasamoto, 2016] but can be defined for large class of such functions and related to general biorthogonal ensembles
[Cafasso - Claeys, 2024].

Remark This kernel can be thought as a *finite temperature* deformation of

$$L_{N,s,\tau}(y, y') = \frac{1}{(2\pi \mathrm{i})^2} \int_{\Sigma} du \int_{\ell} dv \frac{\mathrm{e}^{(y+s)u-(y'+s)v} \mathrm{e}^{\frac{\tau}{2}(v^2-u^2)}}{(v-u)} \frac{W_N(v)}{W_N(u)}$$

since it is recovered by the choice $\sigma(x) = \chi_{(0,+\infty)}(x)$.

Work in progress with M. Cafasso and T. Claeys

We have that the Fredholm determinant

$$M(s,\tau) = \det(1 - \mathcal{L}_{N,s,\tau}\chi_{(0,+\infty)})$$

is characterized by a solution of the coupled nonlinear Schrodinger equations by

$$\frac{\partial^2}{\partial s^2} \log M(s,\tau) = q(s,\tau) r(s,\tau).$$

where

$$\frac{\partial}{\partial \tau} q = q^2 r - \frac{1}{2}\frac{\partial^2}{\partial s^2} q,$$
$$\frac{\partial}{\partial \tau} r = -q r^2 + \frac{1}{2}\frac{\partial^2}{\partial s^2} r.$$

Questions

- boundary value problems?
- comparison with other solutions that already appeared e.g. in relation with weak noise theory of KPZ [Krajenbrink - Le Doussal, 2022].
- New solutions for the finite-temperature case ?

Thank you!